Single-loop System Reliability-Based Design & Topology Optimization (SRBDO/SRBTO): A Matrix-based System Reliability (MSR) Method

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Deterministic Optimization vs. RBDO

A. Deterministic Optimization

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} \quad f(\mathbf{d},\boldsymbol{\mu}_\mathbf{X})$$

$$s.t. \quad g_i(\mathbf{d},\mathbf{X}) \ge 0 \quad i=1,\ldots,n$$

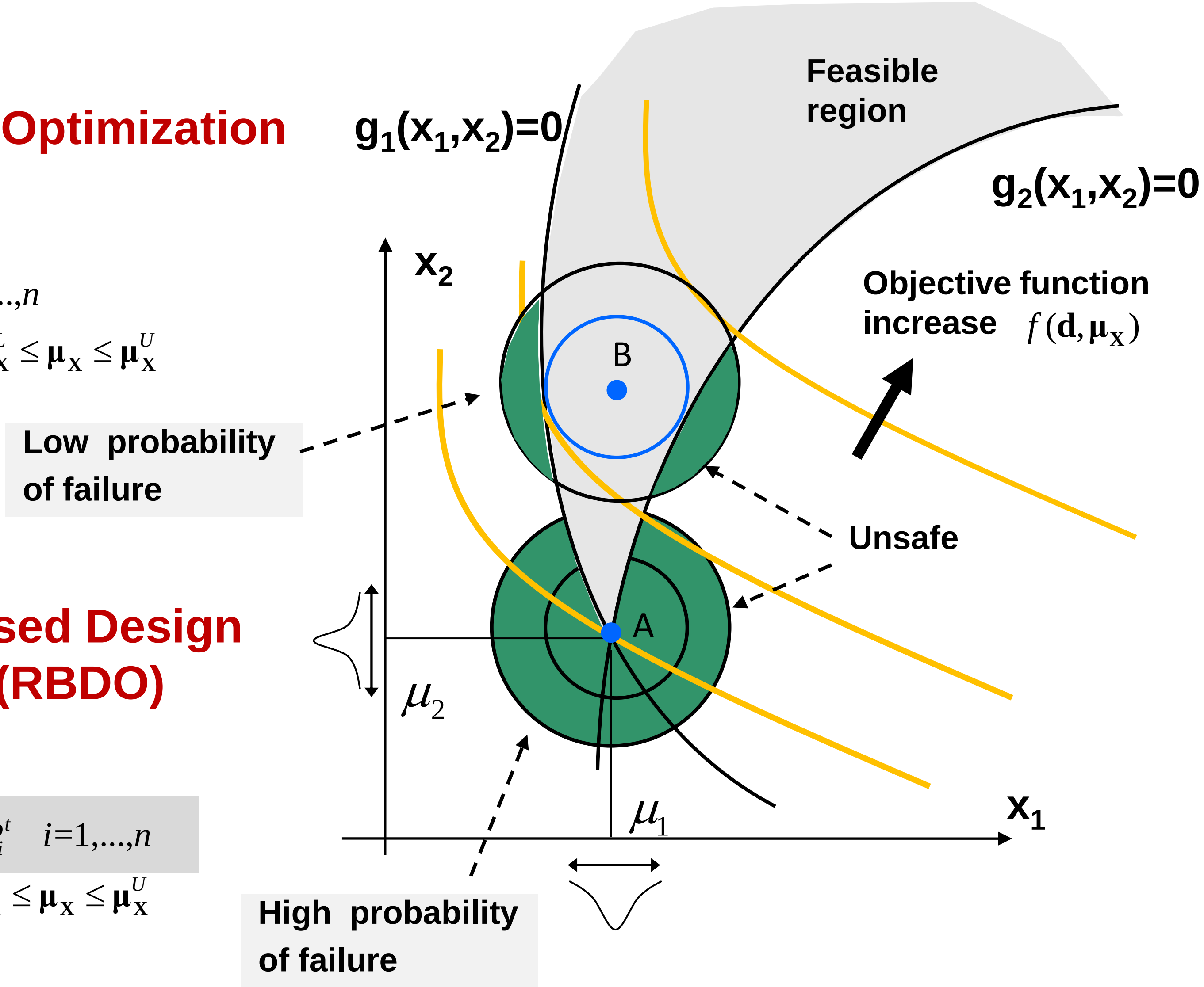
$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U, \qquad \boldsymbol{\mu}_\mathbf{X}^L \le \boldsymbol{\mu}_\mathbf{X} \le \boldsymbol{\mu}_\mathbf{X}^U$$

B. Reliability-Based Design Optimization (RBDO)

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} \quad f(\mathbf{d},\boldsymbol{\mu}_\mathbf{X})$$

$$s.t. \quad P\left[g_i(\mathbf{d},\mathbf{X}) \le 0\right] \le P_i^t \quad i=1,\ldots,n$$

$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U, \qquad \boldsymbol{\mu}_\mathbf{X}^L \le \boldsymbol{\mu}_\mathbf{X} \le \boldsymbol{\mu}_\mathbf{X}^U$$



RBDO Approaches (1): RIA vs. PMA

Reliability Index Approach

Enevoldsen and Sorensen (1994)

