WCSMO-8

Topology Optimization with Polygonal Finite Elements

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Motivation

- In topology optimization, parameterization of shape and topology of the design has been traditionally carried out on uniform grids

- Conventional approaches use uniform meshes consisting of Lagrangian-type finite elements (e.g. linear quads) to simplify domain discretization and the analysis routine

- As a result of this choice, numerical artifacts such as the well-known checkerboard pathology and one-node connections may appear

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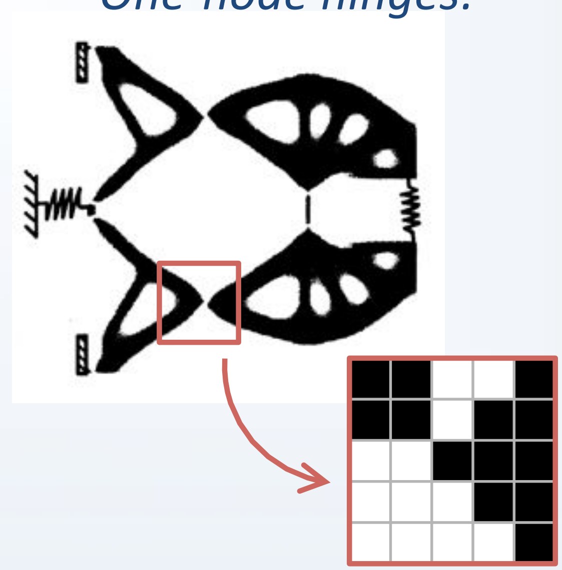
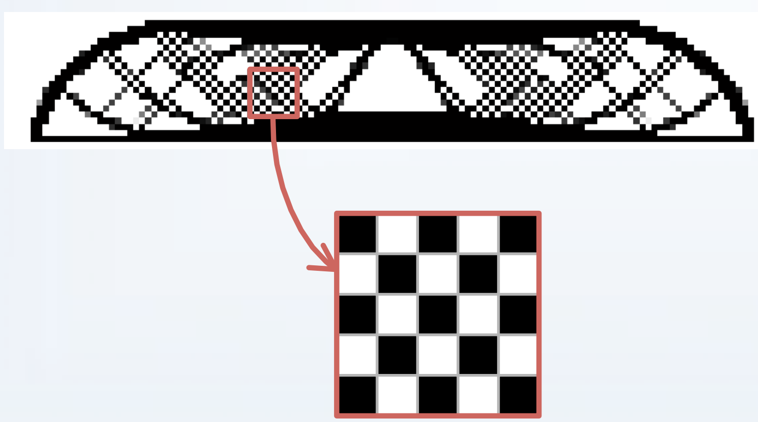
One-node hinges:



Checkerboard:



Motivation

- A problem less often noted is that the constrained geometry of structured grids can cause bias in the orientation of members, leading to mesh-dependent sub-optimal designs

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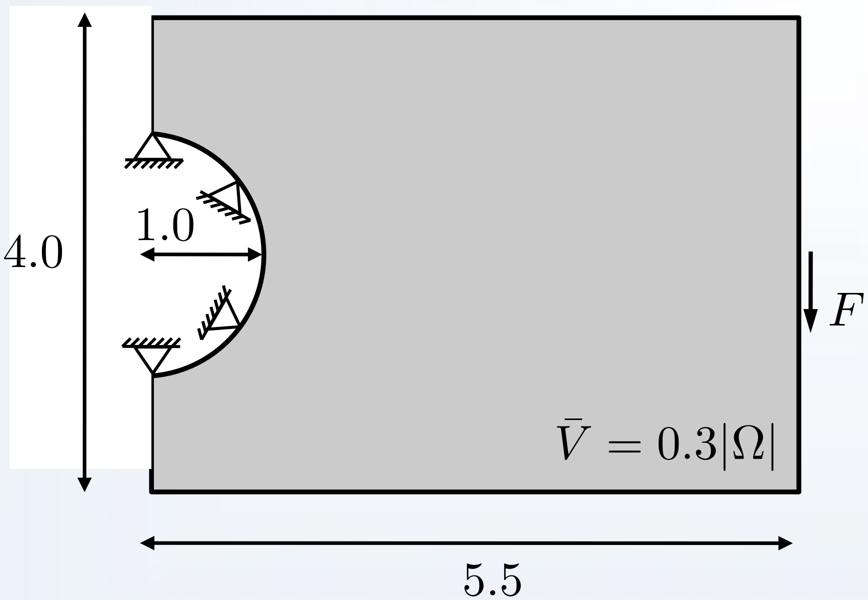
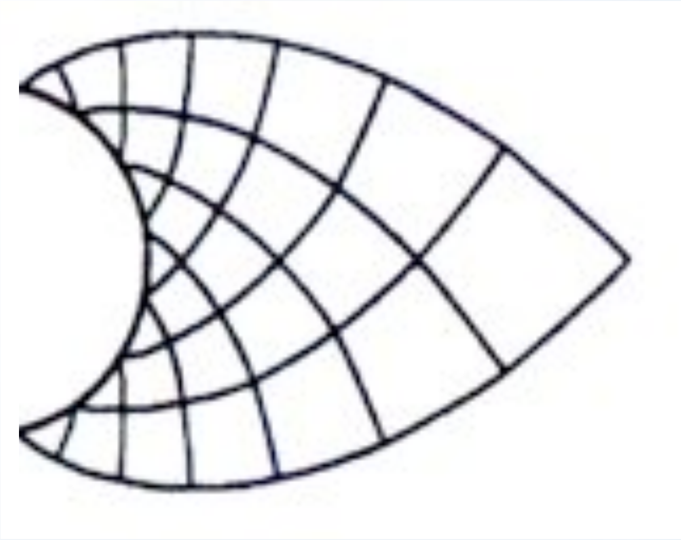
Problem definition:



