Probabilistic Fracture Analysis of Functionally Graded Materials - Part II: Implementation and Numerical Examples

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Abstract. Probabilistic fracture analyses are performed for investigating uncertain fracture response of Functionally Graded Material (FGM) structures. The First-Order-Reliability-Method (FORM) is implemented into an existing Finite Element code for FGM (FE-FGM), which was previously developed at the University of Illinois at Urbana-Champaign [2]. The computational simulation will be used in order to estimate the probability of crack initiation with uncertainties in the material properties only. The two-step probability analysis method proposed in the companion paper (Part I, [1]) is illustrated by a numerical example of a composite strip with an edge crack. First, the reliability index of a crack initiation event is estimated as we vary the mean and standard deviation of the slope and the location of the inflection point of the spatial profile of Young's modulus. Secondly, the reliability index is estimated as we vary the standard deviation and the correlation length of the random field that characterize the random spatial fluctuation of Young's modulus. Also investigated is the relative importance of the uncertainties in the toughness compared to those in Young's modulus.

Keywords: Crack initiation, first-order reliability method, fracture analysis, fracture toughness, J-integral, material uncertainty, probabilistic analysis, random field

INTRODUCTION

In a companion paper (Song et al. [1]), a probabilistic fracture analysis method is presented for predicting uncertain fracture responses of functionally graded material (FGM) structures. By integrating a structural reliability method with finite-element (FE) analysis for fracture mechanics including evaluation of the J-integral, one can estimate the probability of crack initiation efficiently. The companion paper [1] also proposes a two-step procedure to investigate the effects of various material uncertainties on the likelihood of crack initiation using both smooth and fluctuant gradation profile models of material property.

In this paper, we apply the probabilistic method and procedure to an FGM composite strip with an edge crack as a numerical example. We primarily consider the uncertainties in material properties including Young's modulus and fracture toughness (J_k) to investigate their effects on the probability of crack initiation. The limit state function is defined for a Mode I crack initiation event in terms of the J-integral. The First-Order-Reliability-Method (FORM) is used in conjunction with an existing FE code for FGM (FE-FGM), which was previously developed at the University of Illinois at Ur-

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bana-Champaign [2]. As the first step of the proposed two-step uncertainty investigation, the reliability index of a crack initiation event is estimated as we vary the mean and standard deviation of the slope and also the location of the inflection point of the smooth spatial profile of Young's modulus. Secondly, the reliability index is estimated as we vary the standard deviation and the correlation length of the random field that characterize the spatial fluctuation of Young's modulus. Also investigated is the relative importance of the uncertainties in the toughness compared to those in Young's modulus.

IMPLEMENTATION FOR PROBABILISTIC FRACTURE ANALYSIS

Consider the probability of crack initiation (see companion paper, [1]). If FORM is used to estimate the probability, we first transform the original random variables $\mathbf{X}$ to the uncorrelated standard normal random variables $\mathbf{U} = \mathbf{U}(\mathbf{X})$ and find the point on the limit state surface $G(\mathbf{u}) = g[\mathbf{x}(\mathbf{u})] = 0$ that is nearest to the origin – so called "design point" or "most probable point." Then, the probability is estimated approximately as the probability content in the half-space determined by the limit-state surface linearized at the design point. The distance between the design point and the origin is often called "reliability index" and denoted by β. Then, the first-order approximation of the probability of crack initiation is

$$P_f \cong \Phi(-\beta) \tag{1}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function (CDF) of the standard normal distribution. Moreover, FORM finds a design point by employing an efficient optimization algorithm, such as the improved HL-RF algorithm [3]. This requires the value of the limit-state function $g(\mathbf{x})$ and its gradients with respect to random variables at each step of optimization iterations. In this paper, we implement the improved HL-RF algorithm in conjunction with the FE-FGM code for probabilistic fracture analysis.

NUMERICAL EXAMPLE

As a numerical example, we consider an FGM composite strip with an edge crack [4]. Figure 1 in the companion paper [1] shows the configuration and the mesh details. A uniform displacement Δ = 0.002 m is applied to the top edge, resulting in uniform strain $\varepsilon_{22} = 0.001$. Plane strain assumption is used in the finite element analysis. Poisson's ratio, $\nu = 0.3$ is assumed to be deterministic and constant over the thickness.

In the first study, we assume the gradation of Young's modulus follows a smooth, hyperbolic tangent function of X_1:

$$E(X_1) = \frac{E_1 + E_2}{2} + \frac{E_1 - E_2}{2} \tanh\left[b(X_1 - a)\right] \tag{2}$$

where b and a represent the slope and the location of the inflection point of the gradation profile, respectively. For given (a,b), we find E_1 and E_2 such that Young moduli at the left and right edge are 3 MPa and 1 MPa, respectively. Some example gradation profiles are illustrated in the companion paper [1]. We consider three random variables $\mathbf{X} = \{a, b, J_k\}^{\mathrm{T}}$ to represent the uncertainties in the gradation profile and the fracture toughness. Reliability indices are obtained for wide ranges of the mean and standard deviation of the random variables to investigate their effects on the probability. Table 1 summarizes the ranges of the means and standard deviations or coefficient of variations (c.o.v.) of the random variables for which probabilistic fracture analyses are performed.

