Topology Optimization with Stress Constraints: Reduction of Stress Concentration in Functionally Graded Structures

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Abstract. This presentation describes a topology optimization framework to design the material distribution of functionally graded structures with a tailored Von Mises stress field. The problem of interest consists in obtaining smooth continuous material fraction distribution that produces an admissible stress field. This work explores the topology optimization method for minimizing volume fraction of one of the phases considering stress constraints. Existence of inherent material microstructure requires consideration of the micro level stress field, which is computed through a mechanical concentration factor based on the local stress in each phase of the material. Thus, p-norm of the Von Mises stress in the microstructure is considered as a global constraint. To illustrate the method and discuss its essential features, we present engineering examples of axisymmetric FGM structures subjected to body forces.

Keywords: Functionally Grade Materials, Stress Constraint, Topology Optimization.

INTRODUCTION

Functionally graded materials (FGMs) present continuously graded properties and are characterized by spatially varying microstructures created by both nonuniform distribution of the reinforcement phase and by interchanging the role of reinforcement and matrix (base) materials in a continuous way. The smooth variation of properties may offer advantages such as reduction of stress concentration and increased bonding strength. A major advantage of FGMs is the possibility of tailoring its gradation to maximize performance.

Due to the above features, to design the FGM gradation by optimization methods is a promising possibility and has been investigated by some researchers, such as Aboudi et al. [1] and Turteltaub [2] among others. In this sense a generic optimization method to tailor material property gradation was proposed by Paulino and Silva [3], who applied the topology optimization method framework to solve the problem of maximum stiffness design. Following this work, Stump [4] applied this framework to solve a stress constraint problem.

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Topology optimization method is a powerful structural optimization method that seeks an optimal structural topology design by determining which parts of space should be solid and which parts should be void (i.e. no material) inside a given domain.

This problem is solved by defining the *extended design domain* which is a large fixed domain that must contain the whole structure and also by defining the concept of *material model* which is a continuous function that maps the volumetric density of material into the constitutive tensor of the composite material. The definition of the material model is necessary to transform the ill-posed discrete problem into a well posed continuous one, even though the aim of traditional topology optimization problem is to find a discrete solution. In this sense, several material models that aim to provide useful discrete (0-1) solutions have been developed.

This work focuses on applying the theoretical and numerical background developed for the topology optimization method to solve a two-phase material distribution optimization problem considering stress constraints, with emphasis on the design of functionally graded structures.

A FUNCTIONALLY GRADED MATERIAL MODEL

To apply the topology optimization framework to the design of functionally graded structures it is necessary to adopt a realistic material model that given two base materials with constitutive tensors $\mathbf{C}^{+}$ and $\mathbf{C}^{-}$, provides an effective constitutive tensor $\mathbf{C}^{Eff}$ that can be realized in an FGM.

In the literature, there are several material models applied to estimate the effective properties of composite materials that could be used in the topology optimization framework. In addition, those conventional models, such as Mori-Tanaka and self-consistent, have been applied to estimate the effective FGM properties in several works [6,7,8,9,10]. However, these models were originally developed for statistically homogeneous materials and are not able to capture the material gradient nature of the FGMs [11].

Due to the continuous microstructural changes in the FGM, the traditional concept of representative volume element (RVE) applied to conventional models remains to be justified [12]. Thus, some specific FGMs specialized material models have been developed such as Yin et al. [13] and Aboudi et al. [12].

In this work, a rather simple and generic type of material model will be applied to estimate the effective properties of FGMs. The proposed material model was developed given the necessity of a closed form equation that provides the effective constitutive tensor $\mathbf{C}^{Eff}$ in terms of the volumetric densities of each phase.

As motivation to our approach, we recall the work by Reiter and Dvorak [8] in which they propose a combination of the Mori-Tanaka method for the matrix-particle zones, and the self-consistent method for the skeletal zone.

Here we consider simply a non-linear interpolation between the upper and lower Hashin-Strikman (HS) bounds. To consider a continuous transition between the lower and the upper bound, an interpolation of both bounds is applied and given by;

$$K^{Eff}(\rho) = \varphi(\rho)K_{upper}(\rho) + (1-\varphi(\rho))K_{lower}(\rho) \quad (1)$$

$$G^{Eff}(\rho) = \varphi(\rho)G_{upper}(\rho) + (1-\varphi(\rho))G_{lower}(\rho) \quad (2)$$

where K_{upper}, K_{lower}, G_{upper} and G_{lower} are the bulk modulus and shear modulus Hashin-Strikman bounds [5], respectively; and $\varphi(\rho)$ is given by the recursive use of a function $\phi(\rho)$ i.e.:

$$\varphi(\rho) = \phi(\phi(\phi(\phi(\phi(\rho))))) \quad \text{such that} \quad \phi(\rho) = \frac{\cos(\pi\rho)}{2} + \frac{1}{2} \quad (3)$$

With the material model defined above, it is possible to write the material constitutive $\mathbf{C}^{Eff}(\rho)$ tensor as a closed form function of the material volumetric density. The material density (η^{Eff}) is simply given by $\eta^{Eff}(\rho) = \rho\eta^{+} + (1-\rho)\eta^{-}$.