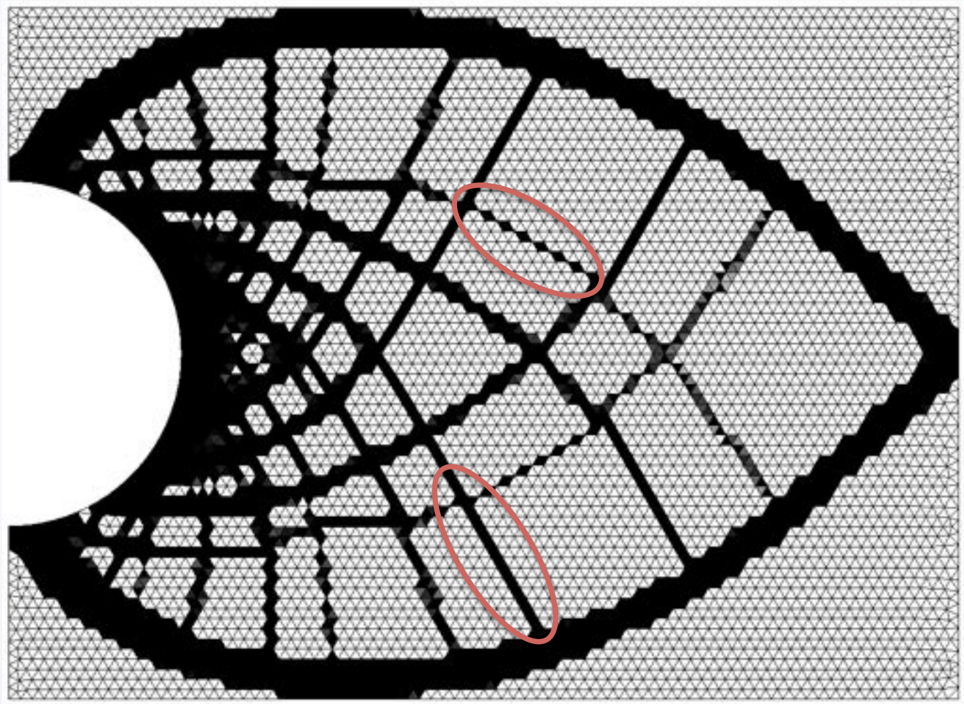
Michell-type solution:



Suzuki , Kikuchi (1991) Comput Methods Appl Mech Engrg

Motivation

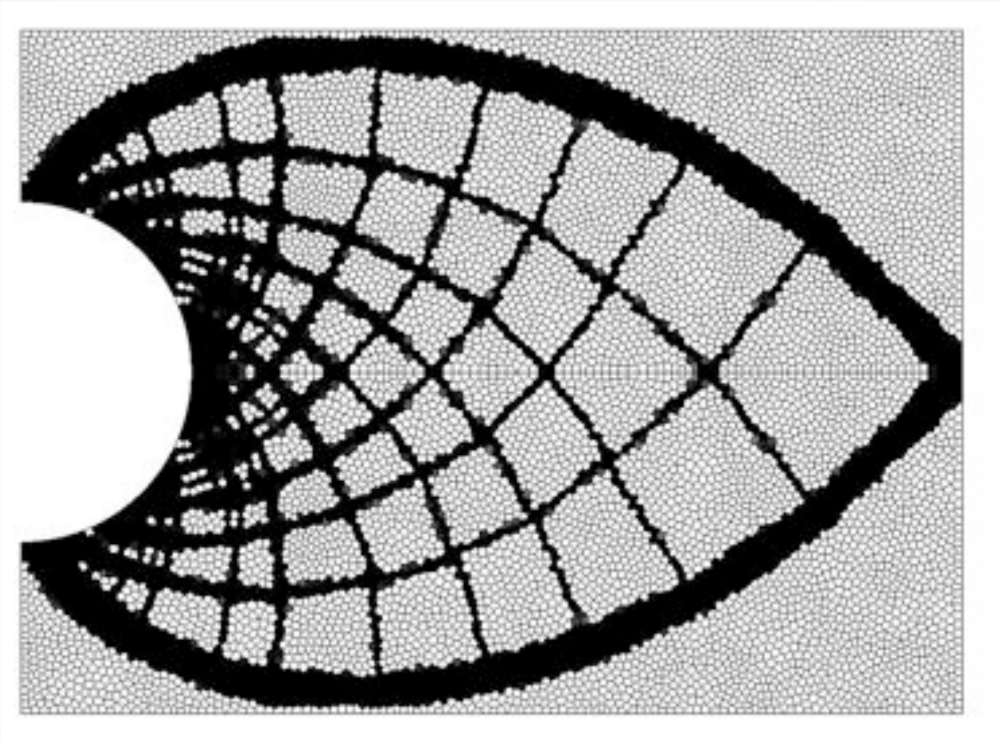
- If the geometric attributes of the mesh are too restricting, certain characteristic patterns of the optimal solution may be excluded from the final design



Solution based on a structured mesh with 8584 T6 elements

Motivation

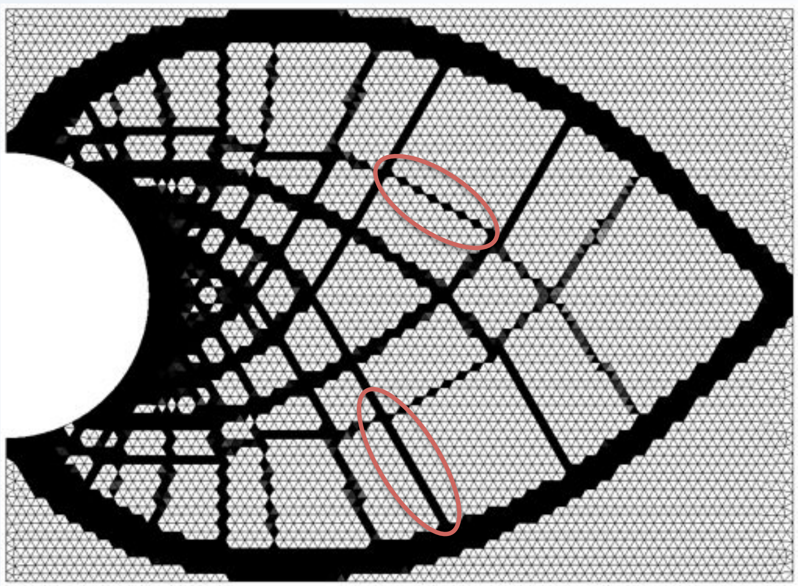
- In this work, we examine the use of irregular meshes consisting of convex polygons in topology optimization to address the abovementioned issues



