

# Single-Loop System Reliability Based Design Optimization (SRBDO) Using Matrix-based System Reliability (MSR) Method

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**ABSTRACT:** This paper proposes a single-loop system reliability based design optimization (SRBDO) approach using the recently developed matrix-based system reliability (MSR) method. A single-loop method was employed to eliminate the inner loop of SRBDO that evaluates probabilistic constraints. The MSR method computes the system failure probability and its parameter sensitivities efficiently and accurately through efficient matrix calculations. The SRBDO/MSR approach proposed in this paper is uniformly applicable to general systems including series, parallel, cut-set and link-set system events. Two numerical examples demonstrate the proposed approach. In the first example, the cross-sectional areas of the members of a statistically indeterminate truss structure are determined for minimum total weight with a constraint on the system failure probability satisfied. The second example demonstrates the application of the proposed approach to topology optimization. The influences of the statistical correlation and the types of constraints, i.e. deterministic, probabilistic (component) and probabilistic (system) on the optimal topology are investigated.

## 1 INTRODUCTION

The main objective of design optimization is to obtain a set of design variables that minimize/maximize objective function(s) of interest while satisfying given constraints. If design optimization is performed in a deterministic manner, i.e. uncertainties are not taken into account during the optimization, the resultant optimal design may have unquantified risk of violating the given constraints. Recently, various reliability based design optimization (RBDO) methods have been developed to achieve optimal designs with acceptable failure probabilities (see Frangopol & Maute 2004 for a state-of-the-art review of RBDO and recent applications to civil and aerospace structural systems). During RBDO, the probability of violating given constraint(s), i.e. the failure probability, is often computed by reliability analysis employing methods such as first order reliability method (FORM), second order reliability method (SORM) or response surface method.

Traditionally, RBDO has been performed by use of a nested or “double loop” approach, that is, each step of the iteration for design optimization involves another loop of iteration for reliability analysis. For example, reliability index approach (RIA; Enevoldsen & Sorensen 1994) and performance measure approach (PMA; Tu et al. 1999) employ FORM as the inner loop to perform the reliability analysis effi-

ciently. If the constraints are active, the two approaches yield the same results. However, it is known that PMA is generally more efficient and stable than RIA (Tu et al. 1999).

In general, the double loop approach is computationally expensive. Recently, a single-loop approach (Liang et al. 2004) was proposed to improve efficiency of RBDO. The Karush-Kuhn-Tucker (KKT) optimality condition is used to approximate the design point (or most probable point, MPP) in the inner loop for each constraint. As a result, the inner loop is replaced by a deterministic constraint, which transforms the double loop RBDO problem into an equivalent deterministic optimization problem.

When multiple failure modes need to be considered as the constraints of a design optimization, RBDO is often formulated such that the optimal structure satisfies each failure mode with predetermined probabilities. This approach is termed as component reliability-based design optimization (CRBDO) in this paper. In some cases, however, the failure event is better described by a system event, i.e. a logical (or Boolean) function of multiple failure modes. In this case, the probabilistic constraint should be given for the system event. This approach is called system reliability-based design optimization (SRBDO). The SRBDO requires system reliability analysis, which is not trivial especially for statistically dependent component events, or for a system event that is not series or parallel system.

Theoretical bounding formulas are applicable to parallel and series systems only (See Song & Der Kiureghian 2003 for a review), and it is difficult to deal with probability bounds during RBDO. Various sampling methods are available, but they may render SRBDO inefficient in practice. Song & Kang (2009) developed a matrix-based system reliability (MSR) method that computes the system reliability by convenient matrix-based framework. The MSR method is uniformly applicable to general system events including series, parallel, cut-set and link-set systems with statistical dependence between component events considered, and provides parameter sensitivities of the system failure probability, which are useful during RBDO.

This paper proposes a single-loop SRBDO approach using MSR method (SRBDO/MSR) to overcome aforementioned challenges in SRBDO. After an overview of existing RBDO formulations and methods, the MSR method is briefly introduced. The MSR method is further developed for integration with a single-loop SRBDO approach. The proposed SRBDO/MSR procedure is demonstrated by two numerical examples.

## 2 SYSTEM RELIABILITY BASED DESIGN OPTIMIZATION

### 2.1 Component reliability based design optimization (CRBDO)

In general, RBDO problems are formulated as follows:

$$\begin{aligned} \min_{\mathbf{d}, \boldsymbol{\mu}_X} & f(\mathbf{d}, \boldsymbol{\mu}_X) \\ \text{s.t.} & P[g_i(\mathbf{d}, \mathbf{X}) \leq 0] \leq P_i^t \quad i=1, \dots, n \\ & \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \quad \boldsymbol{\mu}_X^L \leq \boldsymbol{\mu}_X \leq \boldsymbol{\mu}_X^U \end{aligned} \quad (1)$$