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathrm{X}} \quad f(\mathbf{d},\boldsymbol{\mu}_\mathrm{X})$$

$$s.t \quad \beta_i = -\Phi^{-1}\left[F_{G_i}(0)\right] \geq \beta_i^t$$

$$where\ \beta = \min \|\mathbf{U}\|$$

$$s.t \quad \mathrm{G}(\mathbf{U}) = 0$$

Performance Measure Approach

Tu et al (1999)

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathrm{X}} \quad f(\mathbf{d},\boldsymbol{\mu}_\mathrm{X})$$

$$s.t \quad \mathrm{G}_{p_i} = F_{G_i}^{-1}\left[\Phi(-\beta_i^t)\right] \geq 0$$

$$where\ \mathrm{G}_p = \min\ \mathrm{G}(\mathbf{U})$$

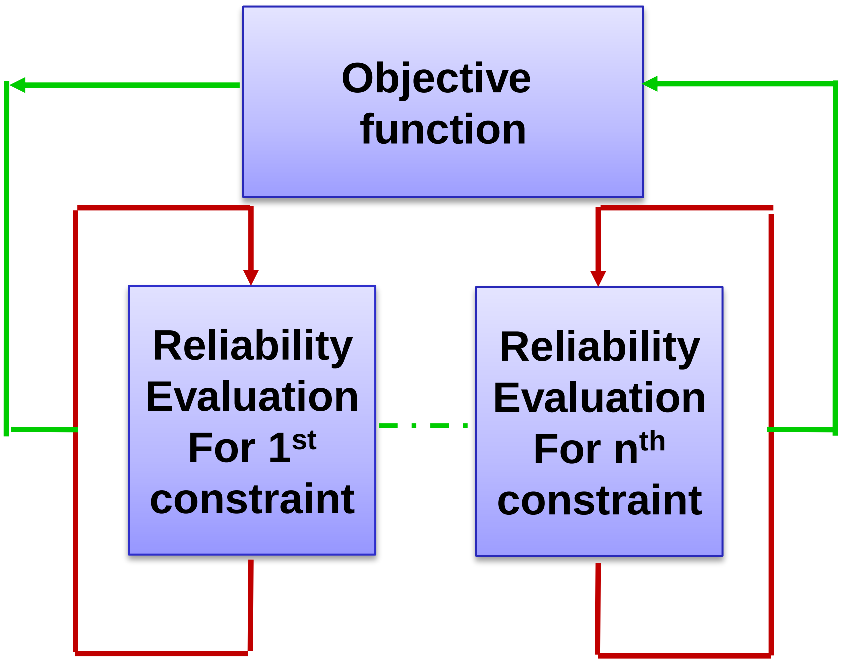
$$s.t \quad \|\mathbf{U}\| = \beta^t$$

- **If constraints are active RIA & PMA yield the same results**
- **PMA is generally more efficient and stable than RIA**

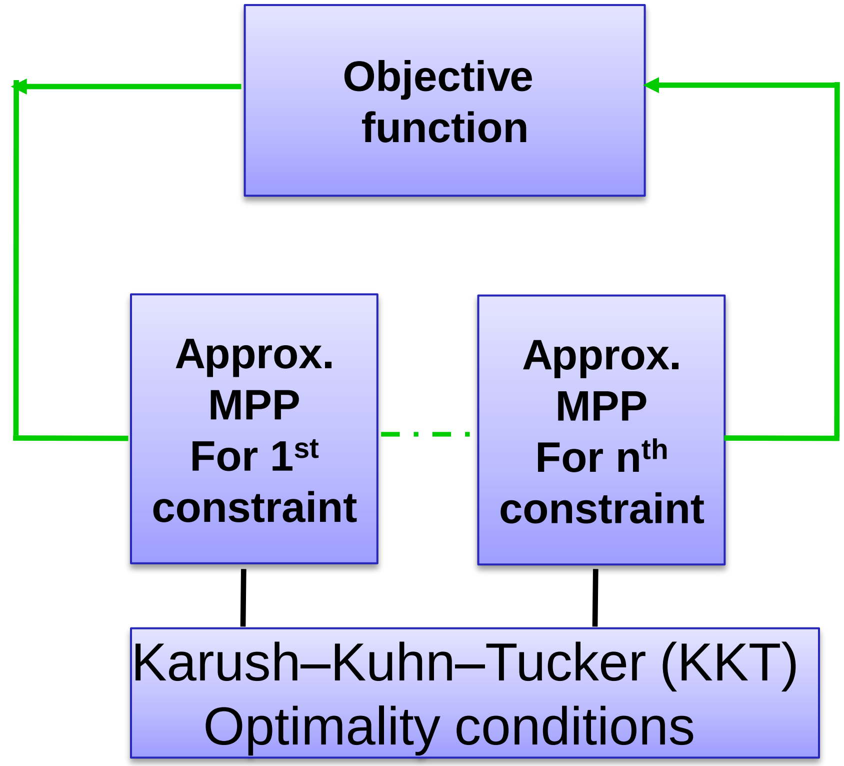
Enevoldsen, I., and Sorensen, J. D., 1994, “Reliability-Based Optimization in Structural Engineering,” Struct. Safety, **15**(3), pp 169–196.
Tu, J., Choi, K. K., and Park, Y. H., 1999, “A New Study on Reliability-Based Design Optimization,” ASME J. Mech. Des., **121**(4), pp 557–564.