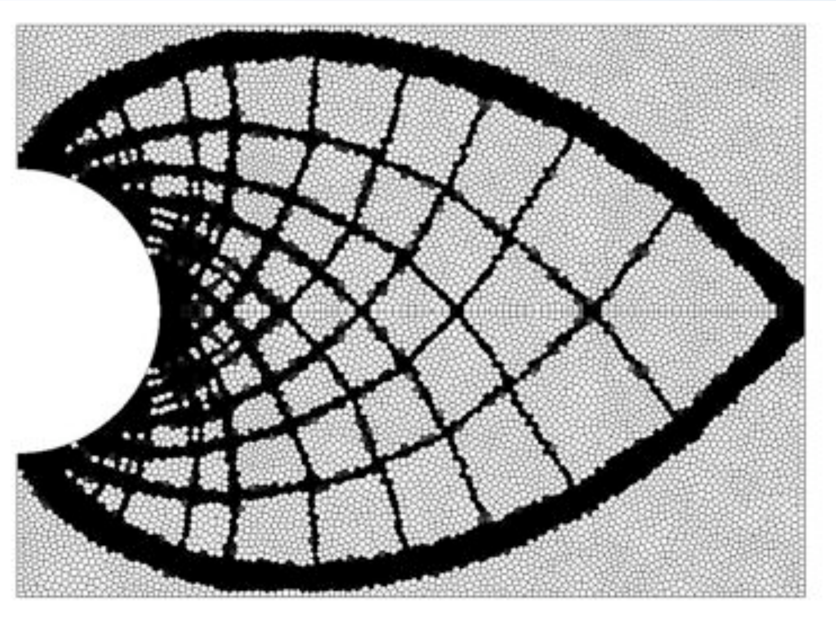
Solution based on a Voronoi mesh with 10000 elements

Motivation

- In addition to possessing higher degree of geometric isotropy, these meshes allow for greater flexibility in discretizing complex domains without suffering from numerical instabilities.



Solution based on a T6 mesh



Solution based on a Voronoi mesh

Outline

- Motivation
- Polygonal mesh generation
- Finite element scheme
- Numerical results
- Concluding remarks

Voronoi diagrams

- Let $P = \{p_i\}_{i=1}^{n}$ denote a set of points in domain Ω. The Voronoi diagram of P corresponds to a partitioning of the domain into cells:

$$V_i = \bigcap_{\forall j, j \neq i} \{\boldsymbol{x} \in \Omega : \delta(\boldsymbol{x}, p_i) \leq \delta(\boldsymbol{x}, p_j)\}, \qquad i = 1, \cdots, n$$

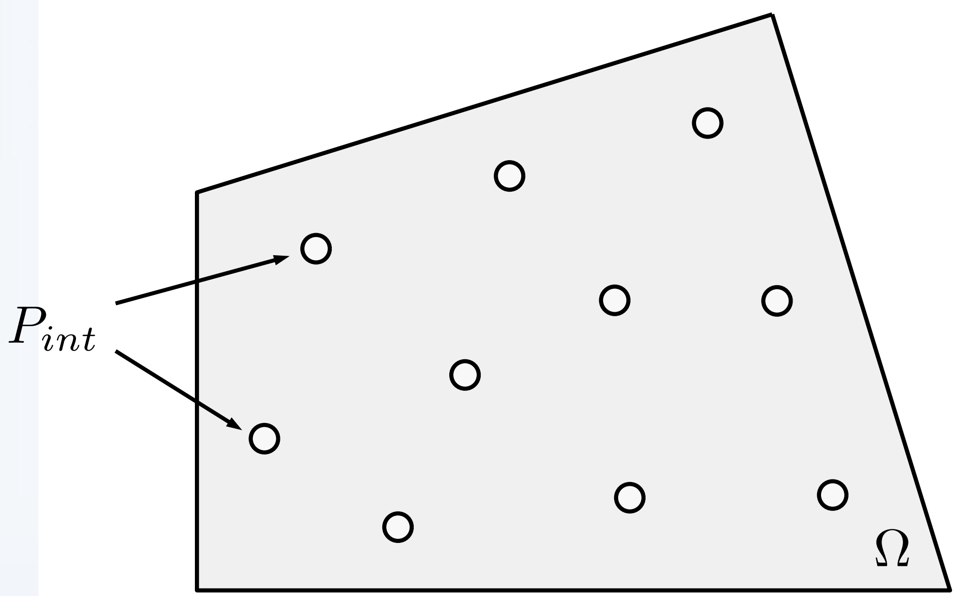
  i.e., each cell consists of points closer to the corresponding seed than any other point in the set

- A polygonal discretization of Ω can be generated from P by including additional points such that the resulting Voronoi diagram incorporates an approximation to the boundary $\partial\Omega$

