Proceedings of the 6th International Conference on
Computation of Shell and Spatial Structures
IASS-IACM 2008: "Spanning Nano to Mega"
28-31 May 2008, Cornell University, Ithaca, NY, USA
John F. ABEL and J. Robert COOKE (eds.)

Wachspress Elements for Topology Optimization

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Abstract

Traditionally, standard Lagrangian-type finite elements, such as linear quads and triangles, have been the elements of choice in the field of topology optimization. However, finite element meshes with these conventional elements exhibit the well-known "checkerboard" pathology in the iterative solution of topology optimization problems. A feasible alternative to eliminate such long-standing problem consists of using hexagonal elements with Wachspress-type shape functions. The features of the hexagonal mesh include two-node connections (i.e. two elements are either not connected or connected by two nodes), and three edge-based symmetry lines per element. In contrast, quads can display 1-node connections, which can lead to checkerboard; and only have two edge-based symmetry lines. In addition, Wachspress rational shape functions satisfy the partition of unity condition and lead to conforming finite element approximations. We explore the Wachspress-type hexagonal elements and present their implementation using three approaches for topology optimization: element-based, continuous approximation of material distribution, and minimum length-scale through projection functions. Examples are presented that demonstrate the advantages of the proposed element in achieving checkerboard-free solutions and avoiding spurious fine-scale patterns from the design optimization process.

1. Introduction

Topology optimization methods seek to find the optimal layout or topology of a fixed amount of material that satisfies a required set of design demands. Despite the maturity of the field, there remains a class of numerical issues such as the well-known checkerboard problem that continues to be the focus of extensive research. This work introduces a new element for the implementation of topology optimization and demonstrates its effectiveness in removing the checkerboard pathology.

The checkerboard solutions appear as a result of inadequate or poor numerical modeling. Diaz and Sigmund [2] attributed the formation of checkerboard as a local instability to the error in the finite element approximation. The checkerboard pattern has an artificially high stiffness when modeled by lower order finite elements so it is economical in the optimization process. In a related investigation, Jog and Haber [4] addressed general numerical instabilities in topology optimization by formulating the corresponding mixed variational problem. They also concluded that insufficient interpolation of the displacement field can lead to unstable modes. Moreover, the discontinuous representation of the material field in the element-based approach (see section 3) is conducive to the appearance of checkerboard. In one approach proposed by Matsui and Terada [6], the continuity of the material field is enforced by using finite element shape functions to interpolate the density throughout the design domain from nodal densities. As a result of this choice of density field representation, the discontinuous checkerboard patches are naturally excluded from the design space. However, other forms of numerical instabilities such as "islanding" and "layering" effects have been observed with these formulations (Rahmatalla and Swan [8]).

It is evident from this discussion that the approximation of the two distinct fields of displacement and density greatly influences the stability of the topology optimization problem. In this work, we address the checkerboard issue by introducing the Wachspress hexagonal element which possesses desirable characteristics in representing both fields (see next section). Thus, checkerboard-free solutions are obtained without any further restrictions or filtering.

2. Features of Hexagonal Wachspress Element

If we restrict ourselves to uniform meshes, there are only three possible regular tessellations in two dimensions, namely those generated by equilateral triangles, squares, and hexagons. We recognize that the hexagonal tessellation is distinguished from the other two in that it does not allow for corner contacts (Figure 1a). Consequently, unlike triangular and quadrilateral grids, the hexagonal tessellation, by the virtue of its geometry, constrains the material layout and naturally excludes the unwanted formation of checkerboard and one-node hinges. Another appealing feature of the hexagonal element is that it has more lines of symmetry per element compared to the triangular and square elements and, consequently, suffers from less directional constraint and allows for a more flexible arrangement of the final layout in the optimization process (Figure 2b).