RBDO (2): Double- vs. Single-loop

Double-loop RBDO



Single-loop RBDO



- Single-loop is generally more efficient than double-loop

RBDO (3): Component vs. System

Component RBDO

$$\min_{\mathbf{d},\,\boldsymbol{\mu}_{\mathbf{X}}} \quad f(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}})$$

$$s.t. \;\; P\left[g_i(\mathbf{d},\mathbf{X}) \le 0\right] \le P_i^t \quad i=1,\ldots,n$$

- n constraints on components
- Target component probabilities: pre-assigned

System RBDO

$$\min_{\mathbf{d},\,\boldsymbol{\mu}_{\mathbf{X}}} \quad f(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}})$$

$$s.t. \;\; P_{\text{sys}} = P\left[\bigcup_k \bigcap_{i\in C_k} g_i(\mathbf{d},\mathbf{X}) \le 0\right] \le P_{sys}^t$$

- ONE constraint on system probability
- Target system probability: pre-assigned
- Comp. Prob.: indirectly constrained

- System performance is better described by SRBDO

Existing SRBDO Methods

Series System

- Ba-abbad et al (2006): Sys. probability ~ sum of comp. probabilities (upper bound)

$$P_{sys} = \mathrm{P}\left[\bigcup_{i=1}^{n} g_i(\mathbf{d},\mathbf{X}) \le 0\right] \cong \min\left(1, \sum_{i=1}^{n} P_i\right)$$

- Liang et al (2007) : Sys. probability: ~ theoretical bound formulas (Ditlevsen, 1979) (upper bound)

$$P_{sys} \cong \sum_{i=1}^{n} P_i^t - \sum_{i=2}^{n} \max_{j<i} P_{ij}^t$$

- Silva et al (2009) : System reliability-based topology optimization (SRBTO) (independent component events)

General System (cut-set)

- MacDonald and Mahadevan (2008): Using cut-sets

$$E_{\text{system}} = (E_1 \cap E_2) \cup (E_3 \cap E_4) \cup (E_3 \cap E_5) = C_1 \cup C_2 \cup C_3 \qquad P_{sys} \cong \min\left(1, \sum_{i=1}^{n} P_{C_i}\right)$$

(upper bound)

We propose using the matrix-based system reliability (MSR) method to overcome the limitations

Song, J., and Kang, W.-H., 2009, "System Reliability and Sensitivity under Statistical Dependence by Matrix-Based System Reliability Method," Struct. Safety, 31(2), pp 148–156.

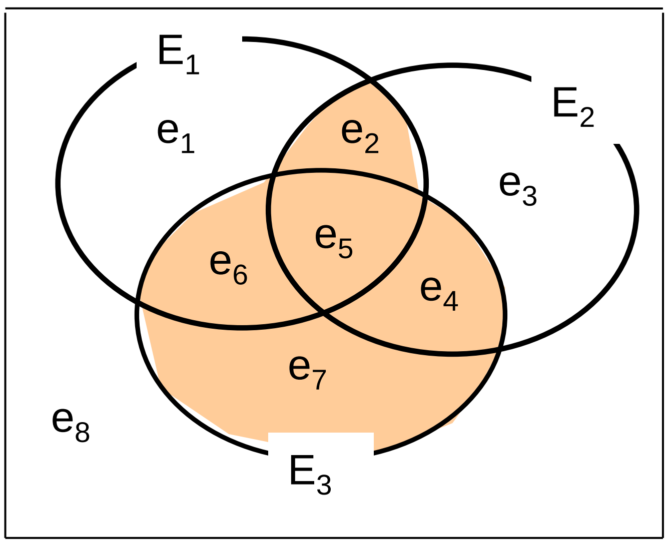
Matrix-based System Reliability (MSR) Method

Matrix-based formulation of system failure probability

$$P(E_{system}) = \mathbf{c}^{\mathrm{T}}\mathbf{p}$$

Example

$$P(E_1E_2 \cup E_3) = \mathrm{p}_2 + \mathrm{p}_4 + \mathrm{p}_5 + \mathrm{p}_6 + \mathrm{p}_7$$
$$= [0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0\ ]\,[\mathrm{p}_1 \quad \mathrm{p}_2 \quad \mathrm{p}_3 \quad \mathrm{p}_4 \quad \mathrm{p}_5 \quad \mathrm{p}_6 \quad \mathrm{p}_7 \quad \mathrm{p}_8]^{\mathrm{T}}$$



mutually exclusive and collectively exhaustive events (MECE)

- **c: "event" vector**
  describe the system event of interest
- **p: "probability" vector**
  likelihood of component joint failures

Applicable to any system event (series, parallel, general) with dependent components

Song, J., and Kang, W.-H., 2009, "System Reliability and Sensitivity under Statistical Dependence by Matrix-Based System Reliability Method," Struct. Safety, 31(2), pp 148–156.