where  $\mathbf{d} \in \mathcal{R}^k$  is the vector of deterministic design variables;  $\mathbf{X} \in \mathcal{R}^m$  is the vector of random variables;  $\boldsymbol{\mu}_X$  is the mean vector of  $\mathbf{X}$ ;  $f(\cdot)$  is the objective function;  $\mathbf{U} = \mathbf{U}(\mathbf{X})$  is the vector of standard normal random variables that correspond to  $\mathbf{X}$ ;  $g(\cdot)$  is the limit-state function indicating the occurrence of the failure by  $g(\cdot) \leq 0$ ;  $P_i^t$  is the constraint on the probability of the  $i$ -th mode;  $\mathbf{d}^L$  and  $\mathbf{d}^U$  are the lower/upper bounds on  $\mathbf{d}$ ;  $\boldsymbol{\mu}_X^L$  and  $\boldsymbol{\mu}_X^U$  are the lower/upper bounds on  $\boldsymbol{\mu}_X$  (these boundary values will be omitted in the following RBDO formulations for simplicity); and  $n$ ,  $k$ ,  $m$  are the number of constraints, deterministic variables, and random variables, respectively.

The constraint in Equation 1 can be given alternatively by use of the cumulative distribution function (CDF) of the limit state function, i.e.

$$P[g_i(\mathbf{d}, \mathbf{X}) \leq 0] = F_{g_i}(\mathbf{d}) \leq \Phi(-\beta_i^t) \quad (2)$$

where  $F_{g_i}(\cdot)$  denotes the CDF of  $g_i(\cdot)$ ;  $\Phi(\cdot)$  is the CDF of the standard normal random variable; and  $\beta_i^t$  is the target reliability index. This RBDO problem has two nested optimization loops: the outer loop for design optimization and the inner loop for reliability analysis.

One of the common double-loop approaches for RBDO is the reliability index approach (RIA; Enevoldsen & Sorensen 1994):

$$\begin{aligned} \min_{\mathbf{d}, \boldsymbol{\mu}_X} & f(\mathbf{d}, \boldsymbol{\mu}_X) \\ \text{s.t.} & \beta_i = -\Phi^{-1}[F_{g_i}(\mathbf{d})] \geq \beta_i^t \quad i=1, \dots, n \end{aligned} \quad (3)$$

where  $\beta_i$  is the distance from the origin of the space of standard normal random variables  $\mathbf{U} = \mathbf{U}(\mathbf{X})$  to the nearest point on the limit state surface  $G_i(\mathbf{d}, \mathbf{U}) = 0$  in which  $G_i(\cdot)$  is the limit-state function  $g_i(\cdot)$  given in terms of  $\mathbf{U}$ . This distance is termed as “reliability index.” The nearest point (“design point”) and the corresponding reliability index are identified by solving a nonlinear constrained optimization:

$$\begin{aligned} \mathbf{U}_i^* &= \arg \min_{\mathbf{U}} \|\mathbf{U}\| \\ \text{s.t.} & G_i(\mathbf{d}, \mathbf{U}) = 0 \end{aligned} \quad (4)$$

where  $\mathbf{U}_i^*$  is the design point vector of the  $i$ -th constraint.

The RIA formulation in Equation 3 can be inefficient if the constraints are inactive. Moreover, the algorithm may not provide an optimal design solution if the failure events  $G_i(\mathbf{d}, \mathbf{U}) \leq 0$  never occur in the given feasible domain. To overcome these issues, Tu et al. (1999) recently proposed the performance measure approach (PMA):

$$\begin{aligned} \min_{\mathbf{d}, \boldsymbol{\mu}_X} & f(\mathbf{d}, \boldsymbol{\mu}_X) \\ \text{s.t.} & g_{p_i} = F_{g_i}^{-1}[\Phi(-\beta_i^t)] \geq 0 \quad i=1, \dots, n \end{aligned} \quad (5)$$

where  $g_{p_i}$  is the performance function, which is computed by solving a constrained optimization problem, i.e.

$$\begin{aligned} g_{p_i} &= \min_{\mathbf{U}} G_i(\mathbf{d}, \mathbf{U}) \\ \text{s.t.} & \|\mathbf{U}\| = \beta_i^t \end{aligned} \quad (6)$$

The constraint in Equation 5, i.e.  $g_{p_i} \geq 0$  implies that the design point is located outside the sphere  $\|\mathbf{U}\| = \beta_i^t$ . Therefore, this is a constraint equivalent to  $\beta_i \geq \beta_i^t$ .

These double-loop RBDO approaches are computationally expensive. To improve efficiency, a sequential optimization and reliability assessment (SORA) method was recently proposed (Du & Chen 2004). Its main idea is to decouple the outer loop optimization from reliability analysis. From the information from previous design iteration, the bounda-

ries of the constraints are shifted to the feasible direction and the design point is updated accordingly. Despite its improved efficiency, however, the updated design points may be inaccurate.

