Topology Optimization Using Wachspress-Type Interpolation with Hexagonal Elements

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Abstract. Traditionally, standard Lagrangian-type finite elements, such as quads and triangles, have been the elements of choice in the field of topology optimization. However, finite element meshes with these elements exhibit the well-known "checkerboard" pathology in the solution of topology optimization problems. A feasible alternative to eliminate this long-standing problem consists of using hexagonal elements with Wachspress-type shape functions. The features of the hexagonal mesh include 2-node connections (i.e. 2 elements are either not connected or connected by 2 nodes), and 3 edge-based symmetry lines per element. In contrast, quads can display 1-node connection, which can lead to checkerboard; and only have 2 edge-based symmetry lines. We explore the Wachspress-type hexagonal elements and show their advantages in solving topology optimization problems. We also discuss extensions of the work to account for material gradient effects.

Keywords: Topology Optimization, Checkerboard, Wachspress Interpolation Functions

INTRODUCTION

Topology optimization is a powerful tool that aims at finding the optimal layout of material in a structural system. Despite much advancement and numerous applications using topology optimization methods, a handful of issues have remained unresolved to this date. One such issue is the presence of checkerboard layouts or regions with alternating void and material elements in the final solution.

Diaz and Sigmund [1] demonstrated that the stiffness of checkerboard patches in the Q4 implementation is overestimated, making them favorable in minimum compliance problems. Jog and Haber [2] also addressed the root causes of checkerboard and investigated the stability of various finite element implementations. In both works, it was discovered that higher order finite elements such as 8 or 9-node quadrilateral elements are less susceptible to the checkerboard problem but do not eliminate it completely. Depending on the severity of the penalization scheme and the convergence criteria used, patches of checkerboard may still appear in the solution of Q8 and Q9 topology optimization implementation. Additionally, these elements suffer from “one-node hinges,” which appear in some topology optimization problems such as compliant mechanism design [3]. “One-node hinge” refers to the case where two structural elements are connected at one node in the final topology. This issue is inherent to quadrilateral elements as a result of their geometry. Due to these drawbacks and also the higher computational cost, the use of these elements (i.e. Q8, Q9) is not practical. Thus an alternative approach is needed.

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In this work, the use of hexagonal finite element with Wachspress basis functions is proposed for topology optimization. The geometry of this hexagonal element only allows for 2-node connections, which makes the occurrence of checkerboard patterns impossible. Furthermore, this eliminates the need to impose any density gradients or local constraints to overcome the checkerboard problem. In the following sections, the properties of this element and its implementation are discussed and numerical results are provided to demonstrate its performance.

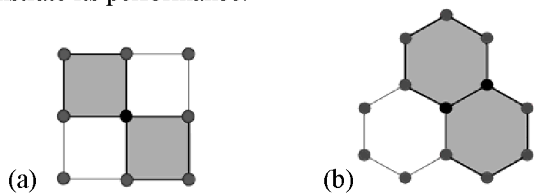

FIGURE 1. (a) Q4 mesh can display one-node connections. (b) In a hexagonal mesh, two connected elements always share two nodes.

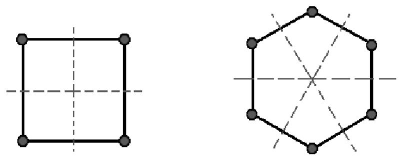

FIGURE 2. Edge-based symmetry lines: while the Q4 element has two symmetry lines, the hexagonal element has three symmetry lines.

HEXAGONAL ELEMENT

The proposed element in this work is a regular hexagon with Wachspress shape functions. Hexagons, along with quadrilaterals and triangles, are the only polygons that can form a regular tessellation or tiling [4]. Unlike triangular and rectangular grids, however, a hexagonal mesh does not allow for single points of contact and therefore can eliminate the "one-node hinge" problem in topology optimization solutions (Figure 1). As mentioned before, the 2-node connections, which are characteristic of the hexagonal geometry, also eliminate the possibility of checkerboard formation. Furthermore, a regular hexagon has three edge-based lines of symmetry compared to two lines for a square (Figure 2). Therefore, a hexagonal mesh provides less directional constraint and allows for more flexible formation of the final solution in the optimization process.