Identification of Event Vector "c"

- **Construct event vector "c"**

$$\mathbf{c}^{\bar{E}} = \mathbf{1} - \mathbf{c}^{E}$$

$$\mathbf{c}^{E_1 \cdots E_n} = \mathbf{c}^{E_1} .* \mathbf{c}^{E_2} .* \ldots * \mathbf{c}^{E_n}$$

$$\mathbf{c}^{E_1 \cup \ldots \cup E_n} = \mathbf{1} - (\mathbf{1} - \mathbf{c}^{E_1}).*(\mathbf{1} - \mathbf{c}^{E_2}).* \ldots *(\mathbf{1} - \mathbf{c}^{E_n})$$

- ➢ **Efficient and easy to implement by matrix-based language Software (Matlab®)**
- ➢ **Can construct directly from event vectors of components and system events**
- ➢ **Can develop specific algorithms to identify event vector**

Computation of Probability Vector "p"

- **Iterative matrix-based procedure for statistically independent (s.i) components**

$$\mathbf{p}_{[1]} = \begin{bmatrix} P_1 & \bar{P}_1 \end{bmatrix}^T$$

$$\mathbf{p}_{[i]} = \begin{bmatrix} \mathbf{p}_{[i-1]}.P_i \\ \mathbf{p}_{[i-1]}.\bar{P}_i \end{bmatrix}$$

- **Example : n =3**

$$\mathbf{p}_{[1]} = \begin{bmatrix} P_1 & \bar{P}_1 \end{bmatrix}^T = \begin{bmatrix} P_1 \\ \bar{P}_1 \end{bmatrix} \quad \mathbf{p}_{[2]} = \begin{bmatrix} \mathbf{p}_{[1]}.P_2 \\ \mathbf{p}_{[1]}.\bar{P}_2 \end{bmatrix} = \begin{bmatrix} P_1 P_2 \\ \bar{P}_1 P_2 \\ P_1 \bar{P}_2 \\ \bar{P}_1 \bar{P}_2 \end{bmatrix} \quad \mathbf{p}_{[3]} = \begin{bmatrix} \mathbf{p}_{[2]}.P_3 \\ \mathbf{p}_{[2]}.\bar{P}_3 \end{bmatrix} = \begin{bmatrix} P_1 P_2 P_3 \\ \bar{P}_1 P_2 P_3 \\ P_1 \bar{P}_2 P_3 \\ \bar{P}_1 \bar{P}_2 P_3 \\ P_1 P_2 \bar{P}_3 \\ \bar{P}_1 P_2 \bar{P}_3 \\ P_1 \bar{P}_2 \bar{P}_3 \\ \bar{P}_1 \bar{P}_2 \bar{P}_3 \end{bmatrix}$$

MSR for System with Dependent Components

System probability with dependent components

$$
\begin{aligned}
P(E_{sys}) &= \int_{\mathbf{s}} P(E_{sys} \mid \mathbf{s}) f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s} \\
&= \int_{\mathbf{s}} \mathbf{c}^{\mathrm{T}} \mathbf{p}(\mathbf{s}) f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s} = \mathbf{c}^{\mathrm{T}} \int_{\mathbf{s}} \mathbf{p}(\mathbf{s}) f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s} \\
&= \mathbf{c}^{\mathrm{T}} \tilde{\mathbf{p}}
\end{aligned}
$$

- Utilize conditional S.I of components given an outcome of random variables “s” causing component dependence, named “Common Source Random Variables” (CSRV).
- Event vector **c** is independent of this consideration
- No need to construct probability vector for new system event

Identification of CSRV's Implicit

- **Approximation by Dunnett-Sobel (DS) correlation matrix**

$$Z_i \sim N(0,R),\ \rho_{ij} = \sum_{k=1}^{m} (r_{ik}.r_{jk})$$

$$Z_i = \left(1 - \sum_{k=1}^{m} r_{ik}^2\right)^{0.5} U_i + \sum_{k=1}^{m} r_{ik} S_k$$

- **Z_i, i=1,..,n are conditional s.i given S=s**
- **Fit the given correlation matrix with DS class correlation matrix with least square error**

SRBDO/MSR: Algorithm

Current SRBDO algorithm

- Using bound formulas (inaccurate)
- NOT applicable to general system
- Sensitivities NOT available.