Recently, Liang et al. (2004) proposed a single-loop RBDO. The key idea is to obtain the point that satisfies Equation 6 approximately by employing the following Karush-Kuhn-Tucker (KKT) condition instead of solving a nonlinear constrained optimization problem:

$$\begin{aligned} \nabla_{\mathbf{U}} G_i(\mathbf{d}, \mathbf{U}) + \lambda \cdot \nabla_{\mathbf{U}} (\|\mathbf{U}\| - \beta_i^t) &= 0 \\ \|\mathbf{U}\| - \beta_i^t &= 0 \end{aligned} \quad (7)$$

The negative normalized gradient vector of the limit-state function at the solution of Equation 6 is approximately obtained by evaluating it at the solution of Equation 7,  $\mathbf{U} = \tilde{\mathbf{U}}_i$ :

$$\hat{\boldsymbol{\alpha}}_i^t \cong \left( -\frac{\nabla_{\mathbf{X}} g_i(\mathbf{d}, \mathbf{X}(\mathbf{U}))}{\|\nabla_{\mathbf{X}} g_i(\mathbf{d}, \mathbf{X}(\mathbf{U}))\|} \mathbf{J}_{\mathbf{X}, \mathbf{U}} \right)_{\mathbf{U}=\tilde{\mathbf{U}}_i} \quad (8)$$

where  $\mathbf{J}_{\mathbf{X}, \mathbf{U}}$  is the Jacobian of the  $\mathbf{X} = \mathbf{X}(\mathbf{U})$  transformation. The solution of Equation 6 is then approximated by scaling this unit vector by the target reliability index, i.e.

$$\mathbf{U}_i^t \cong \beta_i^t \hat{\boldsymbol{\alpha}}_i^t \quad (9)$$

As a result, the RBDO formulation is

$$\begin{aligned} \min_{\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}}} f(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}}) \\ \text{s.t. } g_i(\mathbf{d}, \mathbf{X}(\mathbf{U}_i^t)) \geq 0 \quad i=1, \dots, n \end{aligned} \quad (10)$$

In summary, the inner-loop of the PMA RBDO is replaced by the approximate, non-iterative procedures in Equations 7-9. This single-loop approach can improve the efficiency of RBDO significantly (Liang et al. 2004).

## 2.2 System reliability based design optimization (SRBDO)

In the case when the failure event in the design constraint is better described by a system event, i.e. a logical (Boolean) function of multiple component events, the RBDO requires a system reliability analysis. This system reliability based design optimization (SRBDO) can be formulated as

$$\begin{aligned} \min_{\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}}} f(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}}) \\ \text{s.t. } P_{\text{sys}} = \text{P}(E_{\text{sys}}) = \text{P} \left[ \bigcup_k \bigcap_{i \in C_k} g_i(\mathbf{d}, \mathbf{X}) \leq 0 \right] \leq P^t \end{aligned} \quad (11)$$

where  $P_{\text{sys}}$  is the system failure probability;  $E_{\text{sys}}$  is the system failure event;  $C_k$  is the index set of the components in the  $k$ -th cut set; and  $P^t$  is the target

system failure probability. The Boolean expression in Equation 11 can represent general systems including series, parallel, and cut-set systems.

An SRBDO approach was proposed for series system problems by Ba-abbad et al. (2006). In this approach, the failure probability of a series system is approximated as the sum of the component failure probabilities, i.e..

$$P_{\text{sys}} = \text{P} \left[ \bigcup_{i=1}^n g_i(\mathbf{d}, \mathbf{X}) \leq 0 \right] \cong \sum_{i=1}^n P_i \quad (12)$$

Then, SRBDO problems are formulated as

$$\begin{aligned} \min_{\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}}, P_1^t, \dots, P_n^t} f(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}}) \\ \text{s.t. } \text{P}[g_i(\mathbf{d}, \mathbf{X}) \leq 0] \leq P_i^t \quad i=1, \dots, n \\ P_{\text{sys}} \cong \sum_{i=1}^n P_i^t \leq P^t \end{aligned} \quad (13)$$

Note that the constraints on the component probabilities,  $P_i^t$  are used as design variables. This approach can significantly overestimate the system risk because the approximation in Equation 12 provides a fairly conservative upper bound (See Song & Der Kiureghian 2003). Moreover, this approach cannot account for the effect of the statistical correlation between random variables or component events.

A single-loop SRBDO approach was recently proposed for series systems (Liang et al. 2007). This approach also uses  $P_i^t$ 's as design variables. The inner loop is eliminated by approximating the design points by KKT conditions. The system failure probability is approximated by the upper bound in the bi-component theoretical bounding formula by Ditlevsen (1979). The single-loop SRBDO is formulated as

$$\begin{aligned} \min_{\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}}, P_1^t, \dots, P_n^t} f(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}}) \\ \text{s.t. } g_i(\mathbf{d}, \mathbf{X}(\mathbf{U}_i^t)) \geq 0 \quad i=1, \dots, n \\ P_{\text{sys}} \cong \sum_{i=1}^n P_i^t - \sum_{i=2}^n \max_{j < i} P_{ij}^t \leq P^t \end{aligned} \quad (14)$$

in which  $\mathbf{U}_i^t$  is obtained by Equations 7-9; and  $P_{ij}^t$  is the joint failure probability of the  $i$ -th and  $j$ -th constraints, computed by numerical integration. Despite its improved accuracy in estimating the system failure probability by using a higher-order bounding formula, it still overestimates the system failure probability and is not applicable to non-series system events for which higher-order theoretical bounding formulas are generally not available.