TABLE 1. Random Variables and Probabilistic Information

Random Variables	Mean	Coefficient of Variation	Standard Deviation	Distribution Type
Slope of Gradation, b	1~5	0.05~0.25		Lognormal
Inflexion Point Location, a	–0.4~0.4		0.015~0.2	Normal
Fracture Toughness, J_k	(0.5~1.4) J/m	0.1~1.0		Lognormal
Young's Modulus at x_i	Eq. (2)	0.05~0.25		Normal

For analyses based on fluctuant material gradation model ("2nd step"), the uncertainties in material properties are presented by a random field. This random field consists of functions that describe the spatial distribution of the mean, standard deviation and correlation coefficient of Young's modulus. At selected discretized points whose locations are denoted by $x_1, \ldots, x_n$, we assign random variables $E(x_1), \ldots, E(x_n)$ based on the functions of the random field. In the computer code FE-FGM, Young's moduli at Gauss points in each element are calculated from random moduli at the discretized points using shape functions. As a result, we consider random variables, $\mathbf{X} = \{E(x_1), \ldots, E(x_n), J_k\}^{\mathrm{T}}$ for FORM analyses. The smooth model in Eq. (2) is used as the mean function of the random field while the range of the constant c.o.v. function is listed in Table 1. The constant correlation length $\rho_{\Delta x}$ varies from 0.15 to 0.40.

UNCERTAINTY ANALYSIS BY GRADATION PROFILE

Figure 1(a) shows the reliability indices of crack initiation events for various mean values of a and b, estimated by FORM analyses employing the FE-FGM code. The c.o.v. of the fracture toughness is assumed to be 47% in these analyses. For any given mean value of the slope, reliability indices decrease as the mean location of the inflection point moves toward right-hand side. By contrast, the effect of the mean slope depends on the location of the inflection point. If the inflection point is within a crack zone, i.e., $\mathrm{E}[a] < -0.1$, a stiffer gradation profile reduces the probability of crack initiation. If the inflection point is located outside the crack, i.e., $\mathrm{E}[a] > -0.1$, a stiffer gradation increases the probability. It is seen that the probability increases as profiles by $\mathrm{E}[a]$ and $\mathrm{E}[b]$ leads to a large Young's modulus at the crack tip point. This is due to the fact that a large Young's modulus at the crack tip point increases the J-integral value, which reduces the Mode I limit-state function (see companion paper, [1]).

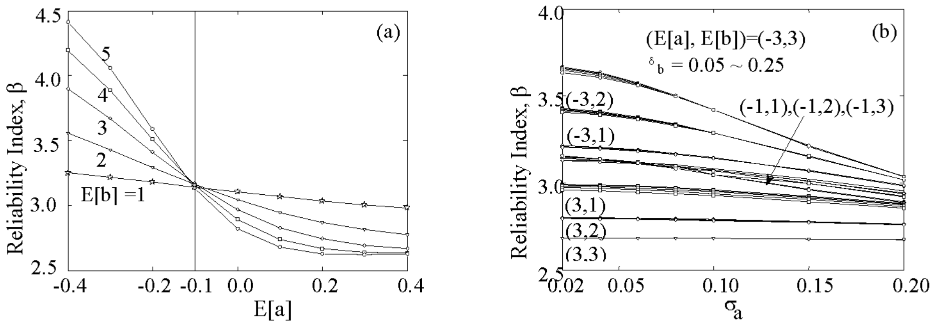

FIGURE 1. Reliability indices of crack initiation events with various (a) means and (b) standard deviations (or c.o.v.'s) of slope and location of smooth gradation profile.

The effect of variability of "a" and "b" are also investigated. Figure 1b shows the reliability indices as we vary the standard deviation of "a" (σ_a) from 0.02 to 0.20 and the c.o.v. of "b" (δ_b) from 0.05 to 0.25. The FORM analyses are performed for nine different combinations of the mean values of "a" and "b" to verify how the effects of σ_a and δ_b depend on the mean values. As shown in Figure 1b, the uncertainty in the slope does not affect the probability significantly. The uncertainty in the location of the inflection point increases the probability of crack initiation.

