Probabilistic Fracture Analysis of Functionally Graded Materials - Part I: Uncertainty and Probabilistic Analysis Method

Junho Song, Tam H. Nguyen and Glaucio H. Paulino

Department of Civil and Environmental Engineering, University of Illinois, Urbana-Champaign
Urbana, Illinois 61801

Abstract. Probabilistic fracture analysis is performed for predicting uncertain fracture responses of Functionally Graded Material (FGM) structures. The uncertainties in material properties including Young's modulus and fracture toughness are considered. The limit state function for a crack initiation event is defined in terms of the J-integral for FGMs. The First-Order-Reliability-Method (FORM) is used in conjunction with a finite element code that computes the J-integral with high accuracy. A two-step probabilistic analysis procedure is proposed to investigate the effects of the uncertainties in the spatial distribution of Young's modulus on the probability of crack initiation in FGMs. First, we investigate the effects of the uncertainties in the shape of the spatial distribution by considering the slope and the location of the inflection point of a spatial distribution profile as random quantities. Second, we investigate the effects of the spatial fluctuations of Young's modulus by making use of a discretized random field. The companion paper (Part II) implements this method into a finite element fracture analysis code and presents numerical examples.

Keywords: Crack Initiation, First Order Reliability Method, Fracture Toughness, Functionally Graded Material, J-Integral, Material Uncertainty, Probabilistic Fracture Analysis, Random Field.

INTRODUCTION

Functionally graded materials (FGMs) possess spatially varied microstructures and macroproperties. Traditionally, fracture of FGMs has been analyzed by means of deterministic approaches [2, 3]. Previous investigations of probabilistic fracture include the work by Grigoriu et al. [4] and Rao and Rahman [5], which emphasize homogeneous materials; and the work by Rahman and Rao [6], and Ferrante and Graham-Brady [8] which emphasizes FGMs.

This paper presents a probabilistic fracture analysis method to investigate the effects of uncertainties on the fracture responses of FGM structures. The uncertainties in material properties including Young's modulus and fracture toughness are considered. The limit state function for a crack initiation event is defined in terms of the J-integral for FGMs. The First-Order-Reliability-Method (FORM) is used in conjunction with a finite element code that computes the J-integral with high accuracy. A two-step probabilistic analysis procedure is proposed to investigate the effects of the uncertainties in the spatial distribution of Young's modulus on the

CP973, *Multiscale and Functionally Graded Materials 2006*
edited by G. H. Paulino, M.-J. Pindera, R. H. Dodds, Jr., F. A. Rochinha, E. V. Dave, and L. Chen

probability of crack initiation in FGMs. The companion paper (Part II, Nguyen et al. [1]) implements this method into a finite element fracture analysis code and presents numerical examples.

UNCERTAINTIES IN FRACTURE ANALYSIS OF FGM

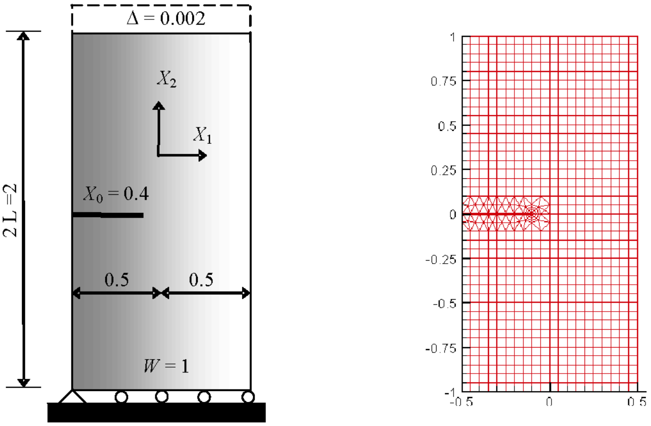
In this study, probability of crack initiation in mode I is considered under uncertain variables. The random vector $\mathbf{X}$ is employed to model uncertainties in loads and material properties. For instance, Young's modulus E, Poisson's ratio ν, far-field applied stress magnitude σ, and mode-I fracture toughness at crack tip J_k are modeled as random variables, i.e. $\mathbf{X} = \{E, \nu, \sigma, J_k\}^{\mathrm{T}}$. In the FE-FGM code, the J-integral (J), which is available through the fracture analysis, is considered as the crack-driving force. In the simulation, a simple failure criterion is considered, i.e. failure occurs when $J > J_k$. This condition cannot be defined deterministically because J is calculated through fracture analysis and depends on $\mathbf{X}$. Therefore, the probability of crack initiation P_f is evaluated as

$$P_f = P[g(\mathbf{X}) \le 0] = \int_{g(\mathbf{x}) \le 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{1}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of $\mathbf{X}$, and

$$g(\mathbf{X}) = J_k - J(\mathbf{X}) \tag{2}$$

is the limit state function.