This paper proposes to use the recently proposed matrix-based system reliability (MSR) method to compute  $P_{\text{sys}}$  in the single-loop SRBDO in Equation 14. The method enables us to compute  $P_{\text{sys}}$  of general system events including series, parallel, cut-set and link-set systems efficiently and accurately

during SRBDO. The sensitivity of  $P_{\text{sys}}$  with respect to design variables further facilitates the use of gradient-based optimization algorithm.

### 3 SYSTEM RELIABILITY BASED DESIGN OPTIMIZATION USING MSR METHOD

#### 3.1 Matrix-based system reliability (MSR) method

Although system reliability analysis is a well established research area, it is still challenging to compute the probability of a general system event and its parameter sensitivity, especially when component events are statistically dependent. Song & Der Kiureghian (2003) introduced a method to compute the bounds on the probability of a general system event by linear programming (LP). This ‘‘LP bounds’’ method subdivides the sample space of component events into the mutually exclusive and collectively exhaustive events (termed as basic MECE events), and the probability of any event is described by use of vectors representing the probabilities of basic MECE events. Then, its upper bounds and lower bounds are obtained by solving the LP problems subjected to the constraints derived from given information such as component probabilities and statistical dependence. This matrix-based framework of system reliability analysis enables the narrowest possible bounds on the probability of any general system and the parameter sensitivities of the bounds (Song & Der Kiureghian 2005) as well.

Song & Kang (2009) recently proposed the Matrix-based System Reliability (MSR) method to compute the probability of general system events in a uniform manner by use of simple matrix calculation instead of solving LP. Consider a system event with  $n$  components each of which has two distinct states, e.g., failure or safe. Then, the sample space can be subdivided into  $N = 2^n$  basic MECE events, denoted by  $e_j$ ,  $j = 1, \dots, N$ . Then any system event can be presented by an ‘‘event’’ vector  $\mathbf{c}$  whose  $j$ -th element is 1 if  $e_j$  belongs to the system event and 0 otherwise. Let  $p_j = P(e_j)$ ,  $j = 1, \dots, N$ , denote the probability of  $e_j$ . Because  $e_j$ 's are mutually exclusive, the probability of system event,  $P_{\text{sys}}$  is the sum of the probability of  $e_j$ 's that belong to the system event  $E_{\text{sys}}$ . Therefore, the system probability is computed by the inner product of the two vectors.

$$P_{\text{sys}} = \sum_{j: e_j \subseteq E_{\text{sys}}} p_j = \mathbf{c}^T \mathbf{p} \quad (15)$$

where  $\mathbf{p}$  is the ‘‘probability’’ vector that contains  $e_j$ 's,  $j = 1, \dots, N$ . Both  $\mathbf{c}$  and  $\mathbf{p}$  are column vectors in this paper, and can be constructed efficiently using matrix-based procedures by Song & Kang (2009).

When component events are statistically dependent, the construction of  $\mathbf{p}$  requires numerous system reliability analysis for each element. In this

case, one can achieve conditional independence between component events given outcomes of a few random variables representing the sources of ‘‘environment dependence’’ or ‘‘common source effects.’’ For example, during a risk analysis of a transportation network based on bridge failure probabilities, the uncertain magnitude of earthquake was considered as such a random variable (Song & Kang 2009). Let  $\mathbf{S}$  denote the vector of such random variables, named ‘‘common source random variables’’ (CSRV). By the total probability theorem, the system failure probability can be then computed as

$$P_{\text{sys}} = \int_{\mathbf{s}} P(E_{\text{sys}} | \mathbf{s}) f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s} \\ = \int_{\mathbf{s}} \mathbf{c}^T \mathbf{p}(\mathbf{s}) f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s} \quad (16)$$

where  $P(E_{\text{sys}} | \mathbf{s})$  is the conditional probability of the system event given an outcome of CSRV,  $\mathbf{S} = \mathbf{s}$ ;  $f_{\mathbf{S}}(\mathbf{s})$  is the joint probability density function (PDF) of  $\mathbf{S}$ ; and  $\mathbf{p}(\mathbf{s})$  is the conditional probability vector given  $\mathbf{S} = \mathbf{s}$ , which can be constructed efficiently by the proposed matrix-based procedure employing conditional probabilities of component events given  $\mathbf{S} = \mathbf{s}$ .

The approach in Equation 16 can be applied even in the case when the CSRVs are not explicitly identified. One way to identify such implicit common source effect is to fit the correlation coefficient matrix of basic random variables or safety margin (or factor) with a special correlation matrix model that allows such an identification. Song & Kang (2009) generalized Dunnett-Sobel (DS) class correlation matrix (Dunnett & Sobel 1955) to identify CSRVs. Consider correlated standard normal random variables  $Z_i$ ,  $i = 1, \dots, n$  whose correlation matrix can be fit with the generalized DS model

$$Z_i = \left( 1 - \sum_{k=1}^m r_{ik}^2 \right)^{0.5} U_i + \sum_{k=1}^m r_{ik} S_k \quad \text{for } i = 1, \dots, n \quad (17)$$

in which  $U_i$ ,  $i = 1, \dots, n$  and  $S_k$ ,  $k = 1, \dots, m$  are uncorrelated standard normal random variables; and  $r_{ik}$ 's are the coefficients of the generalized DS model that determined the correlation coefficient between  $Z_i$  and  $Z_j$ , as  $\rho_{ij} = \sum_{k=1}^m (r_{ik} \cdot r_{jk})$  for  $i \neq j$ . Note  $Z_i$  and  $Z_j$  are conditionally independent of each other given the outcome of CSRVs  $S_k$ ,  $k = 1, \dots, m$ .