To evaluate the stress inside each phase of the FGM, the classical material model framework which assumes a constant macro-stress inside the RVE is applied. Considering the macro-stress ($\langle\boldsymbol{\sigma}\rangle$) given by $\langle\boldsymbol{\sigma}\rangle = \mathbf{C}^{Eff}\langle\boldsymbol{\varepsilon}\rangle$, where $\langle\boldsymbol{\varepsilon}\rangle$ is the macro-strain obtained from the analysis of the equivalent structure made of homogenous material with constitutive relations $\mathbf{C}^{Eff}$. From the relation of $\mathbf{C}^{Eff}$, $\mathbf{C}^{+}$, and $\mathbf{C}^{-}$ the matrices $\mathbf{B}^{+}$ and $\mathbf{B}^{-}$ can be derived such that [7],

$$\langle\boldsymbol{\sigma}^{+}\rangle = \mathbf{B}^{+}\langle\boldsymbol{\sigma}\rangle \quad , \quad \langle\boldsymbol{\sigma}^{-}\rangle = \mathbf{B}^{-}\langle\boldsymbol{\sigma}\rangle \quad (4).$$

With the stress evaluated inside each phase of the microstructure it is possible to define the problem formulation.

PROBLEM FORMULATION

The topology optimization problem with stress constraints, as proposed by Duysinx and Bendsøe [14], presents two main problems, the *stress singularity phenomenon* (SSP), and the large number of constraints. The SSP [15] occurs in layout optimization problems when the material volume fraction of a region in the structure tends to vanish and the micro-stress in this region remains finite. This means that, if the finite micro-stress value is greater than the admissible stress of the problem, the stress constraint will become active and the optimization procedure will not be able to remove the material from this region, and reach the lower bound.

To avoid SSP in this work, it is proposed to replace the stress constraint in each phase of the FGM, as it would be a direct extension of reference [14], by an unique stress index constraint given by:

$$\frac{\langle \sigma^{micro}_{vm^2} \rangle}{\langle \sigma^{micro}_{adm^2} \rangle} < 1 \quad (5)$$

where a unique stress index ($\left\langle \sigma^{micro}_{vm^2} \right\rangle$) is given by:

$$\left\langle \sigma^{micro}_{vm^2} \right\rangle = \rho \left\langle \sigma^{+}_{vm^2} \right\rangle + (1-\rho)\left\langle \sigma^{-}_{vm^2} \right\rangle \tag{6}$$

where $\left\langle \sigma^{+}_{vm^2} \right\rangle$ and $\left\langle \sigma^{-}_{vm^2} \right\rangle$ are the squares of the Von Mises stress in the microstructure plus (+) and minus (-) phase, respectively. The advantage of this unique index $\left\langle \sigma^{micro}_{vm^2} \right\rangle$ is that the stress for a homogenous body made of material plus or material minus is recovered when the density ρ tends to 1 or 0, respectively. Thus the SSP is avoided, since when one phase is vanishing the unique stress index tends to the squared Von Mises on the remaining phase, and thus, the constraint is still physically meaningful.

The unique admissible stress $\left\langle \sigma^{micro}_{adm^2} \right\rangle$ was defined based on the model of Swan and Kosaka [16], and it is given by the equation,

$$\left\langle \sigma^{micro}_{adm^2} \right\rangle = \rho^{s} \left\langle \sigma^{+}_{adm^2} \right\rangle + (1-\rho^{s})\left\langle \sigma^{-}_{adm^2} \right\rangle \tag{7}$$

provided that $\left\langle \sigma^{+}_{adm^2} \right\rangle \geq \left\langle \sigma^{-}_{adm^2} \right\rangle$, where $\left\langle \sigma^{+}_{adm^2} \right\rangle$ and $\left\langle \sigma^{-}_{adm^2} \right\rangle$ are the squares of the reference stress of plus and minus phase, respectively and where s is the penalization parameter that brings the unique admissible stress $\left\langle \sigma^{micro}_{adm^2} \right\rangle$ closer to the lower strength bound or to the upper strength bound. This model guarantees that the admissible stress lies between the Reuss and Voigt model for the material strength, as the model proposed in Reference [16].