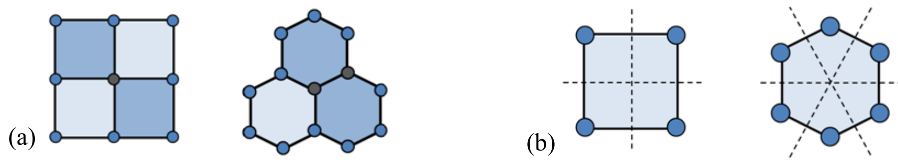

Figure 1: An illustration of the geometric properties of the hexagonal element

In this work, we adopt Wachspress rational interpolation functions for the proposed hexagonal element. Wachspress interpolants were developed using concepts of projective geometry and are the lowest order functions that satisfy the conditions of boundedness, linear precision, and global continuity (Sukumar and Malsch [9]). The geometric construction of these shape functions is based on the algebraic equations of the edges of the polygonal domain and can be found in Wachspress [11].

3. Topology Optimization Formulation

The performance of the proposed hexagonal element is assessed through the implementation of benchmark compliance minimization problems. Using material "density" ρ as the design variable, the minimum compliance problem in the discrete form is formulated as (Bendsøe and Sigmund [1]):

$$\begin{aligned} \min_{\rho,\mathbf{u}}: \quad & c(\rho,\mathbf{u}) = \mathbf{f}^T\mathbf{u} \\ s.t.: \quad & \mathbf{K}(\rho)\mathbf{u} = \mathbf{f} \\ & \int_\Omega \rho dV \leq V_s \end{aligned} \tag{1}$$

Here $c(\rho,\mathbf{u})$ is the objective function (i.e. the compliance of the structure) and $\mathbf{f}$ and $\mathbf{u}$ are the global force and displacement vectors. Moreover, $\mathbf{K}$ represents the global stiffness matrix, which is dependent on the density distribution. The parameter V_s is the specified maximum volume of structural material. In order to solve this optimization problem, we must choose a proper descritization of the design field. We consider the following three different approaches for implementation of the Wachspress hexagonal element:

3.1 Element-Based Approach

In the element based approach, a uniform density parameter ρ_e is assigned to each displacement finite element. The element densities become the design variables, and their sensitivities are calculated using the adjoint method:

$$\frac{\partial c}{\partial \rho_e} = -\mathbf{u}_e^T \frac{\partial \mathbf{K}_e}{\partial \rho_e}\mathbf{u}_e = -p\rho_e^{p-1}\mathbf{u}_e^T\mathbf{K}_e^0\mathbf{u}_e \tag{2}$$

As discussed previously, the element-based implementation using linear triangular and bilinear quadrilateral displacement elements suffer from the checkerboard.

3.2 Continuous Approximation of Material Distribution (CAMD)

Alternatively, we can define the design parameters to be the nodal densities, from which the density through the domain is interpolated. Based on the concept of graded elements (Kim and Paulino [5]), we use shape functions to obtain the density within the each element and subsequently throughout design domain:

$$\rho(\mathbf{x}) = \sum_{e=1}^{n} \sum_{i=1}^{6} N_i^e(\mathbf{x})\, \rho_i^e \tag{3}$$

Here ρ_i^e denotes the nodal density of element e, which is taken to be coincident with the corresponding displacement node. This approach for topology optimization is referred to as the Continuous Approximation of Material Distribution (CAMD) (Matsui and Terada [6]). The sensitivities of the objective function with respect to the nodal densities in the CAMD implementation can be computed as follows:

$$\frac{\partial c}{\partial \rho_i^e} = -\sum_{e \in S_i} \mathbf{u}_e^T \frac{\partial \mathbf{K}_e}{\partial \rho_i^e} \mathbf{u}_e \tag{4}$$

Here S_i is the set of all elements sharing node i. If we let $\mathbf{B}$ denote the strain-displacement matrix and $\mathbf{E}^0$ the constitutive matrix of the solid phase, then the sensitivity of the element stiffness matrix is given by:

$$\frac{\partial \mathbf{K}_e}{\partial \rho_i^e} = \int_{\Omega_e} p N_i^e \left(\sum_{j=1}^{6} N_j^e\, \rho_j^e \right)^{p-1} \mathbf{B}^T \mathbf{E}^0\, \mathbf{B} d\Omega \tag{5}$$