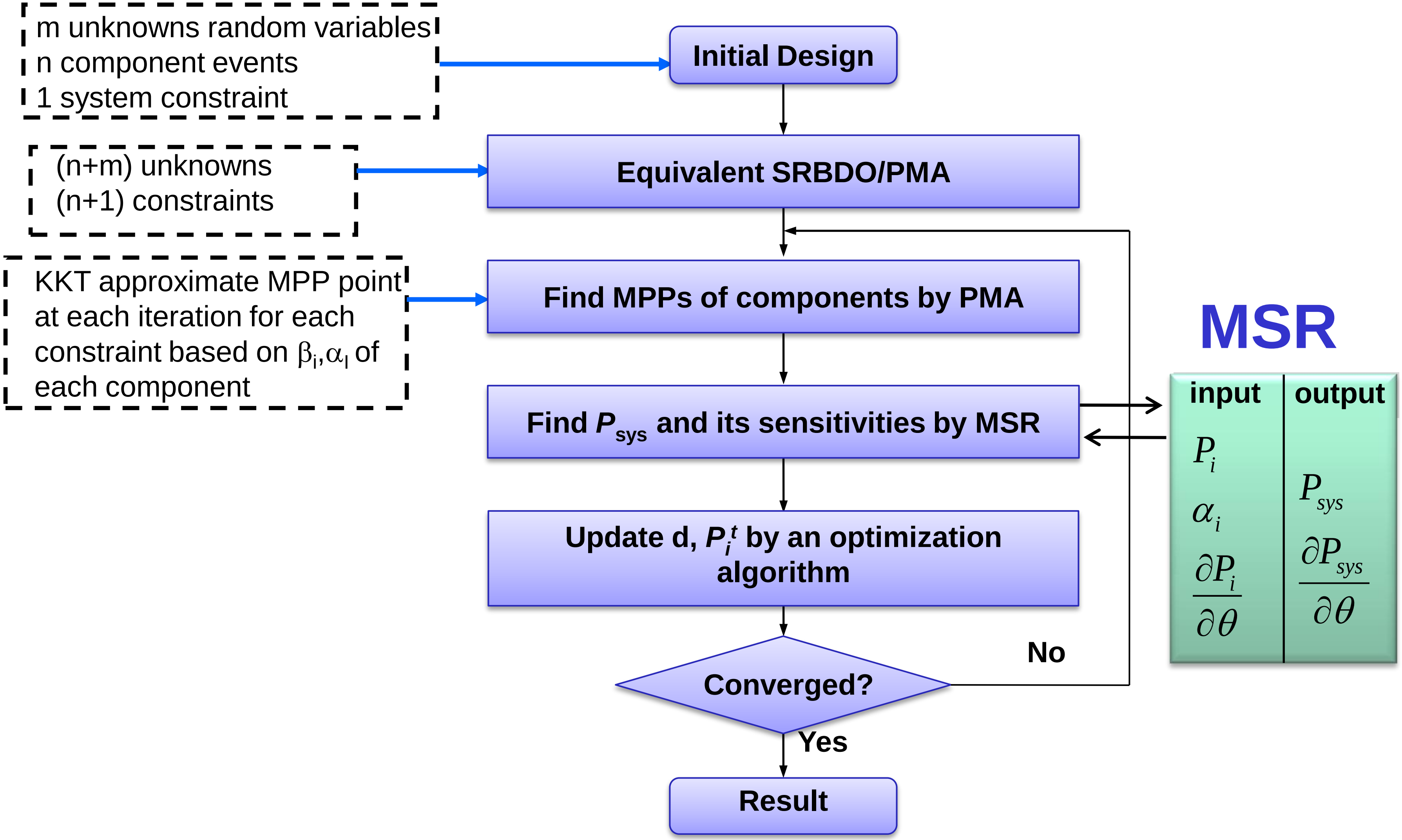
MSR method

- Compute system failure probability
- Applicable to general system
- Independent events and dependent events

Propose: Single-loop PMA SRBDO/MSR algorithm

- Based on Single-loop SRBDO-PMA procedure
- System probability and its sensitivities by MSR

Single-loop PMA SRBDO/MSR: Algorithm



SRBDO/MSR: Parameter Sensitivities

- **Parameter sensitivities** $P_{\text{sys}} = \mathbf{c}^{\mathrm{T}}\mathbf{p}$

$$\frac{\partial P_{\text{sys}}}{\partial \theta} = \mathbf{c}^{\mathrm{T}} \frac{\partial \mathbf{p}}{\partial \theta}$$ Independent

$$\frac{\partial P_{\text{sys}}}{\partial \theta} = \int_{\mathbf{s}} \mathbf{c}^{\mathrm{T}} \frac{\partial \mathbf{p}(\mathbf{s})}{\partial \theta} f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s}$$ dependent

$$\frac{\partial \mathbf{p}}{\partial \theta} = \left[\mathbf{p}^{\langle 1 \rangle} \ \mathbf{p}^{\langle 2 \rangle} \ldots \mathbf{p}^{\langle n \rangle} \right] \frac{\partial \mathbf{P}}{\partial \theta} = \mathbf{P} \frac{\partial \mathbf{P}}{\partial \theta}$$

$$\mathbf{P} = [P_1 \ P_2 \ \cdots P_n]^{\mathrm{T}}$$

- **System probability sensitivities w.r.t component probabilities (DS class)**

$$P_i(\mathbf{s}) = \Phi\left[-\frac{\beta_i - \sum_{k=1}^{m} r_{ik} s_k}{\left(1 - \sum_{k=1}^{m} r_{ik}^2\right)^{0.5}} \right]$$

$$\frac{\partial P_i(\mathbf{s})}{\partial P_i} = \frac{\partial P_i(\mathbf{s})}{\partial \beta_i} \cdot \frac{\partial \beta_i}{\partial P_i} = -\frac{\partial P_i(\mathbf{s})}{\partial \beta_i} \cdot \frac{1}{\varphi(-\beta_i)}$$