#### 3.2 Sensitivity of system failure probability

The MSR method enables us to compute the parameter sensitivity of the probability of a general system event. First, when the component events are statistically independent, the sensitivity of the system failure probability with respect to a parameter  $\theta$  is computed as

$$\frac{\partial P_{\text{sys}}}{\partial \theta} = \mathbf{c}^T \frac{\partial \mathbf{p}}{\partial \theta} \quad (18)$$

The separation of the system event description ( $\mathbf{c}$ ) and the probabilities ( $\mathbf{p}$ ) in the MSR framework allows us to compute the sensitivity in a uniform manner for general system events. The sensitivity of  $\mathbf{p}$  in Equation 18 can be computed by the following matrix-based procedure (Song & Kang 2009):

$$\frac{\partial \mathbf{p}}{\partial \theta} = \begin{bmatrix} \mathbf{p}^{(1)} & \mathbf{p}^{(2)} & \dots & \mathbf{p}^{(n)} \end{bmatrix} \frac{\partial \mathbf{P}}{\partial \theta} = \hat{\mathbf{P}} \frac{\partial \mathbf{P}}{\partial \theta} \quad (19)$$

where  $\mathbf{P} = [P_1 \ P_2 \ \dots \ P_n]^T$  in which  $P_i$  is the probability of the  $i$ -th component event; and  $\mathbf{p}^{(j)}$ ,  $j = 1, \dots, n$  is the probability vector constructed with the probabilities of the  $j$ -th component event and its complementary event replaced by 1 and  $-1$ , respectively. In summary, the MSR framework allows us to compute the system-level parameter sensitivity by use of component probabilities and their sensitivities.

When the components are statistically dependent, the sensitivity is computed as

$$\frac{\partial P_{\text{sys}}}{\partial \theta} = \int_{\mathbf{s}} \mathbf{c}^T \frac{\partial \mathbf{p}(\mathbf{s})}{\partial \theta} f_{\mathbf{s}}(\mathbf{s}) d\mathbf{s} \quad (20)$$

in which the sensitivity in the integral is constructed by Equation 19 using the conditional probability of the component events given  $\mathbf{S} = \mathbf{s}$ , i.e.

$$P_i(\mathbf{s}) = P(\beta_i - Z_i \leq 0 | \mathbf{S} = \mathbf{s}), \quad i = 1, \dots, n \quad (21)$$

Substituting Equation 17 into Equation 21,

$$P_i(\mathbf{s}) = \Phi \left[ - \left( \beta_i - \sum_{k=1}^m r_{ik} s_k \right) / \left( 1 - \sum_{k=1}^m r_{ik}^2 \right)^{0.5} \right] \quad (22)$$

The sensitivity of  $P_i(\mathbf{s})$  with respect to the reliability index  $\beta_i$  is derived as

$$\frac{\partial P_i(\mathbf{s})}{\partial \beta_i} = \frac{\varphi \left[ - \left( \beta_i - \sum_{k=1}^m r_{ik} s_k \right) / \left( 1 - \sum_{k=1}^m r_{ik}^2 \right)^{0.5} \right]}{\left( 1 - \sum_{k=1}^m r_{ik}^2 \right)^{0.5}} \quad (23)$$

in which  $\varphi(\cdot)$  denotes the PDF of the standard normal random variable. The sensitivity of system failure probability with respect to the  $i$ -th component probability is derived as

$$\frac{\partial P_i(\mathbf{s})}{\partial P_i} = \frac{\partial P_i(\mathbf{s})}{\partial \beta_i} \cdot \frac{\partial \beta_i}{\partial P_i} = - \frac{\partial P_i(\mathbf{s})}{\partial \beta_i} \cdot \frac{1}{\varphi(-\beta_i)} \quad (24)$$

This sensitivity is used for computing the sensitivity vector using Equation 19 or 20.

### 3.3 SRBDO/MSR algorithm

Figure 1 shows the flowchart of the proposed SRBDO/MSR algorithm.