Yip, Mohle, Bolander(2005). Computer-Aided Civil and Infrastructure Engineering

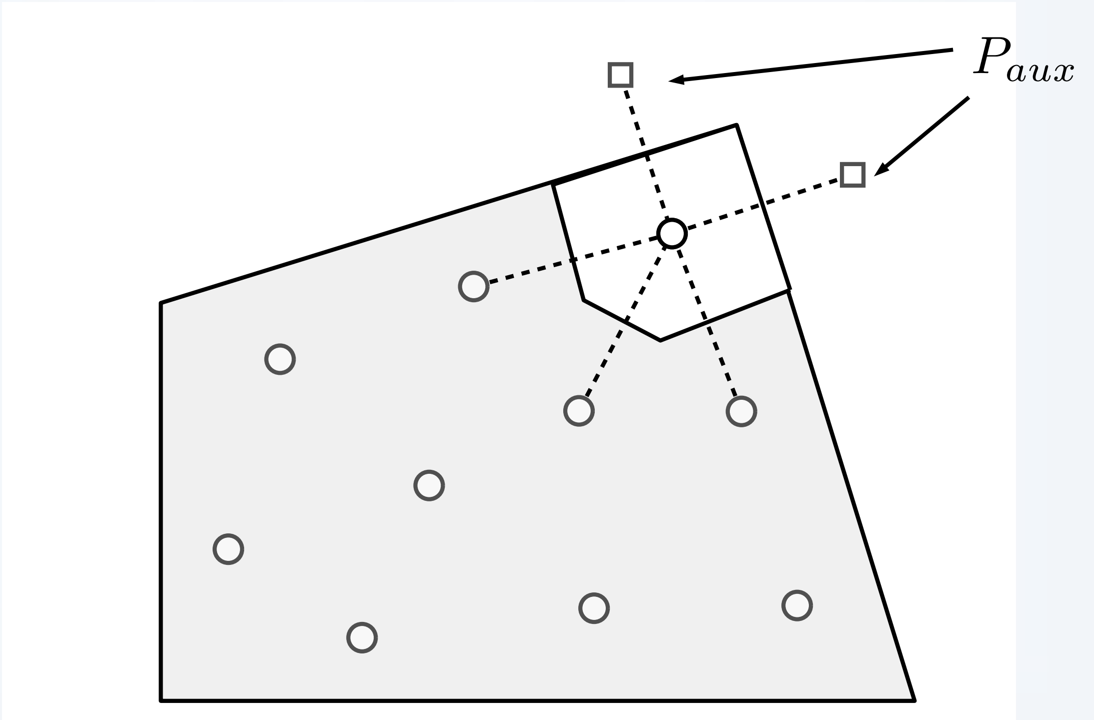
Meshing algorithm

Step 1: The interior of Ω is populated with a desired number of generating seeds. We denote this point set by P_{int}



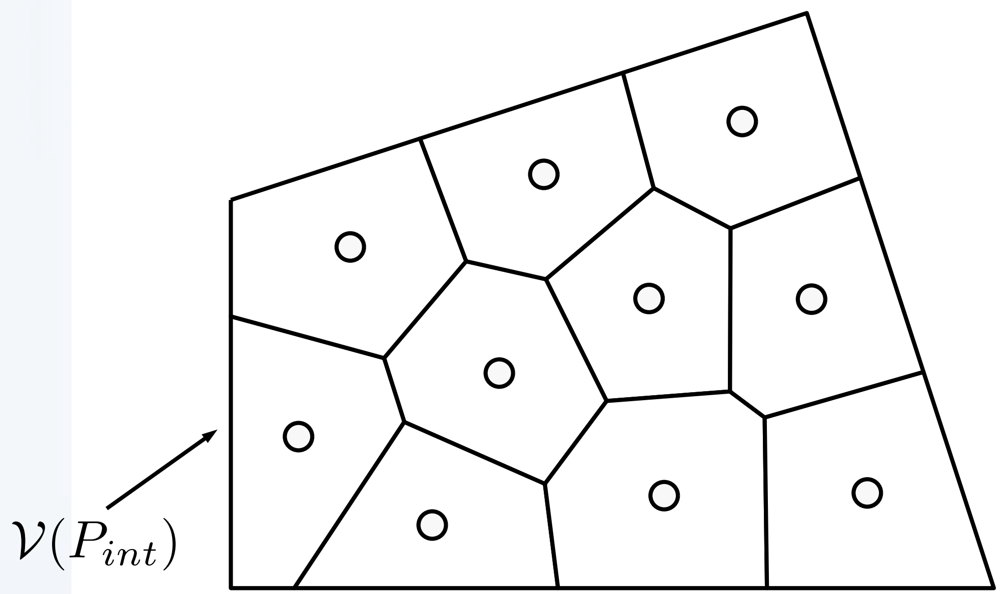
Meshing algorithm

Step 2,3: The interior points are reflected about the edges of the domain and Voronoi diagram of $P = P_{int} \cup P_{aux}$



Meshing algorithm

Step 4: A polygonal discretization of the domain is given by the cells associated to P_{int}



Centroidal Voronoi Tessellations

- A Voronoi tessellation is centroidal if each generating point coincides with the centroid of the corresponding Voronoi cell:

$$p_i = \overline{p}_i \quad \forall i = 1, \cdots, n \quad \text{where} \quad \overline{p}_i := \frac{\int_{V_i} \boldsymbol{x}\mu(\boldsymbol{x})d\boldsymbol{x}}{\int_{V_i} \mu(\boldsymbol{x})d\boldsymbol{x}}$$

- Alternatively, CVT can be thought of as minimizers of energy:

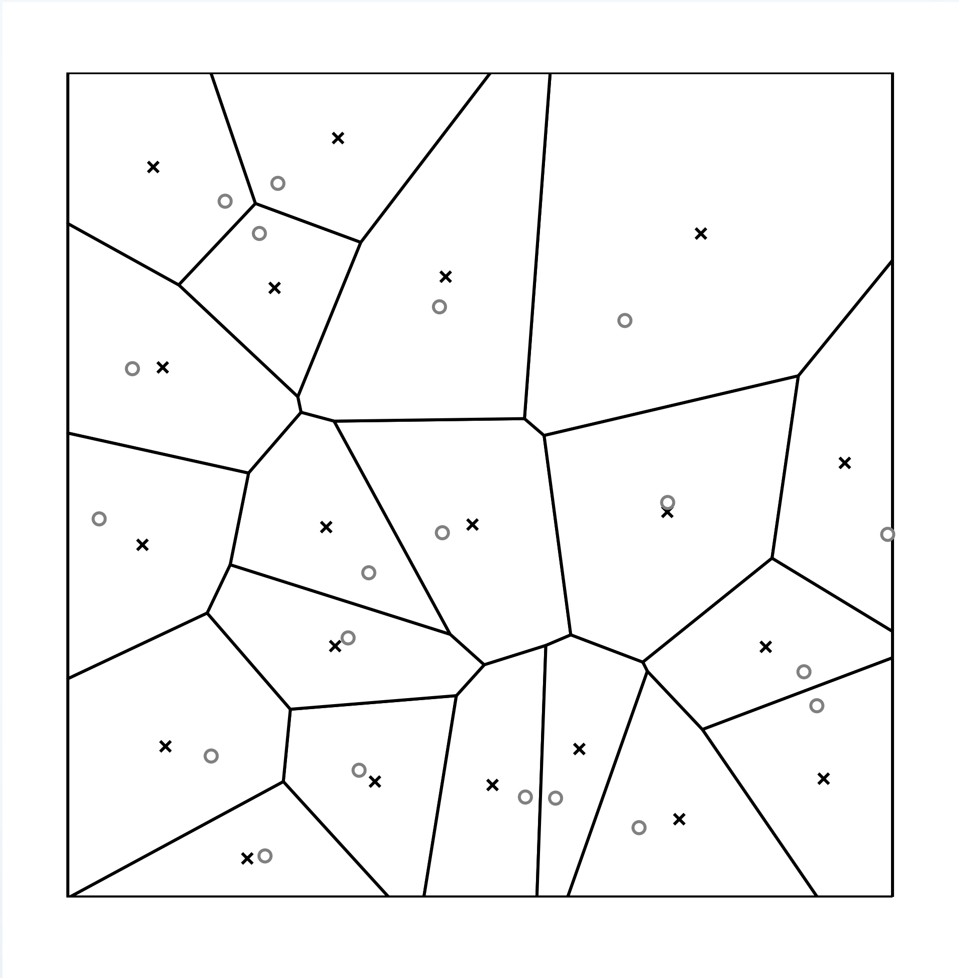
$$\mathcal{E}(P) = \sum_{i=1}^{n} \int_{V_i} |\boldsymbol{x} - p_i|^2 \mu(\boldsymbol{x})d\boldsymbol{x}$$