3.3 Projection Method: A minimum length-scale approach

The other scheme explored in this work is the use of projection functions with a fixed length scale. Proposed by Guest et al. [3] for Q4 discretization, the method assigns to each element a uniform density based on a projection of nodal densities surrounding that element. By choosing a fixed radius r_{min} independent of the mesh, one can obtain mesh-independent designs with prescribed minimum member size. The element density is given by a weighted average of nodal densities that are within radius r_{min} from the centroid of that element:

$$\rho_e = \frac{\sum_{i \in S_e} w_i \rho_i}{\sum_{i \in S_e} w_i} \tag{6}$$

The linear weight functions are given by (here r_i is the distance of the node i from the centroid of element e:

$$w_i = \frac{r_{min} - r_i}{r_{min}}, \quad r_i \leq r_{min} \tag{7}$$

4. Results and Conclusions

The benchmark MBB-beam problem (Olhoff et al. [7]) is solved using the Wachspress hexagonal element and results are compared with the corresponding Q4 implementation. We have solved the optimization problem using the Method of Moving Asymptotes (MMA) developed by Svanberg [10], along with a continuation on the SIMP penalty parameter. Coarse and fine meshes (with mesh sizes 60×20 and 120×40 respectively) were implemented for both elements.

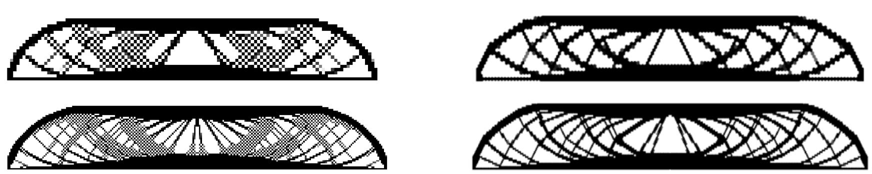

Figure 2. MBB beam design with element-based formulation: Q4 (left) and Wachspress (right) implementation

In Figure 2, the results of the element-based formulation for the Q4 element and the hexagonal element are shown. The solutions with Q4 implementation contain patches of checkerboard while no such fine scale patterns are observed with the Wachspress implementation. Note that no filtering technique or density gradient was imposed and thus the checkerboard-free property of the hexagonal element is attributed essentially to its geometric features and interpolation characteristics. The CAMD results in Figure 3 shows that Q4/Q4 designs suffer spurious islanding and layering patterns, which are absent in designs with Wachspress elements:

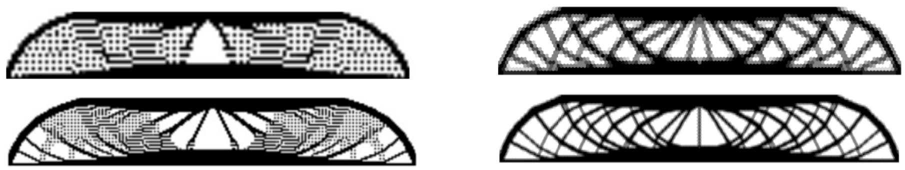

Figure 3. MBB beam design with CAMD formulation: Q4/Q4 (left) and H6/H6 (right) implementation

Finally, the results using projection scheme are presented in Figure 4. The radius of the projection r_{min} is taken to be 0.15 of the height of the beam and independent of the mesh size. We can see that despite the change in the level of mesh refinement, the same design is recovered. The length scale imposed on the optimization through r_{min} guarantees mesh-independent solutions that satisfy the required minimum member size.

Figure 4. MBB beam design with projection scheme and Wachspress hexagonal element. Note that the designs with coarse mesh (left) and fine mesh (right) are identical unlike the other two approaches.

Acknowledgement

We acknowledge the support by the Department of Energy Computational Science Graduate Fellowship Program of the Office of Science and National Nuclear Security Administration in the Department of Energy under contract DE-FG02-97ER25308. We are grateful to Prof. Krister Svanberg for providing his MMA (Method of Moving Asymptotes) code, which was used to generate the examples in this paper.

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