The hexagonal element proposed here utilizes Wachspress shape functions. These rational interpolation functions were constructed for general convex polygons using concepts from projective geometry [5, 6]. The construction of first order shape functions and their properties for the hexagonal element are discussed below.

Let $\mathbf{\Omega}$ denote the hexagonal domain (See Figure 3(a)). The shape function ϕ_i, corresponding to node i, is given by:

$$\phi_i(\mathbf{x}) = c_i \frac{\lambda_{i+2}(\mathbf{x})\lambda_{i+3}(\mathbf{x})\lambda_{i+4}(\mathbf{x})\lambda_{i+5}(\mathbf{x})}{q(\mathbf{x})}, \tag{1}$$

where $\lambda_{i+1} = 0$ represents the straight line going through nodes "$i+1$" and "i" while q is the equation of the circle encompassing the points of intersection of extensions of the edges. It is understood that $\lambda_7 = \lambda_1$, $\lambda_8 = \lambda_2$ and so on. The coefficient c_i is a normalizing factor that forces the interpolated value at the node to be identical to the nodal data and is given by:

$$c_i = \frac{q(\mathbf{x}_i)}{\lambda_{i+2}(\mathbf{x}_i)\lambda_{i+3}(\mathbf{x}_i)\lambda_{i+4}(\mathbf{x}_i)\lambda_{i+5}(\mathbf{x}_i)}, \tag{2}$$

where $\mathbf{x}_i$ represents the coordinates of node i. A typical shape function is shown in Figure 3(b).

Wachspress rational shape functions satisfy the necessary conditions for conforming Galerkin approximations. First, these shape functions are bounded, non-negative and form a partition of unity. In other words, they satisfy the discrete maximum principle:

$$\sum_{i=1}^{6} \phi_i(\mathbf{x}) = 1, \quad 0 \le \phi_i(\mathbf{x}) \le 1 \tag{3}$$

Furthermore, they exhibit the Kronecker-delta property which is required to impose necessary boundary conditions:

$$\phi_i(\mathbf{x}_j) = \begin{cases} 0 & \text{for } i \ne j \\ 1 & \text{for } i = j \end{cases} \tag{4}$$

Also, these shape functions can exactly reproduce a linear function and therefore satisfy linear precision, the sufficient condition for convergence of second-order partial differential equations:

$$\sum_{i=1}^{6} \phi_i(\mathbf{x})\mathbf{x}_i = \mathbf{x} \tag{5}$$

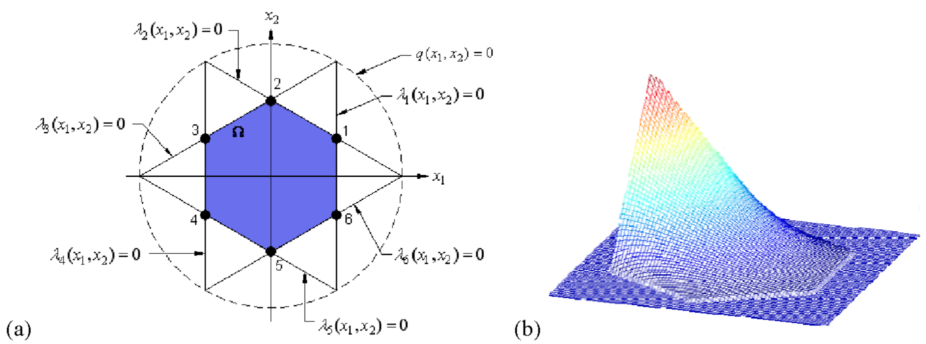

FIGURE 3. (a) Construction of Wachspress shape functions for a hexagonal domain. (b) Typical shape function for the hexagonal element (values along the edges have been raised to enhance visualization).