- Lloyd's algorithm constructs CVTs by replacing the seeds by the centroids of the corresponding cells

Du, Faber, Gunzburger. (1999) SIAM

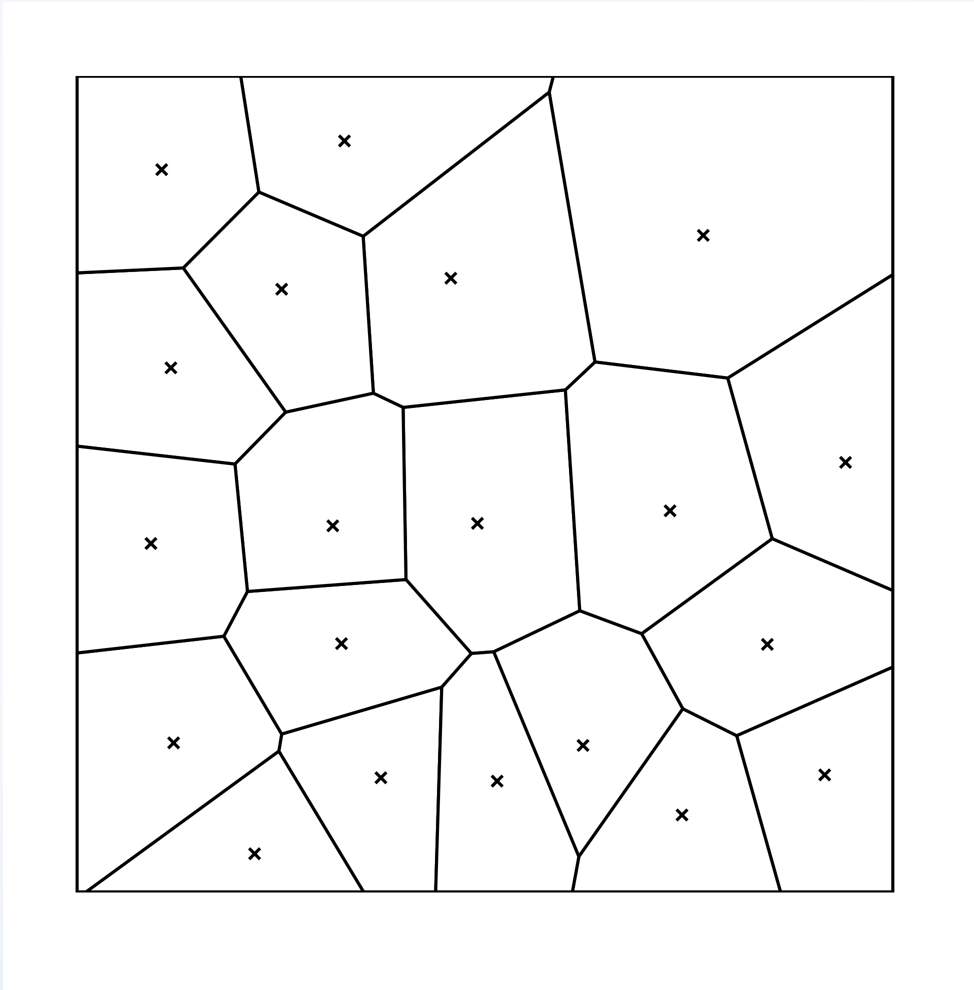
Lloyd's algorithm

- Random seeds (o's) and centroids of the cells (x's):



