EngOpt 2008 - International Conference on Engineering Optimization
Rio de Janeiro, Brazil, 01 - 05 June 2008.

Topology Optimization with Wachspress and Voronoi Finite Elements

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1. Abstract
Traditionally, standard Lagrangian-type finite elements, such as linear quads and triangles, have been the elements of choice in the field of topology optimization. In general, finite element meshes with these elements exhibit the well-known *checkerboard* pathology in the iterative solution of topology optimization problems. Voronoi and Wachspress-type finite elements are less susceptible to such anomalies. Moreover, these elements provide more flexibility in mesh generation and are suitable for applications involving significant changes in the topology of the material domain. In particular, hexagonal Wachspress meshes include two-node connections (i.e. two elements are either not connected or connected by two nodes), and three edge-based symmetry lines per element. In contrast, quads can display one-node connections, which favor checkerboard configurations; and only have two edge-based symmetry lines. Thus checkerboard-free solutions are obtained without any further restrictions on the local variation of material density or filtering techniques (e.g. filter of sensitivities). We explore general Voronoi-type elements and present their implementation using a couple of approaches for topology optimization: e.g. element-based, and minimum length-scale control through projection functions. Examples are presented that demonstrate the advantages of the proposed elements in achieving checkerboard-free solutions and avoiding spurious fine-scale patterns from the design optimization process. Potential extensions and impact of this work will also be discussed.
2. Keywords: Topology Optimization, Compliance Minimization, Wachspress Elements, Voronoi Elements, Length-scale.

3. Introduction
This work addresses numerical instabilities in topology optimization that appear as a result of poor modeling of the response field by an inappropriate choice of finite element discretization. In particular, we are concerned with the well-known checkerboard phenomenon in which alternating void and material regions (resembling checkerboard patches) emerge in the optimization process. We are proposing a new topology optimization formulation consisting of polygonal Wachspress and Voronoi-type finite elements that not only eliminates the checkerboard problem but also provides more flexibility for the topology design.

The formation of checkerboard has been attributed to the use of lower order elements that make the checkerboard pattern artificially stiff [14, 4]. This high stiffness makes the pattern favorable in the compliance minimization problems, especially considering that the checkerboard also avoids the penalization that is imposed on the intermediate densities. This observation is confirmed by the fact that high order elements are less susceptible to the appearance of the checkerboard anomaly (e.g., Sigmund and Petersson [7]). Similarly, nonconforming quadrilateral elements can model the vanishing stiffness of checkerboard, so they can lead to solutions free of such patches (Jang et. al. [3]).

The discretization of the design field also plays a role in the appearance of spurious features such as the checkerboard in topology optimization solutions. For example, designs using triangular or quadrilateral meshes allow for one-node connections (dominant in checkerboard patches), while hexagonal meshes naturally avoid such features [15]. Two connected hexagonal elements necessarily share two nodes: since checkerboard layouts contain corner-contacts, they are excluded from the design space. In Talischi et. al. [15], it is also shown that the use of Wachspress shape functions makes the hexagonal element numerically stable for topology optimization.

In this work, we extend the previous investigation to consider general convex polygonal discretization. In addition to addressing the checkerboard problem, the proposed approach provides more flexibility in mesh generation and less constraint in the formation of optimal topology. The performance and robustness of the proposed formulation is assessed through benchmark minimum compliance problems.

4. Formulations

In this section, we discuss the topology optimization formulation for the minimum compliance designs. In this class of problems, the goal is to find the stiffest design that is subject to a set of traction and displacement boundary conditions. The final structure must lie entirely in a predefined extended design domain and it needs to satisfy a constraint on its volume. The discrete form of the problem is mathematically given by:

$$\begin{aligned} \min_{\rho} \quad & c = \mathbf{f}^T \mathbf{u} \\ \text{s.t.:} \quad & \mathbf{K}(\rho)\mathbf{u} = \mathbf{f} \\ & \int_{\Omega_S} dV \leq V_s \end{aligned} \tag{1}$$

Here c is the compliance of the structure; $\mathbf{f}$ and $\mathbf{u}$ are the global force and displacement vectors; $\mathbf{K}$ denotes the global stiffness matrix, which is dependent on the design variable ρ; and V_s is the upper bound on the volume of the design denoted by Ω_S.

