ON RESTRICTION METHODS FOR TWO PHASE OPTIMAL DESIGN PROBLEMS

Cameron Talischi, Glaucio Paulino

Department of Civil and Environmental Engineering
University of Illinois at Urbana-Champaign

MOTIVATION:

- Recently there has been great interest in implicit function and level set methods for topology optimization
- Since the definition and motion of the interface is restricted to have certain regularity, these methods only make sense for "restriction" formulations
- In general it is not clear what continuum problem is being solved and its ill-posedness has certain implications for the numerical algorithm

PROBLEM STATEMENT:

- The two phase optimal design problem is given by:

$$\inf_{\chi \in \mathcal{H}} I(\chi, \mathbf{u}) \quad \text{where } \mathbf{u} \in \mathcal{V} \text{ solves} \quad \mathcal{B}(\mathbf{u}, \mathbf{v}; \chi) = \ell(\mathbf{v}), \ \forall \mathbf{v} \in \mathcal{V}$$

Here $\Omega \subset \mathbb{R}^d$ is open and smooth, $\mathcal{V} = \left\{ \mathbf{u} \in H^1\left(\Omega; \mathbb{R}^d\right) : \mathbf{u}|_{\Gamma_D} = \mathbf{0} \right\}$,

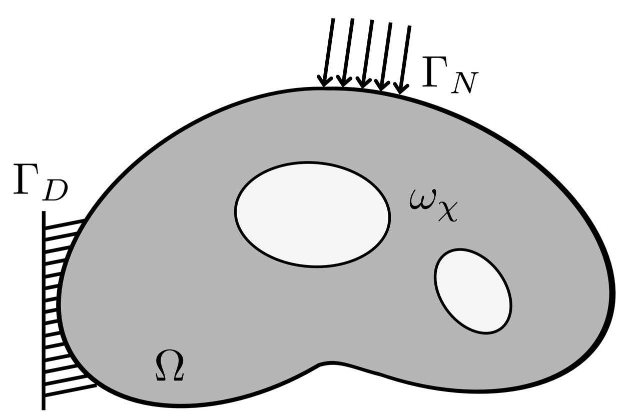
$$\mathcal{B}(\mathbf{u}, \mathbf{v}; \chi) = \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \left[\chi \mathbf{C}_+ + (1-\chi)\, \mathbf{C}_-\right] : \boldsymbol{\varepsilon}(\mathbf{v}) d\mathbf{x}, \quad \ell(\mathbf{u}) = \int_{\Gamma_N} \mathbf{t} \cdot \mathbf{v} ds$$

and the objective function $I(\chi, \mathbf{u})$ is continuous on strongly topology of $L^p(\Omega) \times H^1\left(\Omega; \mathbb{R}^d\right)$

- The classical space of admissible designs is:

$$\mathcal{H}_C = \left\{ \chi \in L^\infty(\Omega; \{0, 1\}) : \int_\Omega \chi d\mathbf{x} \leq V_+ \right\}$$

Each $\chi \in \mathcal{H}_C$ is the characteristic function for the set occupied by the solid phase $\mathbf{C}_+$



- Note that each $\chi = H(\varphi)$ for some implicit function $\varphi \in \mathcal{F} = L^\infty\left(\Omega, [-\alpha, \alpha]\right)$

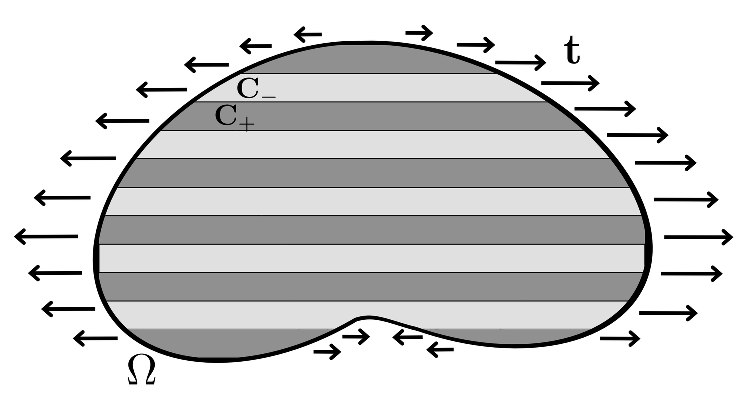
ILL-POSEDNESS:

- This problem is in general ill-posed since it admits no solutions in the classical space of admissible designs
- Consider the counterexample:

$$V_+ = \frac{1}{2}\mu(\Omega), \quad \Gamma_D = \emptyset, \quad \mathbf{t} = (\mathbf{e}_d \otimes \mathbf{n}) \cdot t_0 \mathbf{e}_d, \quad I(\chi, \mathbf{u}) = \ell(\mathbf{u})$$

Let $\varphi_n(\mathbf{x}) = \sin(nx_1)$. Then $\chi_n = H(\varphi_n)$ is a *minimizing sequence* that does not converge to an element of $\mathcal{H}_C$

- The optimal design for problem is a rank-1 laminate with laminations in $\mathbf{e}_1$ direction and constant volume fraction of the phases



REFERENCES:

- Allaire, *Shape Optimization by the Homogenization Method*, Springer (2002)
- Liu, Neittaanmaki, Tiba, Existence for shape optimization problems in arbitrary dimension, *SIAM J. Control Optim.*, 41, 1440-1454 (2003)

RESTRICTION OF $\mathcal{F}$:

- To exclude such *oscillations*, $\mathcal{H}_C$ must be replaced by a smaller space with the necessary compactness property
- If $\chi_n, \hat{\chi} \in L^\infty(\Omega; [0, 1])$ and χ_n converges to $\hat{\chi}$ in $L^p(\Omega)$, then up to a subsequence, $\mathbf{u}_n \to \hat{\mathbf{u}}$ strongly in $H^1(\Omega)$
- It follows that compactness in strong topology of $L^p(\Omega)$ is a *sufficient* condition for existence of solutions
- One such choice is for the implicit functions $\varphi \in \mathcal{F} \subset W^{1+\theta,p}$ to satisfy:

$$\text{H1}: \quad \|\varphi\|_{W^{1+\theta,p}(\Omega)} \leq M$$
$$\text{H2}: \quad |\varphi(\mathbf{x})| + |\nabla\varphi(\mathbf{x})| \geq c \quad \text{a.e. in } \Omega$$

for some positive constants θ, M and c

- It can be shown that the space $\mathcal{H}_R = \{\chi \in \mathcal{H}_C : \chi = H(\varphi), \varphi \in \mathcal{F}\}$ is compact in $L^p(\Omega)$

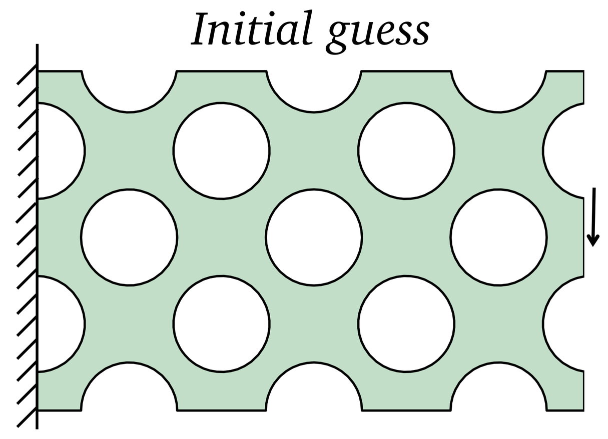
SIGNIFICANCE OF THE CONSTRAINTS:

- H1 excludes the possible rapid oscillations of minimizing sequences:
  - Note that in the counterexample, $\|\varphi_n\|_{W^{1+\theta,p}(\Omega)} \to \infty$.
- H2 ensures that the domain boundary $\{\varphi = 0\}$, where the Heaviside is discontinuous, has zero measure:
  - Without it, $\varphi_n(\mathbf{x}) = (1/n)\sin(nx_1)$ gives a minimizing sequence that satisfies H1 but does not converge
- In essence, these conditions together introduce a minimum length scale into the problem