To avoid the large number of constraints necessary to discretize the local failure function constraint, a global constraint is considered based on the failure function p-norm. Thus, the problem formulation is given by

$$\min_{\rho} \int_{\Omega} \rho(x) d\Omega \quad s.t. \left[\frac{1}{\Omega} \int_{\Omega} \left(\frac{\left\langle \sigma^{micro}_{vm^2} \right\rangle}{\left\langle \sigma^{micro}_{adm^2} \right\rangle} \right)^{p} \right]^{1/p} < 1 \tag{8}$$

where p is a parameter, such that in the limit when it tends to infinity, the local constraint is recovered. In the numerical approach p must be increased to certain value that is defined on a case-by-case basis.

To solve the optimization problem proposed in Eq. 8 numerically the stress field of extended design domain is obtained by the Finite Element Method. The constraint integral is computed by considering the stress constant over each element and equal to the central value. The sensitivity of the constraint is obtained by the adjoint method and the optimization problem is solved by Sequential Linear Programming.

NUMERICAL EXAMPLE

To illustrate the proposed method the conceptual design of a functionally graded turbine disk is presented. The geometry and boundary conditions of the disk are based on the reference [17] and Figure 1 depicts the section of the turbine disk structure which axisymmetric about the (y) axis.

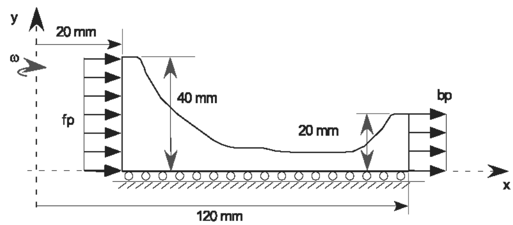
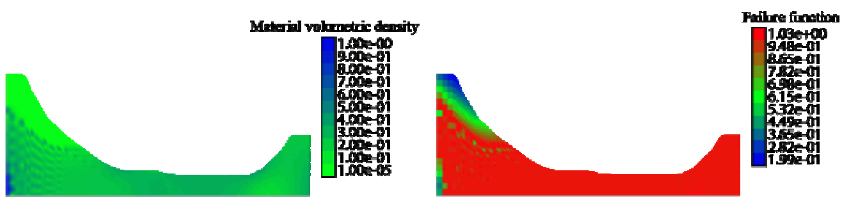


FIGURE 1. Geometry of the turbine disk section in the (x,y) plane, and boundary conditions (fp=40MPa, bp=200MPa, ω=2000rpm).

The material named plus (+) has mechanical properties close to a ceramic material, and thus, it is called material "C". The properties of this fictitious material are: density 4000 kg/m^3, Young's modulus 350GPa, Poisson's ratio 0.25, strength 3500MPa. The material named minus (-) represents a fictitious metallic material, and it is called material "M". Its properties are the following: density 9000 kg/m^3, Young's modulus 110GPa, Poisson's ratio 0.3, strength 300MPa.

After applying the method, an optimized material distribution presented in Figure 2(a) is obtained. Analyzing the material distribution, it is possible to conclude that the method provides a smooth material "C" distribution (blue) inside mostly a material "M" matrix (green). In some small regions, the volumetric density of material "C" reaches its upper bound (100%) by exchanging the role of inclusion with the role of matrix in the composite to satisfy the stress constraints.



(a) (b)

FIGURE 2. Optimized material distribution and failure function of the turbine disk.

Figure 2(b) depicts the failure function (Eq. 5) for the optimized material distribution. It is worth noting that the particular material distribution generated by the optimization method is able to yield a stress field that numerically satisfies the global constraint and is approximately 8 times lower than the stress field of the same disk made of homogeneous material "M". In this problem, the maximum value of the failure function is 1.03, which means that the Von Mises stress in the microstructure is only 3% above the defined reference stress. This situation is numerically admissible for a global stress constraint such as the p-norm.

CONCLUDING REMARKS

This paper presented a powerful and general technique to obtain preliminary design of Functionally Graded Structures. The technique is based on the topology optimization method with stress constraint. In this framework, the two more important drawbacks, i.e. the *stress singularity phenomenon* (SSP) and the large number of constraints, were addressed. The numerical result has demonstrated that the method is able to define a material distribution that satisfies the adopted stress constraint.

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