Finally, these shape functions are continuous throughout the domain $\mathbf{\Omega}$ and are linear on the edges, allowing for necessary linear boundary conditions. This implies that the Wachspress hexagonal element is a conforming element.

NUMERICAL IMPLEMENTATION

The performance of the proposed element is assessed through a standard minimum compliance problem. The minimum compliance problem in discretized form is formulated as:

$$\begin{aligned} &\min \mathbf{f}^T \mathbf{u} \\ &\text{s.t. } \mathbf{Ku} = \mathbf{f} \\ &\sum_{e=1}^{N} \rho_e V_e \le V \end{aligned} \tag{6}$$

where $\mathbf{f}$ and $\mathbf{u}$ are the global force and displacement vectors and $\mathbf{K}$ represents the global stiffness matrix. The parameter V is the specified volume of material, ρ_e is the density design variable assigned to each element, and V_e is the element volume. The topology of the structure is determined through the element densities: $\rho_e = 0$ indicates a void element, while $\rho_e = 1$ corresponds to a material point. To avoid numerical singularities, however, the problem is generally relaxed by allowing ρ_e to continuously vary so that:

$$0 < \rho_{\min} \le \rho_e \le 1 \tag{7}$$

The standard SIMP (Solid Isotropic Material with Penalization) method [7] is used here to penalize the intermediate values of density:

$$\mathbf{K}_e = \rho_e{}^p \mathbf{K}_e^0 \tag{8}$$

where $\mathbf{K}_e$ is the element stiffness matrix, $\mathbf{K}_e^0$ represents the stiffness of a solid element, and p is the penalization factor. Through this power-law relation, the stiffness of the elements with intermediate densities is small compared to their contribution to total volume of the structure. This makes the intermediate densities unfavorable and leads to 0-1 (void-solid) type solutions. The optimization problem is solved using the Optimality Criteria (OC) method [8] with the element sensitivities given as:

$$\frac{\partial c}{\partial \rho_e} = p \rho_e{}^{p-1} \mathbf{u}_e{}^T \mathbf{K}_e \mathbf{u}_e \tag{9}$$

Finally, in order to avoid local minima, the continuation method described by Petersson and Sigmund [9] is employed. The value of p is increased from 1 to 5 with increments of 0.5 after the solution has converged in each step. A discussion on penalization and continuation schemes can be found in reference [10].

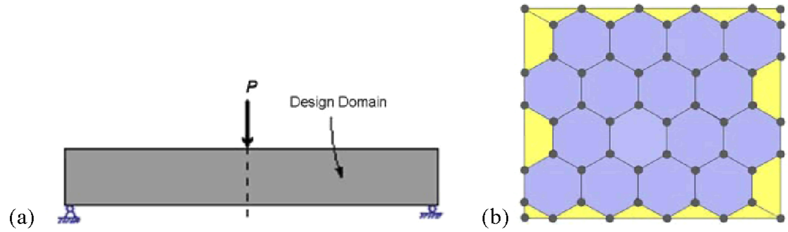

FIGURE 4. (a) Schematic representation of the design domain, loading and boundary conditions for MBB (Messerschmitt-Bölkow-Blohm) beam problem. (b) A rectangular domain with a 4x4 hexagonal grid. Notice that the non-hexagonal elements on the boundary are either triangles or quads.

NUMERICAL RESULTS

The benchmark MBB-beam problem [11] is solved using the hexagonal element with Wachspress shape functions and results are compared with the Q4 implementation. Due to the symmetry of the problem, only half of the MBB-beam is considered (Figure 4(a)). The beam has an aspect ratio of 6:1, and three levels of mesh discretization are used. In order to obtain a rectangular domain for the hexagonal mesh, it is necessary to insert triangular and quadrilateral elements along the boundary. Linear triangular (T3) and bilinear quadrilateral elements (Q4) are used in this example. Note that the difference in element size must be considered when enforcing the volume constraint, and the parameter V_e must be adjusted accordingly. A typical rectangular mesh with a 4×4 grid of hexagons is shown in Figure 4(b). The Poisson's ratio is taken to be 0.3, while V is 50% of volume of the design domain.