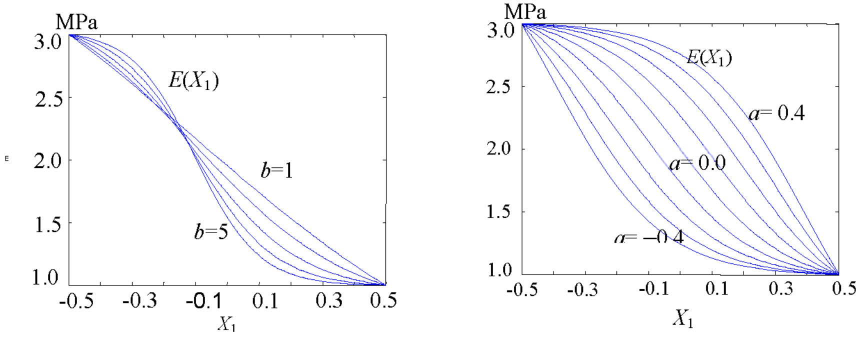
(a) (b)

FIGURE 1. Example FGM strip: (a) geometry and boundary conditions; and (b) mesh detail.

Figure 1 illustrates the geometry and mesh details of an example FGM strip. In this study, we assume that the gradation of Young's modulus follows a smooth, hyperbolic tangent function of the spatial coordinate X_1:

$$E(X_1) = \frac{E_1 + E_2}{2} + \frac{E_1 - E_2}{2}\tanh[b(X_1 - a)] \tag{3}$$

where b and a represent the slope and the location of the inflection point of the gradation profile, respectively. For given (a,b), we find E_1 and E_2 such that the Young's modulus at the left and right edge are 3 MPa and 1 MPa, respectively. Figure 2 illustrates smooth gradation profiles with different slopes and inflection points. In reality, Young's modulus profile is not smooth but it can be described by fluctuation curve. Figure 3 illustrates the fluctuation of Young modulus in the FGM strip and its relationship with the mean function, standard deviation function and correlation length of the random field model by use of a simulated fluctuation. Figure 4 shows the effect of the correlation length on the correlation coefficient function of the random field.



(a) (b)

FIGURE 2. Smooth gradation profile of Young's modulus: (a) $E(X_1)$ with b=1 to 5 when a = –1; and (b) $E(X_1)$ with a =–0.4 to 0.4 when b = 3.

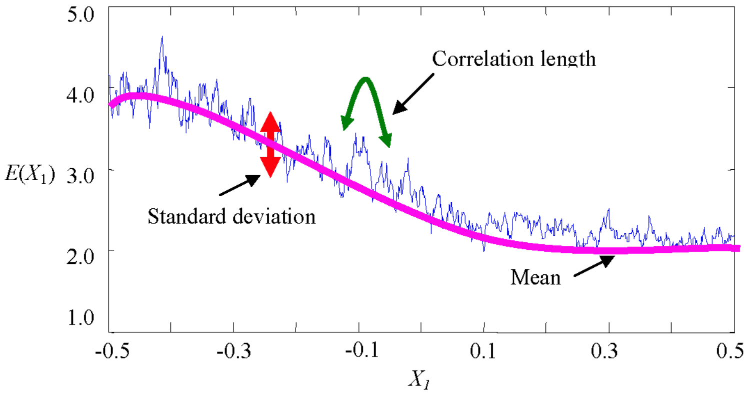


FIGURE 3. Fluctuation of Young modulus in an FGM strip.

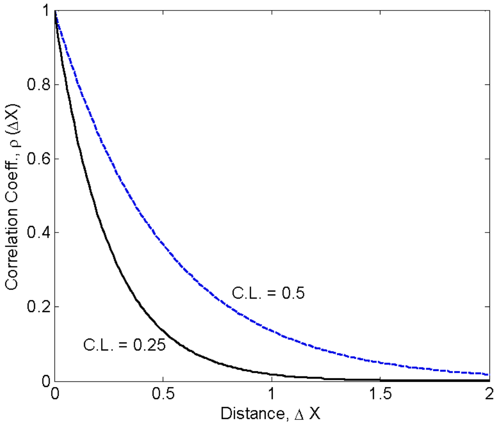
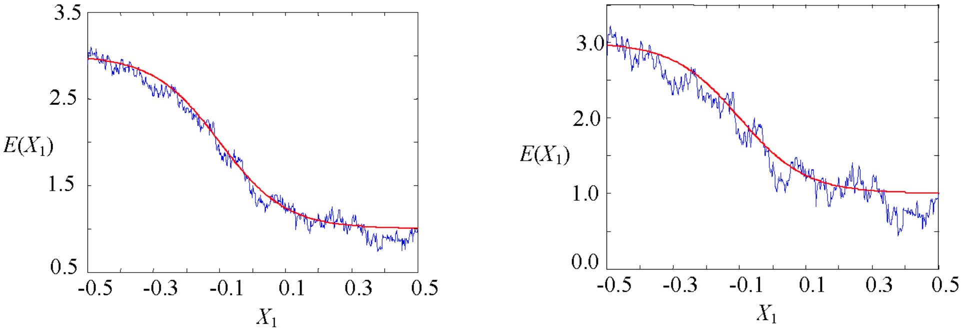
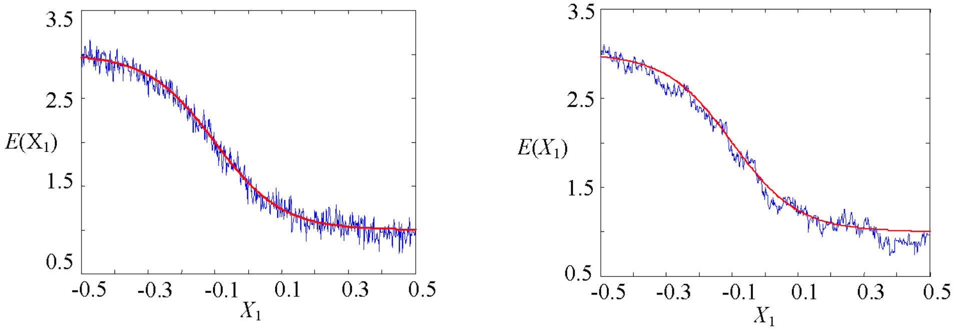