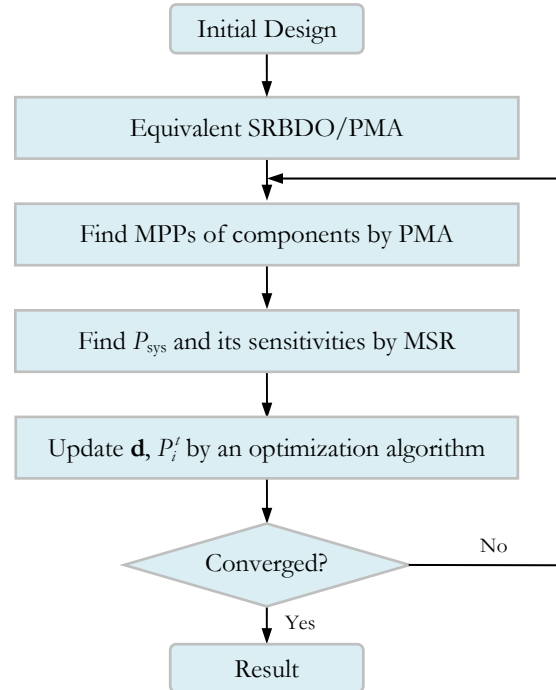


Figure 1. Flowchart of the proposed SRBDO/MSR algorithm

## 4 NUMERICAL EXAMPLES

### 4.1 SRBDO of an indeterminate truss structure

The proposed SRBDO/MSR is demonstrated through application to an SRBDO example of a six-bar statistically indeterminate truss (Figure 2; MacDonald & Mahadevan 2008).

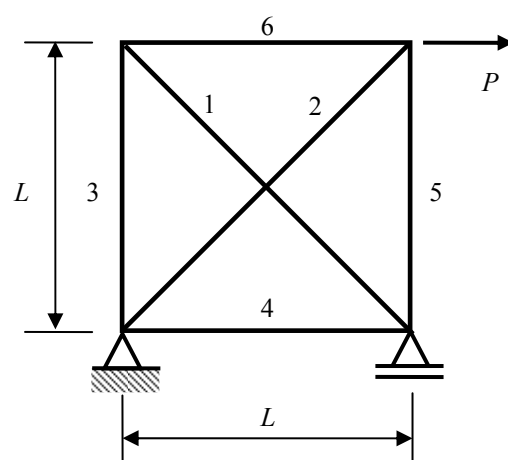


Figure 2. A six-bar indeterminate truss example.

The yielding failures of six members are modeled as component events. When the buckling failure modes, the dynamic effect of member damages, and the influence of the load re-distribution on progressive failures (Song & Kang 2008, 2009) are neglected, the system fails when at least two bars yields. Therefore, the system event is described by 15 minimal cut sets:  $\{C_k\} = \{(1,2), (1,3), (1,4), (1,5),$

(1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)}.

To minimize the weight of the structure, the objective function is defined such that it is proportional to the total weight. The design variables are the cross sectional area of the bars,  $A_i$ ,  $i=1, \dots, 6$ . The target system failure probability is given as 0.001. The load  $P$  is normally distributed with the mean of 1,000 kips and a standard deviation of 100 kips while the yield strength of each bar (in stress) is assumed to be a normal distribution with the mean 36 ksi and the standard deviation 3 ksi. As a result, the SRBDO problem is formulated as

$$\begin{aligned} \min_{\mathbf{d}=\{A_1, \dots, A_6\}} f(\mathbf{d}) &= 1.414(A_1 + A_2) + A_3 + A_4 + A_5 + A_6 \\ \text{s.t.} \quad P_{\text{sys}} &= \mathbb{P} \left[ \bigcup_k \bigcap_{i \in C_k} g_i(\mathbf{d}, \mathbf{X}) \leq 0 \right] \leq 0.001 \\ g_i(\mathbf{d}, \mathbf{X}) &= \begin{cases} A_i F_i - 0.707P & i=1, 2 \\ A_i F_i - 0.500P & i=3, \dots, 6 \end{cases} \\ A_1, A_2, A_3, A_4, A_5, A_6 &\geq 0 \end{aligned} \quad (25)$$

where  $F_i$  is the yield stress of the  $i$ -th bar,  $i=1, \dots, 6$ . In the study by MacDonald & Mahadevan (2008), a single-loop SRBDO approach as the one in Equation 14 was used except that the system failure probability was computed as follows. First, the failure probability of each cut set was calculated as a parallel system using the product of conditional marginals (PCM) approach (Pandey 1998). Then the system failure probability was approximated by the first order bounding formula.

Table 1. Results of SRBDO of indeterminate truss

Bar/ Comp.	Area ( $A_i$ )		Reliability Index ( $\beta_i$ )	
	MM*	SRBDO/ MSR	MM*	SRBDO/ MSR
1	28.57	27.73	2.89	2.67
2	28.32	27.73	2.83	2.67
3	20.94	20.46	3.16	2.99
4	20.83	20.46	3.12	2.99
5	20.66	20.46	3.06	2.99
6	20.29	20.46	2.92	2.99

\* MacDonald & Mahadevan (2008)

The SRBDO/MSR approach proposed in this paper was applied to this example. During SRBDO formulated in Equation 14, the system failure probability and its sensitivities with respect to  $P_i'$  are computed by MSR method without approximation, as explained in Section 3. The computed sensitivities facilitate the use of a gradient-based optimization algorithm. Table 1 compares the results by the two

approaches. The minimum objective function value of the proposed approach is 160.25, which is less than that by the approximation method, 163.16. This is due to the overestimate of the system failure probability by the first-order bounding method, which results in more conservative design than required. This is also evidenced by the lower reliability indexes of the component events by the proposed approach. It is also noteworthy that the accurate system reliability estimates during the SRBDO/MSR reflect the symmetric conditions between diagonal members (1 and 2) and between non-diagonal members (3-6) in the optimal design (area) and the component failure probability (reliability index).