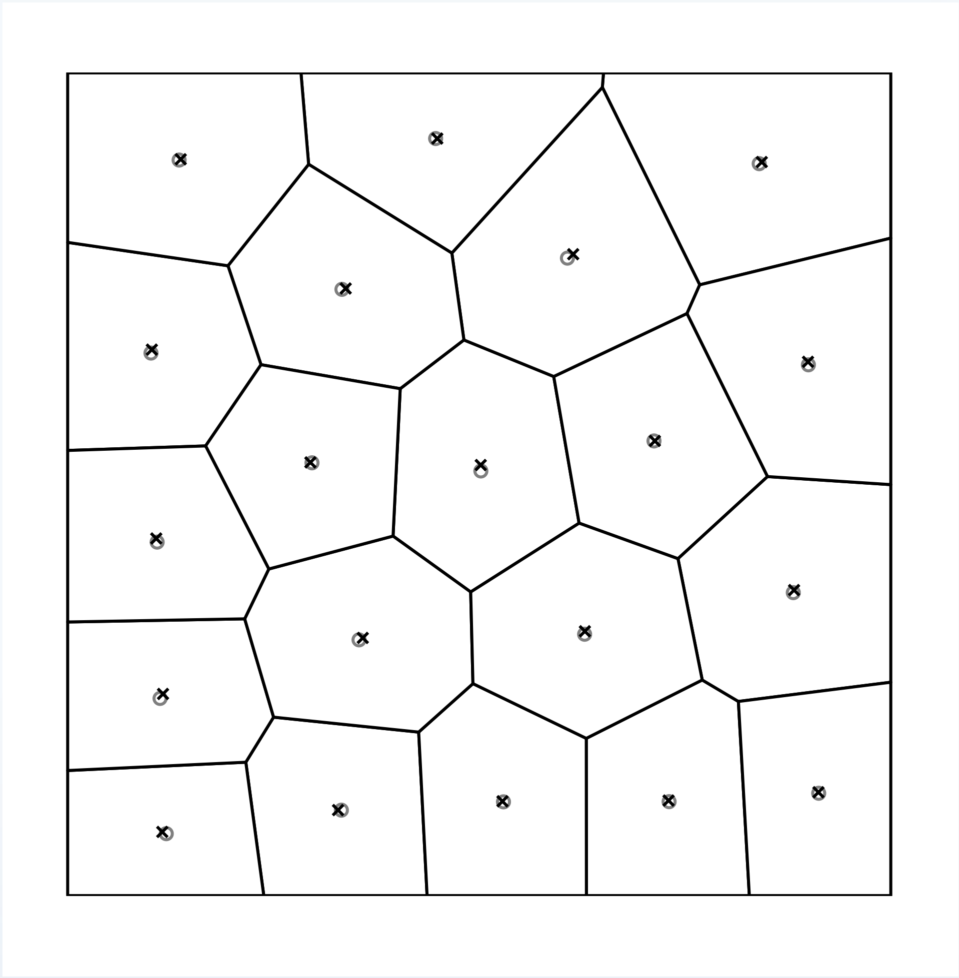
Lloyd's algorithm

- After one iteration (i.e., diagram generated by centroids)