Example – Indeterminate Truss Structure

- **Obj. func:** Min. the weight of truss
- **Opti parameter:** bar areas: A_i
- **Constraint:** System. Prob.
- **Failure:** System fails if two members fail system event (cut-set)

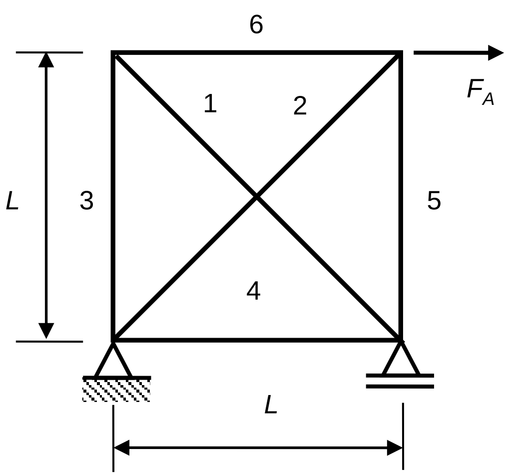
$$\min_{\mathbf{d}=\{A_1,\ldots,A_6\}} f(\mathbf{d}) = \sqrt{2}(A_1 + A_2) + A_3 + A_4 + A_5 + A_6$$

$$s.t. \quad P_{sys} = P\left[\bigcup_{k=1}^{15} \bigcap_{i\in C_k} g_i(\mathbf{d},\mathbf{X}) \le 0\right] \le P_{sys}^t = 0.001$$

$$g_i(\mathbf{d},\mathbf{X}) = \begin{cases} A_i F_i - 0.707 F_A & i = 1,2 \\ A_i F_i - 0.500 F_A & i = 3,\ldots,6 \end{cases}$$

$$A_1, A_2, A_3, A_4, A_5, A_6 \ge 0$$

$\{C_k\}$ ={(1,2), (1,3), (1,4), (1,5), (1,6), (2,3),(2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)}



Means and Standard deviations of the random variables

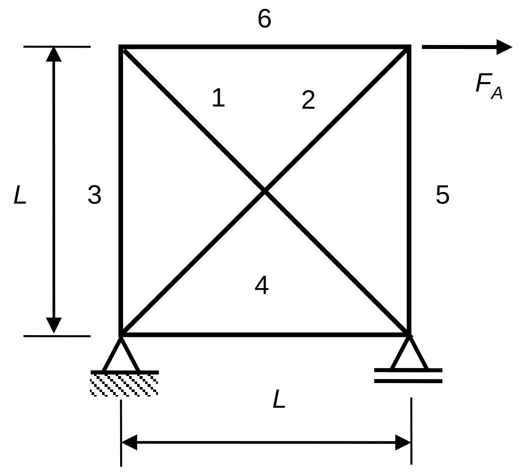
Random Variables (Gaussian distribution)	***Mean***	***Std dev***
Member strength F_i , i=1,..6 (Mpa)	**745**	**62**
Applied load F_A (kN)	**4450**	**45**

MacDonald M., and Mahadevan S., 2008, “Design Optimization with System-Level Reliability,” ASME J. Mech. Des., 130(2), p. 021403

Indeterminate Truss Structure

- MacDonald and Mahadevan (*) vs. SRBDO/MSR

Members	Area: A_i ($\times 10^{-3}$ mm^2)		Reliability Index: β_i	
	SRBDO by (*)	SRBDO/MSR	SRBDO by (*)	SRBDO/MSR
1	18.43	17.89	2.89	2.67
2	18.27	17.89	2.83	2.67
3	13.51	13.20	3.16	2.99
4	13.44	13.20	3.12	2.99
5	13.33	13.20	3.06	2.99
6	13.09	13.20	2.92	2.99
$f(x)$	105.24	103.36		



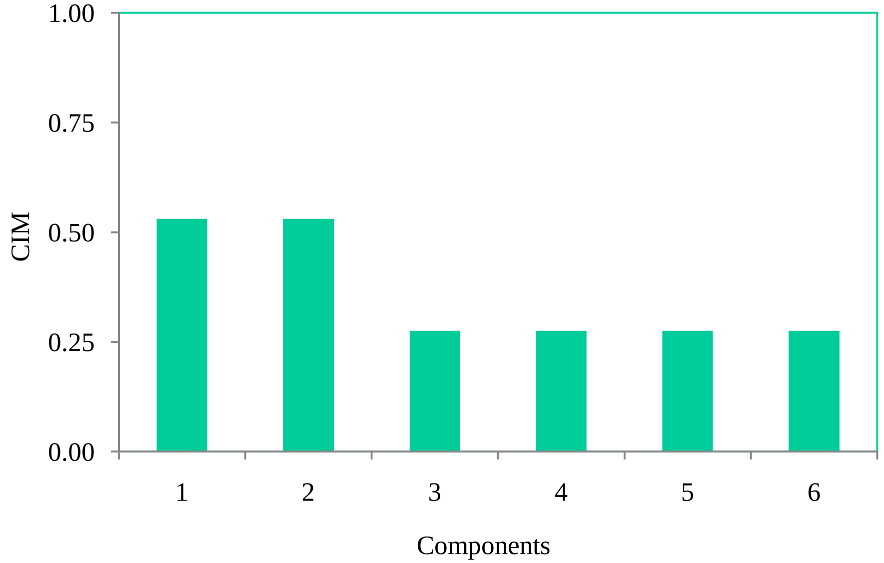
- SRBDO/MSR: better optimal design + symmetric results
- MCS: P_{sys}=0.00107 (10^6 times, c.o.v. = 0.03) ~ SRBDO/MSR: 0.001