The results for the Q4 element and the hexagonal element for various levels of mesh refinement are compared in Figure 5. The results for the Q4 implementation contain patches of checkerboard while no such instabilities are observed with the hexagonal implementation. Note that no filtering technique or density gradient was imposed for the hexagonal element and thus the checkerboard-free property of this element must be attributed to its geometric features and interpolation characteristics.

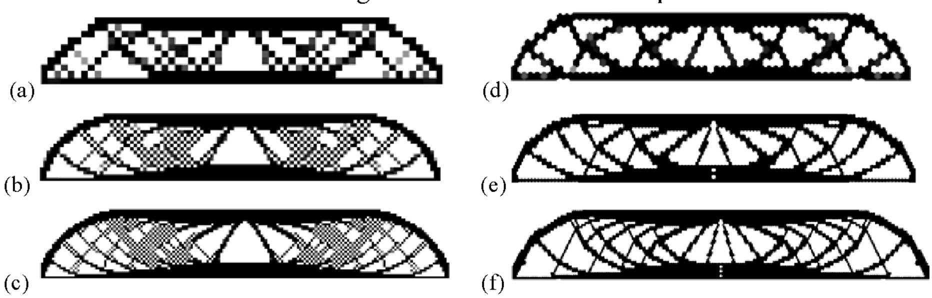

FIGURE 5. (a)-(c) Results for MBB-beam with Q4 elements. (d)-(f) Results for MBB-beam with hexagonal elements. The mesh discretization is 30x10, 60x20, and 90x30 from top to bottom, respectively.

CONCLUSION AND EXTENSIONS

In this work, the checkerboard pathology in topology optimization is addressed and the use of a new element is proposed. The hexagonal element with Wachspress-type shape functions is shown to possess advantages over conventional finite elements. Geometric properties of the hexagonal element such as two-node connections and edge-based symmetry lines in three directions are the distinguishing features of this element. As discussed and demonstrated by example, the use of hexagonal elements eliminates the formation of checkerboard and provides a robust and stable means for solving topology optimization problems. The hexagonal element with the underlining topology optimization formulation has potential and promising extensions, such as compliant mechanism and functionally graded material (FGM) systems. The latter case can be accomplished by means of the FGM-SIMP formulation by Silva and Paulino [12]. Work is in progress to address such extensions.

ACKNOWLEDGMENTS

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REFERENCES

1. A. R. Diaz and O. Sigmund, *Structural Optimization*, **10,** 40–45 (1995).
2. S. Jog, R. B. Haber, *Computer Methods in Applied Mechanics and Engineering,* **130**, 203–226 (1996).
3. O. Sigmund, *Mechanics of Structures and Machines,* **25**, 495 –526 (1996).
4. R. Williams, *The Geometrical Foundation of Natural Structure: A Source Book of Design.* New York: Dover, 1979, p35-43.
5. E. L. Wachspress, *A Rational Finite Element Basis*. Academic Press: New York, NY, 1975.
6. N. Sukumar and E. A. Malsch, *Archives of Computational Methods in Engineering,* **11**(4), 1–38 (2005).
7. M. P. Bendsøe and O. Sigmund, *Arch. Appl. Mech.,* **69**, 635–654 (1999).
8. M. P. Bendsøe, *Optimization of structural topology, shape and material,* Berlin, Heidelberg, New York, Springer, 1995
9. J. Peterson and O. Sigmund, *International Journal for Numerical Methods in Engineering,* **41**, 1417–1434 (1998).
10. O. Sigmund and J. Peterson, *Structural Optimization,* **16**, 68-75 (1998).
11. N. Olhoff, M. P. Bendsøe, J. Rasmussen, *Computer Methods in Applied Mechanics and Engineering,* **89**, 259–279 (1991).
12. E. C. N. Silva and G. H. Paulino, "Topology Optimization Design of Functionally Graded Structures", *Proceedings of the 6th World Congress of Structural and Multidisciplinary Optimization,* 2005.