The common choice of design parametrization is to take ρ as the material "density": by convention, $\rho = 1$ at a point signifies a material region while $\rho = 0$ represents void. The intermediate values are penalized according to the following scheme:

$$E(\rho) = \rho^p E^0, \quad p > 1 \tag{2}$$

Here $E(\rho)$ is the material stiffness of a point with density ρ, while E^0 denotes the stiffness of the solid phase (corresponding to $\rho = 1$). For values of p greater than 1 (usually we take $p \geq 3$), the stiffness of intermediate densities is penalized through the power law relation, so they are not favored. As a result, the final design consists primarily of solid and void regions. This approach is known as the Solid Isotropic Material with Penalization (SIMP), and readers are referred to Bendsoe [12], Zhou and Rozvany [2], and Bendsoe and Sigmund [1] for more information.

We have considered the following discretizations of the density field: (1) Element-based (2) Projection scheme.

4.1. Element-based approach:

In this approach, a constant density value is assigned to each displacement finite element. These element densities ρ_e are then used as the design variables for the optimization problem (1). It is in this context that the checkerboard appears: the density of adjacent elements alternate between zero and one, while the patch of element maintains the connectivity resembling that of a checkerboard.

For the present formulation, as mentioned before, convex polygonal elements are used to construct the finite element discretization. Therefore, the element-based approach with such a discretization does not favor spurious checkerboard-like patterns. Furthermore, polygonal meshes can remove the restrictions on the orientation of the structural members and the final topology as arbitrary polygonal elements have less directional bias when compared to quadrilateral elements. For example, hexagonal element has more lines of symmetry per element compared to the triangular and square elements.

The polygonal finite element mesh can be constructed using a Voronoi diagram of the nodes that cover the design domain. In this paper, we have considered a regular distribution of nodes which produces a regular hexagonal tessellation (similar to that of Talischi et. al. [15]). As shown in Figure 1, the boundary of a rectangular domain consists of one layer of triangular and quadrilateral elements.

The interpolation space on the polygonal mesh is constructed using Laplace (natural neighbor) shape functions as described in Tabarraei and Sukumar [8], and Sukumar and Tabarraei [9]. These shape functions yield a conforming finite element, and satisfy the necessary approximability conditions of constant and linear precision, and exhibit desirable properties such as partition of unity. Moreover, they provide an isoparametric transformation map that allows the computations to be carried out on a parent element. For more details on the implementation of these elements, we refer the reader to reference mentioned above.

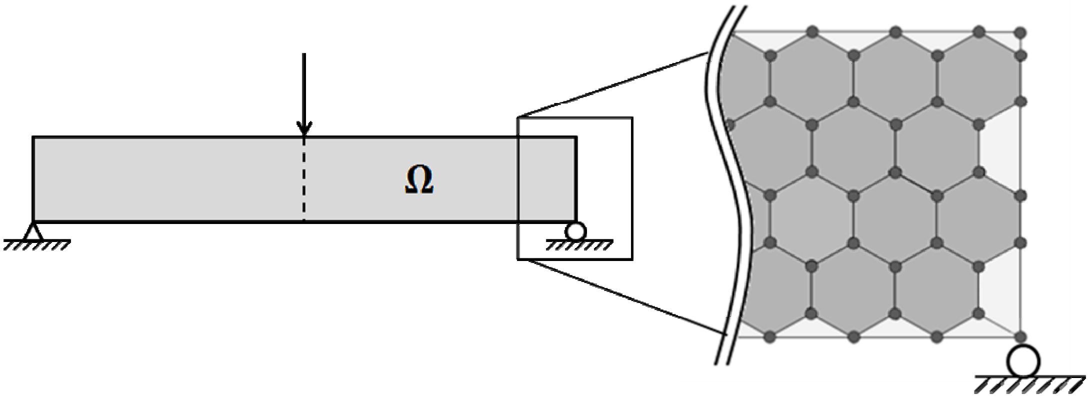


Figure 1: Schematic representation of the design domain, loading and boundary conditions for MBB (Messerschmitt-Bölkow-Blohm) beam problem. Notice that the non-hexagonal elements on the boundary are either triangles or quads [15].