FIGURE 4. Correlation functions with different correlation lengths (C.L.)



(a) Small magnitude of fluctuation

(b) Large magnitude of fluctuation



(c) Small correlation length

(d) Large correlation length

FIGURE 5. Spatial fluctuation of Young's modulus

TWO-STEP PROBABILISTIC FRACTURE ANALYSIS

The first order reliability method (FORM) is investigated in the present work. This is a work in progress and other methods may be used in the near-future, such as second order reliability method [9] and importance sampling method. The deterministic fracture analysis is based on the work by Song and Paulino [3].

A two-step probabilistic fracture analysis procedure is proposed to clearly identify the effects of different uncertainties in material uncertainties. As a first step, we investigate the effects of the uncertainties in the shape of the spatial distribution of Young's modulus by considering the slope and the location of the inflection point of the spatial distribution profile as random quantities. The range of parameters considered is illustrated in Figure 2. Second, we investigate the effects of the spatial fluctuations of Young's modulus by making use of a discretized random field, as illustrated by Figure 5.

SUMMARY

A probabilistic fracture analysis framework is proposed for predicting uncertain fracture responses of FGM structures. The uncertainties in material properties include Young's modulus and toughness. The limit state function for a crack initiation event is defined in terms of the J-integral for FGMs. The First-Order-Reliability-Method (FORM) is used in conjunction with a finite element code that computes the J-integral with high accuracy. We remark that these conclusions are done with respect to the specific framework adopted herein. For instance, a relevant situation in fracture mechanics problems consists of investigating the crack length as a random variable, which has not been done in this work and is a topic for future research. The present analysis framework is explored in a companion paper (Part II), which implements this method into a finite element fracture analysis code and includes numerical examples.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the Vietnam Education Foundation (VEF) for providing a Fellowship to the second author. We also acknowledge the contributions of the Midwest Structural Sciences Center (MSSC). The Center is supported by the U.S. Air Force Research Laboratory Air Vehicles Directorate under contract number FA8650-06-2-3620.

REFERENCES

1. T.H. Nguyen, J. Song and G.H. Paulino, "Probabilistic fracture analysis of functionally graded materials – Part II: implementation and numerical examples" *Proceedings of the Multiscale and Functionally Graded Materials Conference 2006* (M&FGM2006), Kapolei, HI, October 15-18 (2006).
2. J. H. Kim and G.H. Paulino, *International Journal for Numerical Methods in Engineering*, **58**, 1457-1497 (2003).
3. S.H. Song and G.H. Paulino, *International Journal of Solids and Structures*, **43**(16), 4830-4866 (2006).
4. M. Grigoriu, M.T.A. Saif, S. EL Borgi, and A.R. Ingraffea, *International Journal of Fracture*, 45, 19-34 (1990).

5. B.N. Rao and S. Rahman, *Computational Mechanics*, **28**, 351-364 (2002).
6. B.N. Rao and S. Rahman, *Computational Mechanics*, **28**, 365-374 (2002).
7. S. Rahmann, H. Xu and B.N. Rao, "Probabilistic fracture of isotropic functionally graded material," *Proceeding of the ASME Pressure Vessels and Piping Conference 2005–Material and Fabrication, PVP2005*, 435-444 (2005).
8. F.J. Ferrante and L.L. Graham-Brady, *Computer Methods in Applied Mechanics and Engineering*, **194**, 1675-1692 (2005).
9. A. Der Kiureghian, H.S. Lin, and S.-J. Hwang, *Journal of Engineering Mechanics*, ASCE, **113**(8), 2908-1225 (1987).