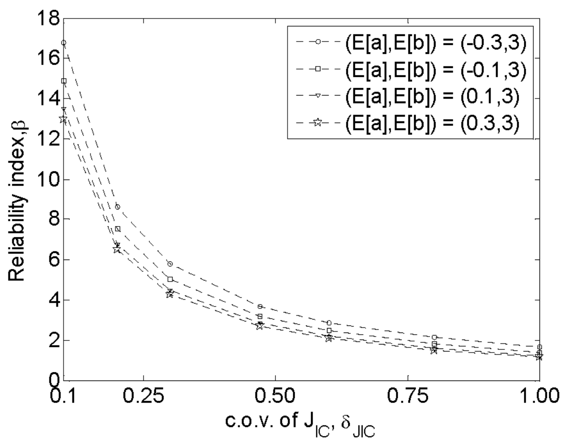

Figure 2. Reliability indices with c.o.v. of fracture toughness

We also investigate the effect of the uncertainty in the fracture toughness on the crack initiation probability as well. Figure 2 shows the reliability indices with the c.o.v. of the fracture toughness between 0.1 and 1.0 while E[J_k] = 1.24 J/m. The probability decreases rapidly as the uncertainty in J_k decreases. It is seen that the crack initiation probability is much more sensitive to the uncertainty in J_k than to the uncertainties in "a" and "b". Therefore, the uncertainty and probability of the crack initiation event can be reduced efficiently if the uncertainty in the fracture toughness can be reduced.

UNCERTAINTY ANALYSIS BY FLUCTUATING PROFILE

Young's modulus profile is fluctuated around it's mean value. The random variables are $X = \{E(x_1), E(x_2), ..,E(x_{21}), J_k\}$. The effects of cov_E and correlation length $\rho(\Delta x)$ on reliability index β are investigated. The Figure 3a shows the relationship between β and cov_E. It can be seen that when cov_E varies from 0.05 to 0.25, with certain values of (a,b), β decreases. This is reasonable since higher cov_E causes more extreme value of E, then β decreases. The correlation length varies from 0.15 to 0.40. It can be seen from Figure 3b that the correlation length has little effect on β. It is due to the fact that the effect of J_k is dominant.

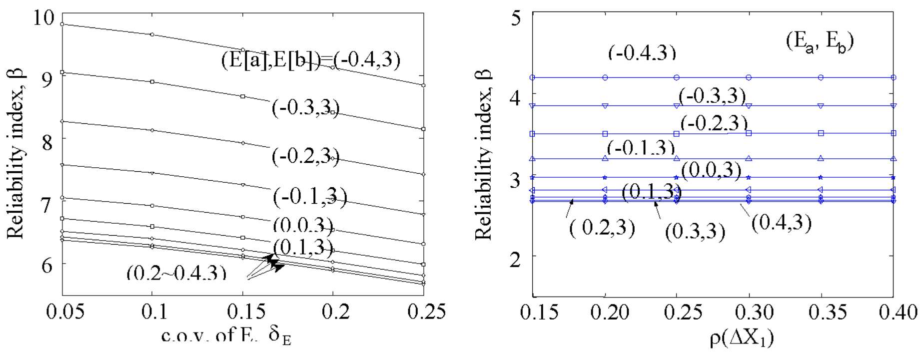

FIGURE 3. (a) Reliability indices with c.o.v. of random field for Young's modulus; (b) Reliability indices with correlation length

6. CONCLUSION

Preliminary probabilistic fracture analyses are performed for investigating uncertain fracture response of Functionally Graded Material (FGM) structures. The First-Order-Reliability-Method (FORM) is implemented into an existing Finite Element code for FGM (FE-FGM) by Song and Paulino [2]. The computational simulation is used to estimate the probability of crack initiation with uncertainties in the material properties only. The two-step probability analysis method proposed in the companion paper (Part I) is illustrated by a numerical example of a composite strip with an edge crack. Also investigated is the relative importance of the uncertainties in the toughness compared to those in Young's modulus. Future work includes consideration of uncertainty on the crack length and consideration of other methods of analysis such as importance sampling, or second-order reliability method.

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REFERENCES

1. J. Song, T. H. Nguyen, and G. H. Paulino, "Probabilistic fracture analysis of functionally graded materials – Part I: uncertainty and probabilistic analysis method," *Proceedings of the Multiscale and Functionally Graded Materials Conference 2006* (FGM2006), Kapolei, HI, October 15-18 (2006).
2. S. H. Song and G. H. Paulino, *International Journal of Solids and Structures,* **43**(16), 4830-4866 (2006).
3. Y. Zhang, and A. Der Kiureghian, "Two improved algorithms for reliability analysis," *Reliability and Optimization of Structural Systems, Proceedings of the 6th IFIP WG 7.5 working conference on reliability and optimization of structural systems* (eds. R. Rackwitz, G. Augusti, A. Borri), 297-304 (1995).
4. J. H. Kim and G.H. Paulino, *International Journal for Numerical Methods in Engineering*, **58**, 1457-1497 (2003).