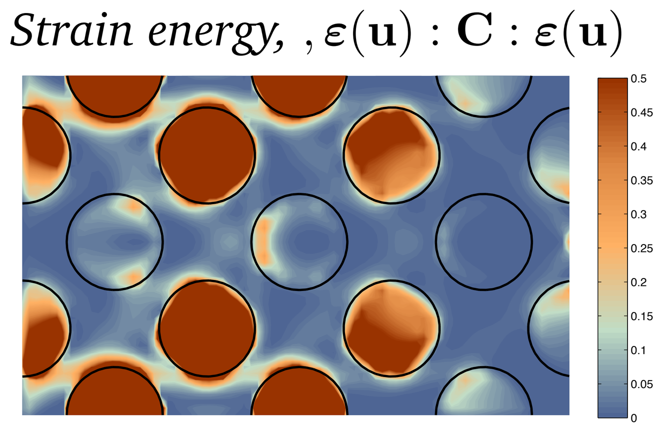
IMPLICATIONS FOR NUMERICAL SIMULATIONS:

- Without H2, the usual approximation of the Heaviside would transform the problem into the variable thickness problem regardless of width of the smeared Heaviside:
  - Thus fattening of the level set function and results with large regions of the grey are expected unless reinitialization are performed frequently

Initial guess

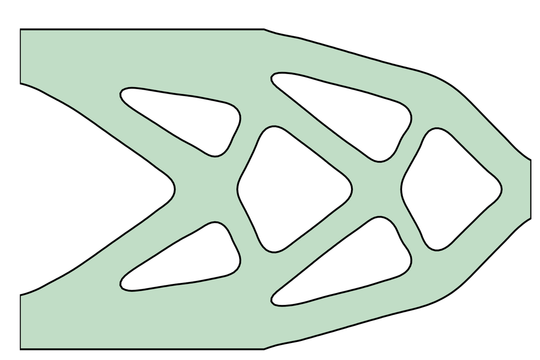


Strain energy, $, \boldsymbol{\varepsilon}(\mathbf{u}) : \mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{u})$

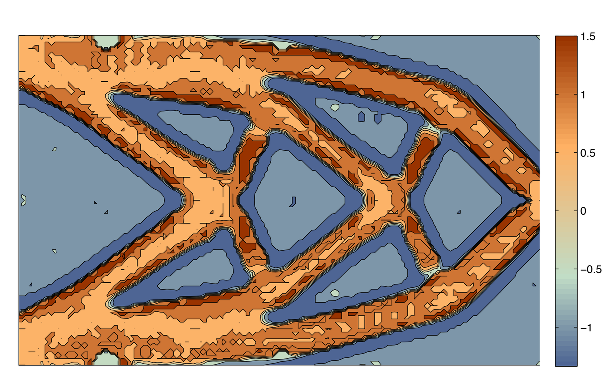


$$\frac{\partial\varphi}{\partial t} + v_n(\varphi)\,|\nabla\varphi| = 0 \quad \xrightarrow{|\nabla\varphi| = 1} \quad \frac{\partial\varphi}{\partial t} = \boldsymbol{\varepsilon}(\mathbf{u}) : \mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{u}) - \lambda$$

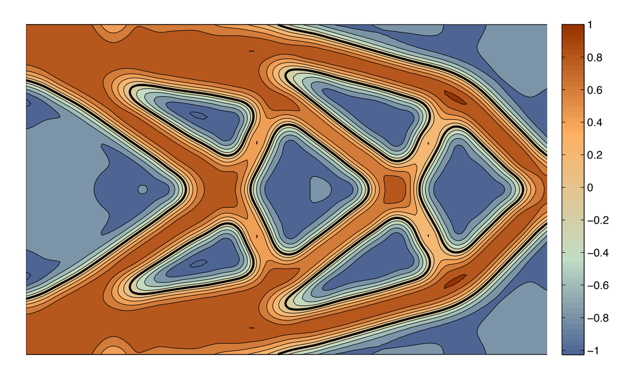
- This observation sheds light on the appropriate choice of velocity extension in level set methods and opportunities for mathematical programming techniques
- H1 can be imposed via convolution of the design field with a smooth *filter* function, as is common in the density methods



Final design



Design field, ψ



Filtered field, $\varphi = \mathcal{S} \star \psi$