#### 4.2 Example of system reliability based topology optimization (SRBTO)

In many practical design problems, an optimal distribution of the material in a certain domain is of interest. This so-called topology optimization (TO) is of importance in various applications since it may lead to a suitable structure layout with cost saving and design improvement. One might seek for minimum compliance within a constraint on the volume or minimum volume with constraints on compliance or displacement (Kim et al. 2006). During TO using finite element analysis, main design variables are the element densities  $\rho$ . An element with  $\rho=1$  is solid while  $\rho=0$  indicates a void. The TO algorithms include the solid isotropic material with penalization (SIMP) method and the "projection" method (Guest et al. 2004; Almeida et al. 2008).

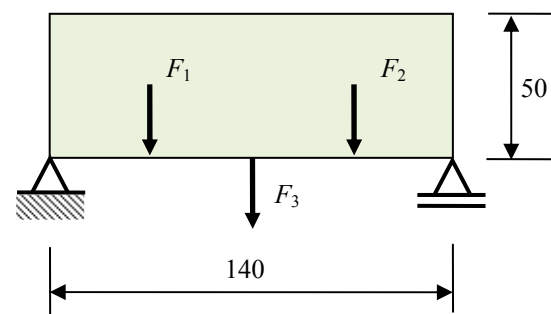


Figure 3. Domain for topology optimization.

In order to examine the influence of the uncertainty treatment during TO on the optimal topology, deterministic TO (DTO), component based RBTO (CRBTO) and system reliability based TO (SRBTO) have been performed to minimize the volume of a structure distributed within the domain of simple beam in Figure 3. The material properties, loads and displacements are given as dimensionless quantities for simplicity. The isotropic material is assumed to have Young's modulus of  $E_0 = 2 \times 10^8$  and Poisson ratio of  $\nu = 0.3$ . The beam has the thickness of 0.05. These material properties are assumed to be deterministic since the uncertainties in material



properties have minimal effect on reliability-based optimal topologies for a structure under linear elastic behavior. The three stochastic loads  $F_1$ ,  $F_2$  and  $F_3$  are applied as shown in Figure 3. They are modeled as Gaussian random variables with the means 300,000 and standard deviations  $\sigma_{F_1} = \sigma_{F_2} = 15,000$  and  $\sigma_{F_3} = 100,100$ . The three random variables are assumed to be uncorrelated. The constraints on the displacements at the locations of the applied forces are described by the limit-state functions

$$g_i(\boldsymbol{\rho}, \mathbf{F}) = d_i(\boldsymbol{\rho}, \mathbf{F}) - d_i^t \quad i=1,2,3 \quad (26)$$

where  $\boldsymbol{\rho}$  denotes the vector of element densities  $\rho$ ;  $\mathbf{F}$  is the vector of the random applied forces;  $d_i(\boldsymbol{\rho}, \mathbf{F})$  is the vertical displacement at the  $i$ -th location predicted by a finite element analysis; and  $d_i^t$  is the limit on the displacement (given as 1.5).

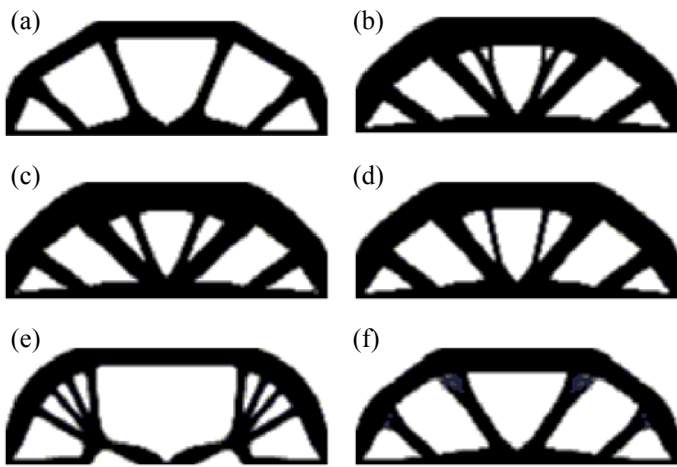


Figure 4. Optimal topologies by DTO, CRBTO and SRBTO.

First, a DTO is performed with the loads equal to the given mean values. Figure 4a shows the optimal design. The ratio of the optimal volume to the initial domain volume,  $r_v$  is 40.9%. Next, a CRBTO is conducted with target reliability index  $\beta_i^t = 2$ , (or  $P_i^t = 0.02275$ ). The optimal topology in Figure 3b has  $r_v = 60.4\%$ . The optimal volume is higher than that by the DTO since the topology that avoids the failure under the mean loads exceeds the constraints on the component failure probabilities. After the optimization is completed, the system failure probability (series system) is estimated by the MSR method as  $P_{\text{sys}} = 0.0434$ . Suppose one now seeks for an optimal topology whose system failure probability is lower than the system failure probability value of the topology in Figure 4b, i.e.  $P_{\text{sys}} = 0.0434$ . If CRBTO is performed in lieu of SRBTO, one possible way to determine the target failure probabilities of components manually is to distribute the system failure probability equally to the components based on the approximation shown in Equation 12, i.e.  $P_i^t = P_{\text{sys}} / 3 = 0.0145$ . This leads to the topology in Figure 4c whose volume ratio is 62.7%, which is higher than that by the previous CRBTO because the

actual target system failure probability is smaller than 0.0434 due to the overestimate in Equation 12.