4.2. Projection scheme:
In this approach, first proposed by Guest el. al. [2], "nodal" densities are the design variables in the optimization. The element densities (again constant for each displacement FE element) are extracted from these nodal design variables through a projection scheme. The projection is carried out as follows:

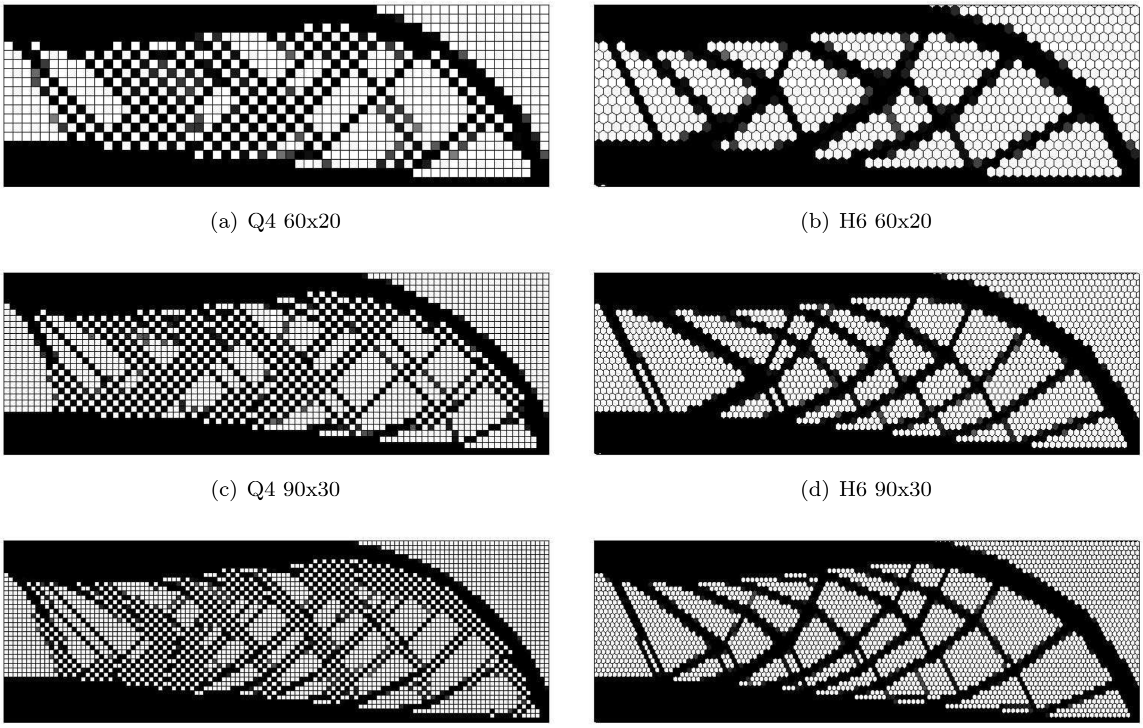
$$\rho_e = \frac{\sum_i w_i \rho_i}{\sum_i w_i} \tag{3}$$

As before, ρ_e is the element density; ρ_i is the design variable associated to node i, and w_i are the weighting functions defined by:

$$w_i = \max\left(\frac{r_{min} - r_i}{r_{min}}, 0\right) \tag{4}$$

Here r_i denotes the distance of node i to the centroid of element e, and r_{min} is a prescribed radius of projection. We can see that the projection has an embedded physical length scale r_{min} that is independent of the mesh size. As such, this scheme addresses the issue of mesh-dependency in topology optimization by limiting the space of admissible solutions to the design having members larger than a minimum physical size.

5. Numerical Results
In this section, the numerical results for the benchmark MBB beam problem (Olhoff et al. [11]) are presented. Due to the symmetry of the problem, only half of the beam is considered in the optimization algorithm (see Figure 1). The Poisson's ratio is taken as 0.3 and the volume fraction V_s is 50% of the volume of the extended design domain. The optimization problem is solved using the Method of Moving Asymptotes (MMA) developed by Svanberg [13]. Also, to avoid getting trapped at local minima, a continuation method is used on the value of SIMP penalty exponent: p is increased (with increment of 0.5) from 1 to 4 after the solution has sufficiently converged for each value.