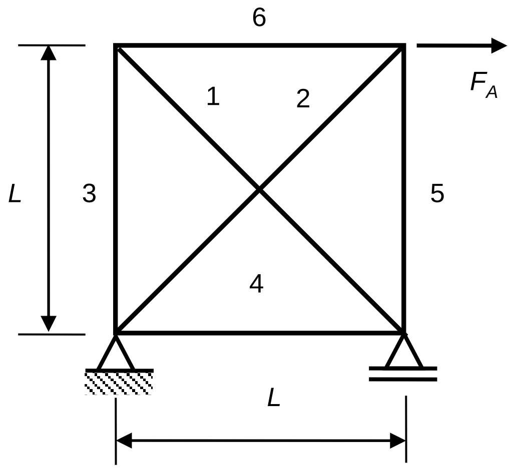
Indeterminate Truss Structure

Conditional probability important Measure (CIM)

$$\mathrm{CIM}_i = P(E_i|E_{sys}) = \frac{P(E_i E_{sys})}{P(E_{sys})} = \frac{\mathbf{c}'^{\mathrm{T}}\mathbf{p}}{\mathbf{c}^{\mathrm{T}}\mathbf{p}}$$

- Only additional task is to identify the event vector of the new system



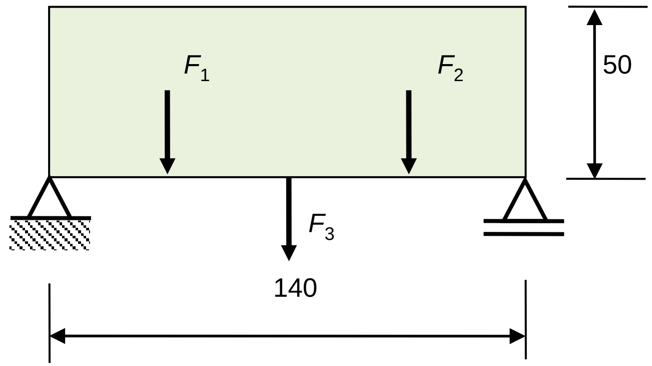


- Component important ranking: (1,2) -> (3,4,5,6)

Topology Optimization: DTO, CRBTO, SRBTO

$$g_i(\boldsymbol{\rho}, \mathbf{F}) = d_i^t - d_i(\boldsymbol{\rho}, \mathbf{F})$$

$$i = 1,2,3$$



Random Variables

$F_1 \sim N(100 \quad 10)$

$F_2 \sim N(100 \quad 10)$

$F_3 \sim N(100 \quad 10)$

$F = (F_1, F_2, F_3)$

$d_i^{max} = 1.5$

DTO

1

$$\underset{\rho}{Min} \quad V(\rho)$$

$$s.t \quad g_i(\rho, F) \le 0, \ \ i = 1,2,3$$

$$g_i(\rho, F) = d_i(\rho, F) - d_i^{max}$$

CRBTO

2

$$\underset{\rho}{Min} \quad V(\rho)$$

$$s.t \quad P[g_1(\rho, F) \le 0] \le P_1^t$$

$$P[g_2(\rho, F) \le 0] \le P_2^t$$

$$P[g_3(\rho, F) \le 0] \le P_3^t$$

$$where: \ P_1^t = P_2^t = P_3^t = 0.02275$$

$$g_1(\rho, F) = d_1^{max} - d_1(\rho, F) \le 0$$

$$g_2(\rho, F) = d_2^{max} - d_2(\rho, F) \le 0$$

$$g_3(\rho, F) = d_3^{max} - d_3(\rho, F) \le 0$$

SRBTO

3

$$\underset{\rho}{Min} \quad V(\rho)$$

$$s.t$$

$$P_{sys} = P[g_1(\rho, F) \cup g_2(\rho, F) \cup g_3(\rho, F) \le 0] \le P_{sys}^t$$