An SRBTO/MSR is performed with the same target system failure probability  $P_{\text{sys}} = 0.0434$ . The volume ratio is 58.2% (See Figure 4d), which is lower than both CRBTO results. The reason why it is lower than even the first CRBTO with the same system failure probability is that CRBTO is more constrained than SRBTO by manually assigned component probabilities, even if they lead to the same level of the system failure probability. For the optimal topology, the component failure probabilities are  $P_1^t = P_2^t = 0.0084$  and  $P_3^t = 0.0389$ , which indicates that the third mode is more critical than the others when achieving the optimal design.

Taking advantage of the uniform applicability of the MSR method to general system problems, SRBTOs are performed for a parallel system, i.e.  $E_{\text{sys}} = E_1 E_2 E_3$  in which  $E_i$  denotes the occurrence of the  $i$ -th failure mode,  $i = 1, 2, 3$  (Figure 4e). The target system failure probability is 0.005 and the volume ratio is 44.5%. Figure 4f shows the optimal topology when the system failure event is defined as  $\bar{E}_{\text{sys}} = \bar{E}_1 \bar{E}_2 E_3$  and the target failure probability is 0.005. The volume ratio is 47.0%. The different topologies in Figure 4 confirm that the system event definition can make significant influence on the optimal shapes.

We also investigate the effect of the statistical correlation between random variables on TO. Figure 5 shows the volume ratios of the optimal topologies as the correlation coefficient between the three loads is varied. The target failure probability of a series system is given as 0.0424. It is seen that the higher correlation between the loads, the more volume is required to achieve the target reliability. In this problem, therefore, if the positive correlation is ignored, RBTO may lead to an unsafe design. This is because the uncertainty in the displacement is increased by the positive correlation between the applied loads.

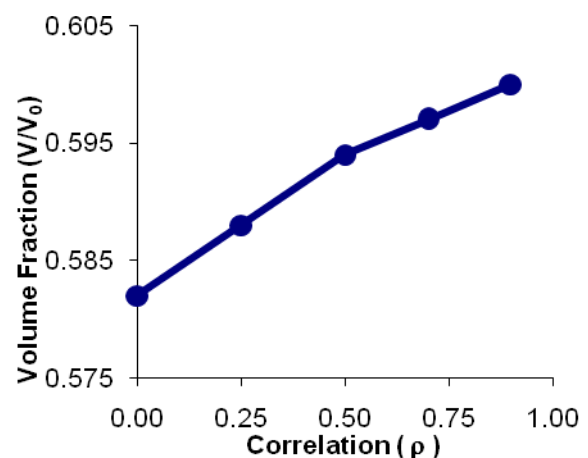


Figure 5. Volume ratio of SRBTO results versus correlation coefficient between applied loads.

The impact of approximations in system reliability analysis during SRBTO is also investigated using the same TO example with series system event. In Case 1,  $P_{sys}$  is approximated by the sum of the three mode probabilities. In Case 2,  $P_{sys}$  and its sensitivities are computed after components are assumed to be statistically independent of each other. In Case 3,  $P_{sys}$  is computed accurately but the sensitivities are calculated approximately assuming that the components are statistically independent. Table 2 shows the constraint on the system failure probability,  $P^f$ , the system failure probability of the optimal topologies computed by the MSR method (afterward for Case 1-3),  $P_{sys}$ , and the volume ratio. In this example, all the approximate SRBTO result in unnecessary conservatism. The close results by Case 3 indicate that the approximation in the sensitivities do not make significant impact on the optimal topology.

Table 2. Approximation of SRBTO by CRBTO

$\rho^*$	Results	System Reliability Approx.			SRBTO /MSR
		Case 1	Case 2	Case 3	
0	$P^f$		0.0434		
	$P_{sys}$	0.0377	0.0378	0.0434	0.0434
	$V_r$	0.591	0.590	0.583	0.582
0.5	$P^f$		0.0434		
	$P_{sys}$	0.0365	0.0365	0.0434	0.0434
	$V_r$	0.611	0.610	0.597	0.594

\* Correlation coefficient between applied loads.

## 5 CONCLUSION

In this study, an efficient and accurate system reliability based design optimization (SRBDO) approach was proposed by integrating a single-loop RBDO algorithm and the recently developed matrix-based system reliability (MSR) method. The MSR method enables accurate calculation of system probability and its sensitivities for general system problems including series, parallel, cut-set and link-set systems. Two numerical examples demonstrate the merits of the proposed SRBDO/MSR approach. The impact of errors in system reliability evaluations, system event definitions and correlation between random variables on the optimal topology were investigated as well.

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