(a) Q4 60x20

(b) H6 60x20

(c) Q4 90x30

(d) H6 90x30

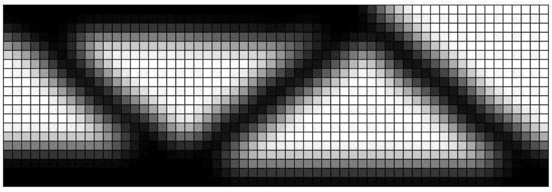
(e) Q4 120x40

(f) H6 120x40

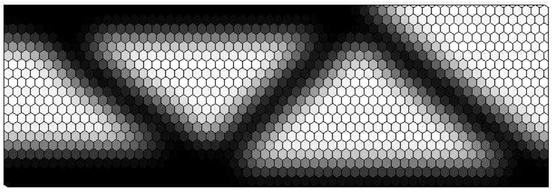
Figure 2: MBB beam design with element–based formulation.

In Figure 2, the results of the element-based formulation for the Q4 element and the hexagonal element are shown. The solutions with Q4 implementation contain patches of checkerboard while no such fine scale patterns are observed with the polygonal implementation. Note that no filtering technique or density gradient was imposed and thus the checkerboard-free property of the hexagonal element is attributed essentially to its geometric features and interpolation characteristics. We should emphasize that the checkerboard solution are unphysical and do not correspond to the optimal structure. Note that for a regular polygonal mesh, Laplace interpolation is identical to Wachspress interpolation. Indeed, the results obtained here match those obtained by Talischi et. al. [15].

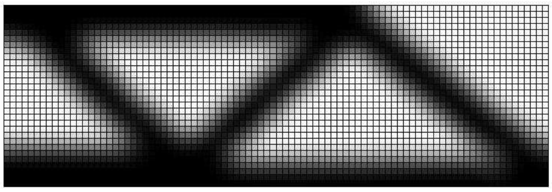
The results using projection scheme are presented in Figure 3. The radius of the projection r_{min} is taken to be 0.15 of the height of the beam and independent of the mesh size. We can see that despite the change in the level of mesh refinement, the same design is recovered. The length scale imposed on the optimization through r_{min} guarantees mesh-independent solutions that satisfy the required minimum member size.



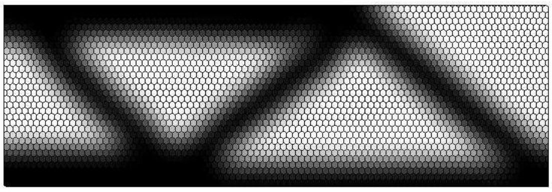
(a) Q4 60x20



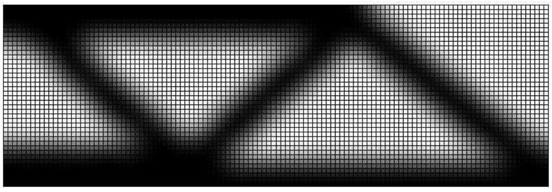
(b) H6 60x20



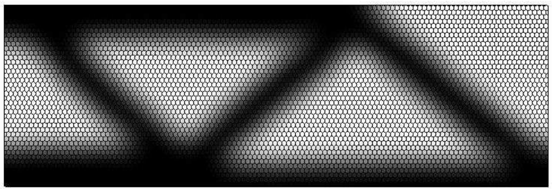
(c) Q4 90x30



(d) H6 90x30



(e) Q4 120x40



(f) H6 120x40

Figure 3: MBB beam design with projection approach ($r_{min} = 0.15h$ where h is the height of the beam.)

6. Conclusions

In this work, the checkerboard pathology in topology optimization is addressed and the use of polygonal finite elements is proposed. As discussed and demonstrated by example, the use of such elements eliminates the formation of checkerboard and provides a robust and stable means for solving topology optimization problems. The future work will include extensions to general irregular polygonal meshes and other density presentations such as Continuous Approximation of Material Distribution [10, 6].

7. Acknowledgement

The authors Cameron Talischi and Glaucio Paulino acknowledge the support by the Department of Energy Computational Science Graduate Fellowship Program of the Office of Science and National Nuclear Security Administration in the Department of Energy under contract DE-FG02-97ER25308. We are grateful to Prof. Krister Svanberg [13] for providing his MMA (Method of Moving Asymptotes) code, which was used to generate the examples in this paper. Anderson Pereira and Ivan Menezes acknowledge the financial support provided by Tecgraf (Group of Technology in Computer Graphics), PUC-Rio, Rio de Janeiro, Brazil.

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