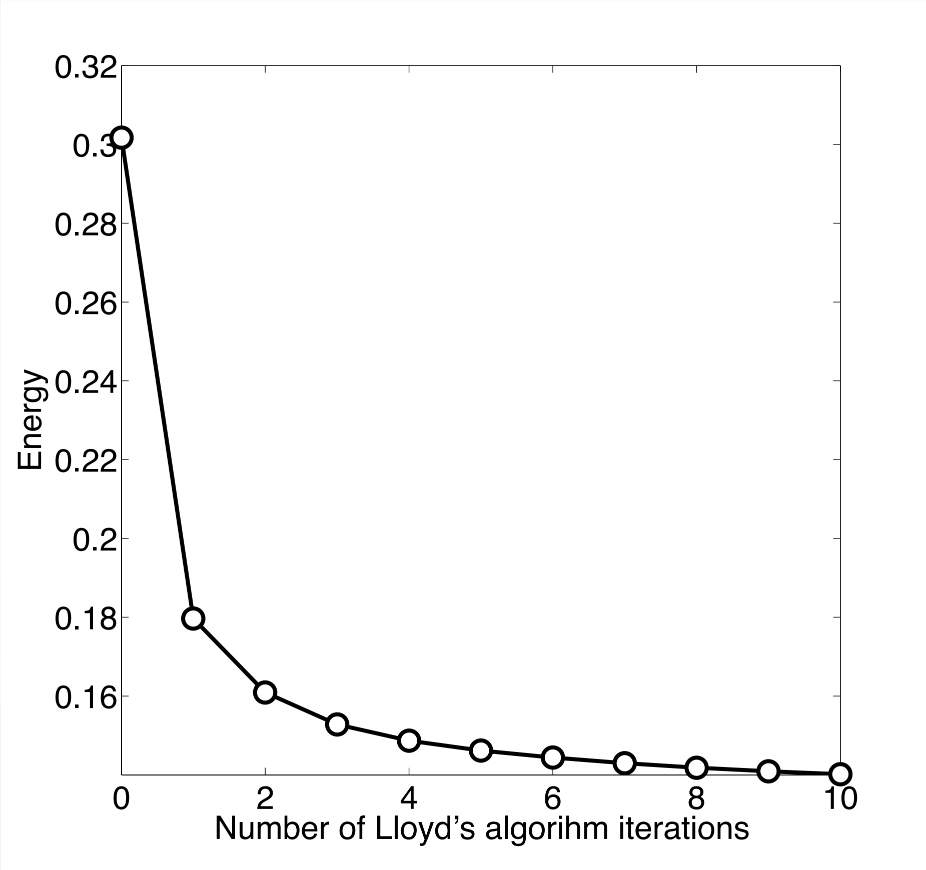
Lloyd's algorithm

- After ten iterations, the diagram is nearly centroidal



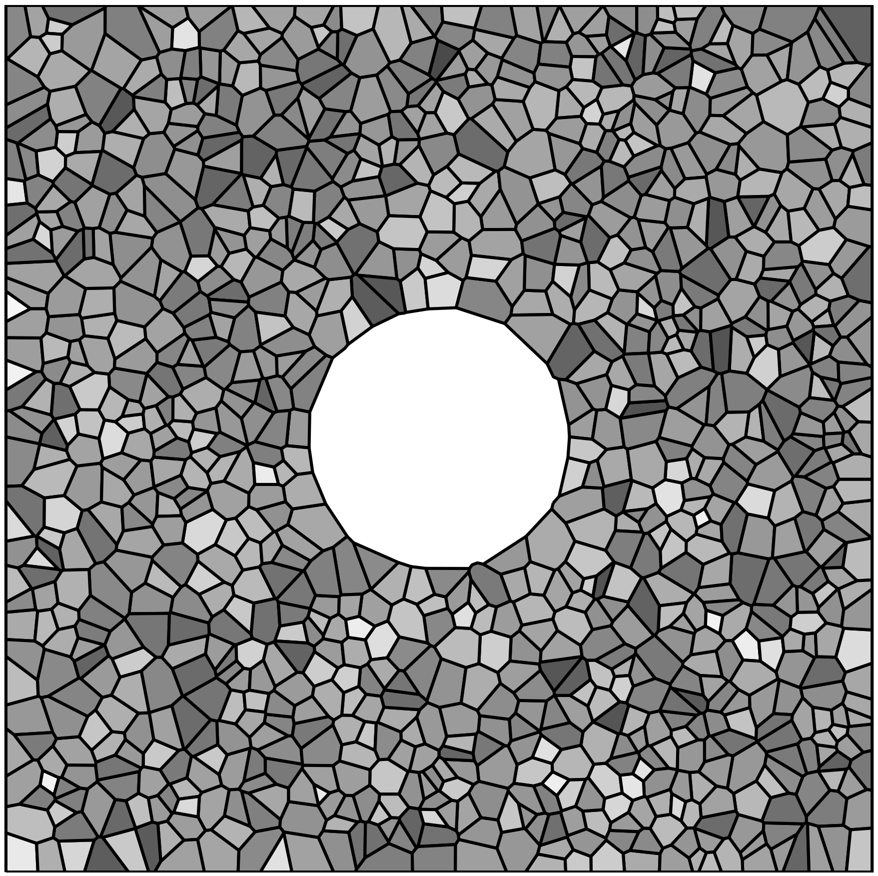
Lloyd's algorithm

- The Lloyd's algorithm is locally convergent in energy



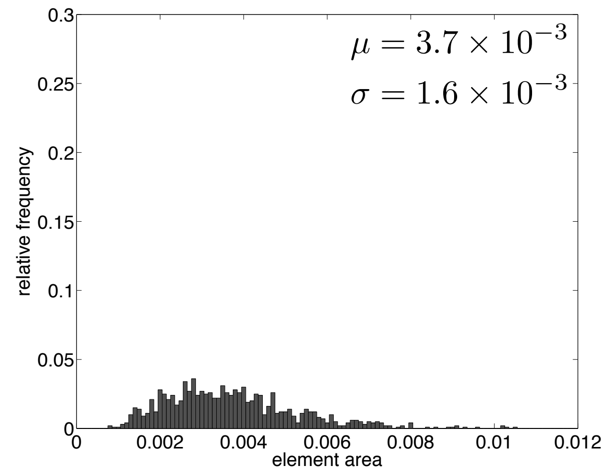
Mesh quality

- Random

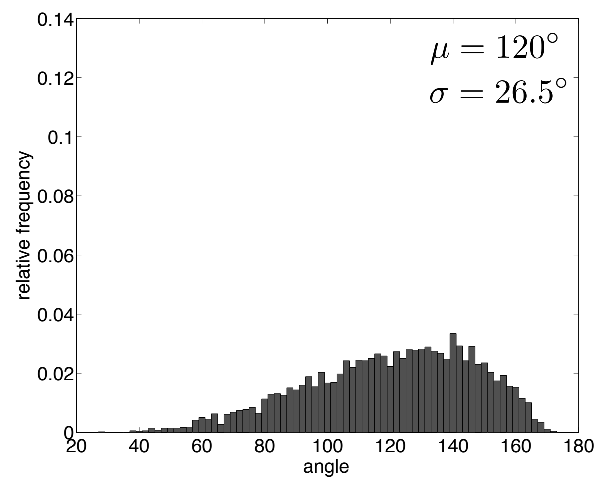


c.o.v of edge length

element area

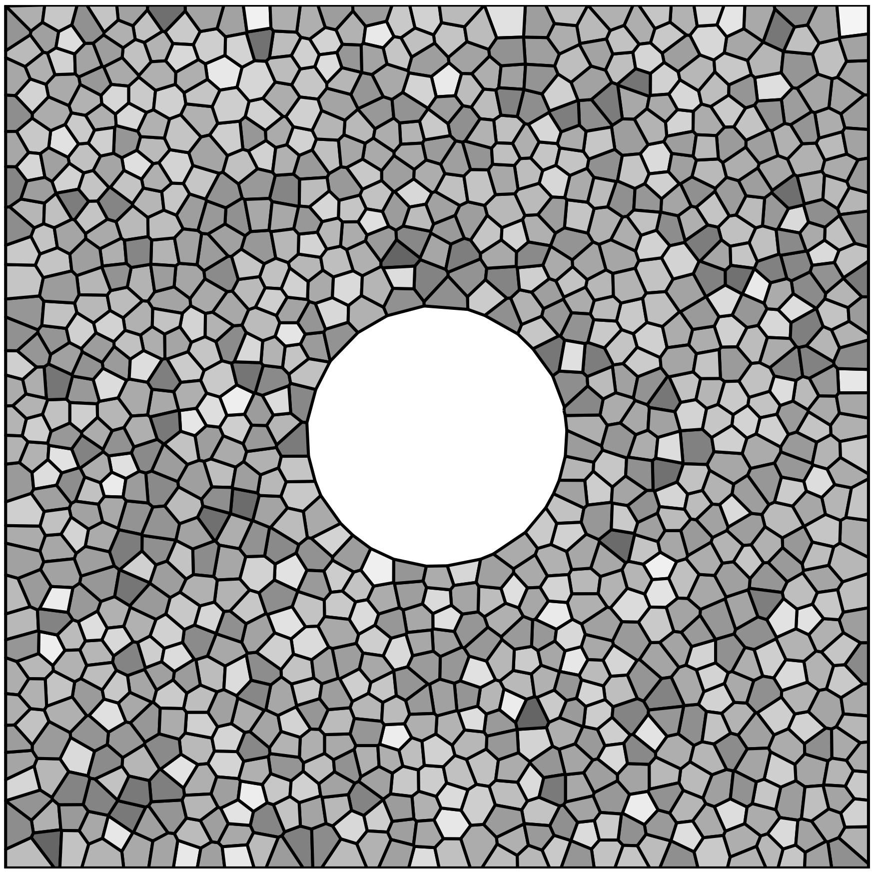