$$where$$

$$g_1(\rho, F) = d_1^{max} - d_1(\rho, F) \le 0$$

$$g_2(\rho, F) = d_2^{max} - d_2(\rho, F) \le 0$$

$$g_3(\rho, F) = d_3^{max} - d_3(\rho, F) \le 0$$

SRBTO/PMA+MSR

4

$$\underset{\rho, P_1^t, P_2^t, P_3^t}{Min} \quad V(\rho)$$

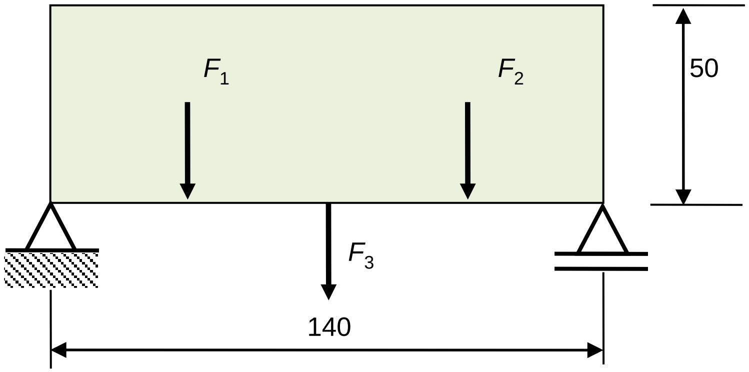
$$s.t \quad g_1(\rho, F) \ge 0$$

$$g_2(\rho, F) \ge 0$$

$$g_3(\rho, F) \ge 0$$

$$P_{sys} = P^{MSR}(P_1^t, P_2^t, P_3^t) \le P_{sys}^t$$

Topology Optimization: DTO, CRBTO, SRBTO



DTO (volfrac = 0.40)

CRBTO (Pi = 0.02275

P_{sys} = 0.0434

volfrac = 0.60)

(P_{sys} = 0.0434

volfrac = 0.58)

$E_{sys} = \bigcup E_i$ **Series**

(P_{sys} = 0.005

volfrac = 0.44)

$E_{sys} = E_1 E_2 E_3$ **Parallel**

(P_{sys} = 0.005

volfrac = 0.47)

$E_{sys} = \bar{E}_1 \bar{E}_2 E_3$ **General**

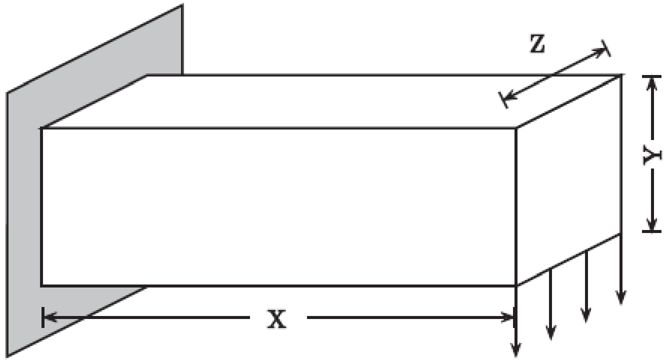
SRBTO/MSR

Summary & Conclusion

- Single-loop PMA SRBDO/MSR algorithm is proposed.
- The SRBDO/MSR is applicable to general system in a uniform manner.
- Sensitivities for gradient-based optimization
- Handle dependence between component events
- Numerical examples: SRBDO & SRBTO examples
- Future work: 3D topology optimization problem

Future work: Large-scale SRBTO

Problem configuration



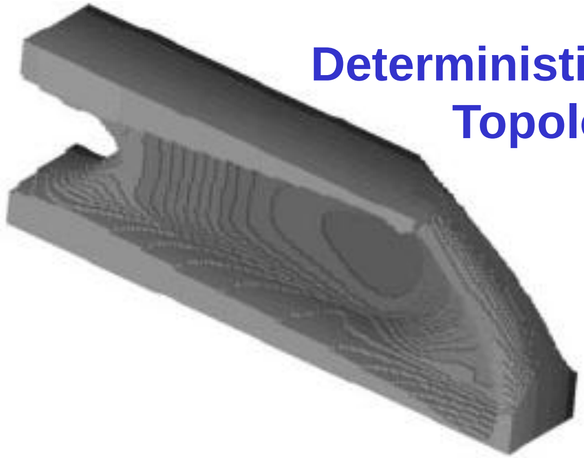
Size (full domain): 180x60x60 elements

Number of unknowns (half domain: ≈ 1.0 mil.)

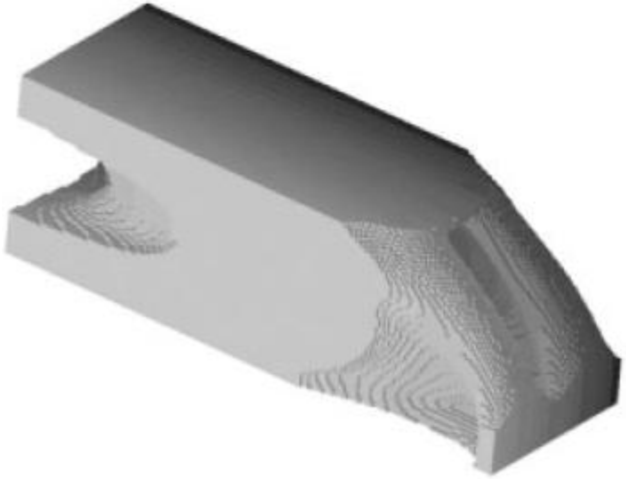
Run time: ~ 45.7 hours

Filtering radius: 6 elements

Deterministic Optimal Topology



(half)



(full)

High resolution SRBTO is computationally expensive

Wang, S., Sturler, E., and Paulino, G.H. 2007, "Large-scale Topology Optimization using Preconditioned Krylov Subspace Methods with Recycling," Int. J. Numer. Meth. Engrg., 69, pp 2441–2468.