interior angles



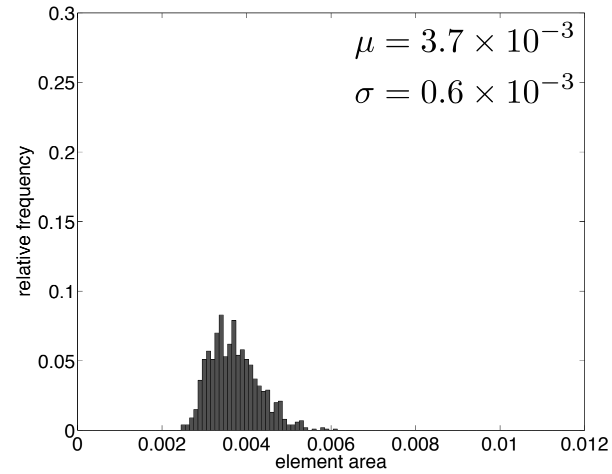
Mesh quality

- Quasi-random

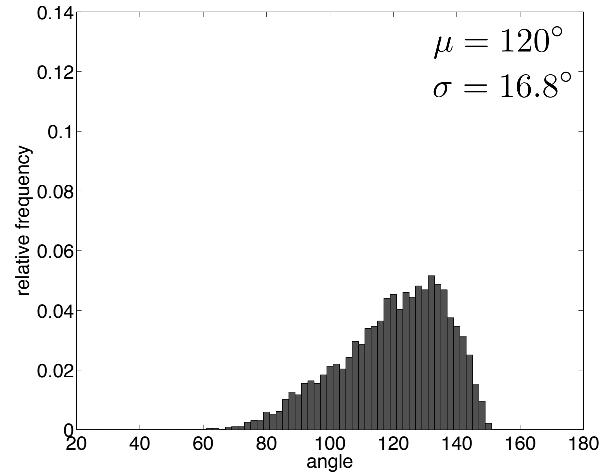


c.o.v of edge length

element area



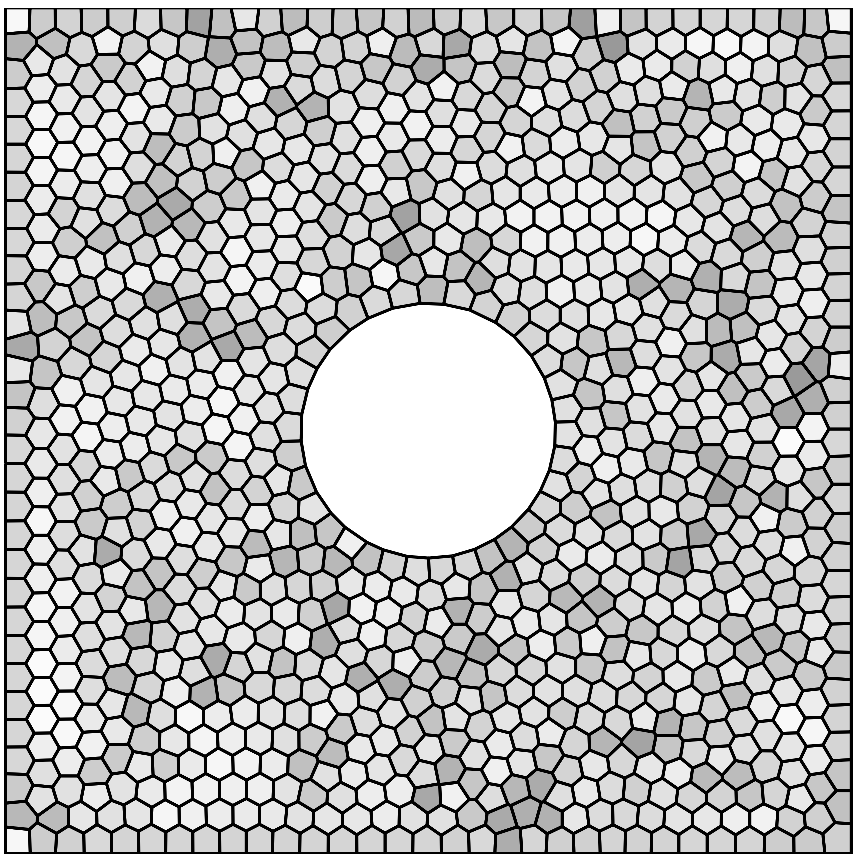
interior angles



Bolander, Saito (1998). Eng Fract Mech

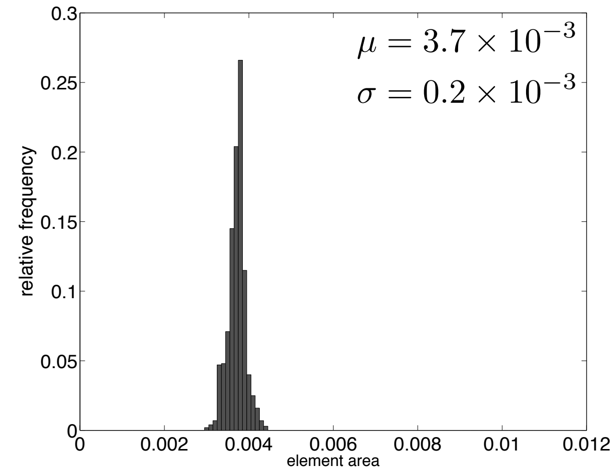
Mesh quality

- CVT

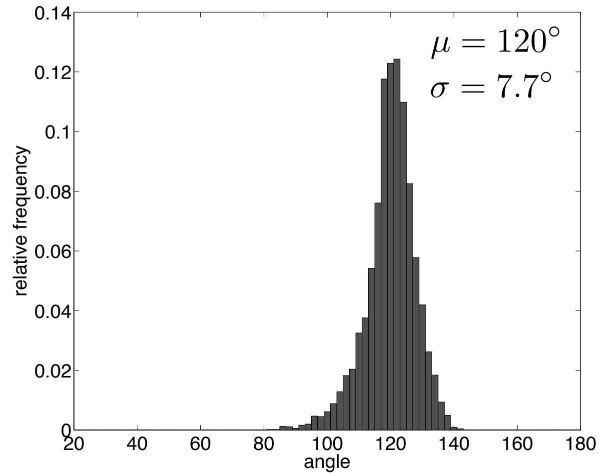


c.o.v of edge length

element area

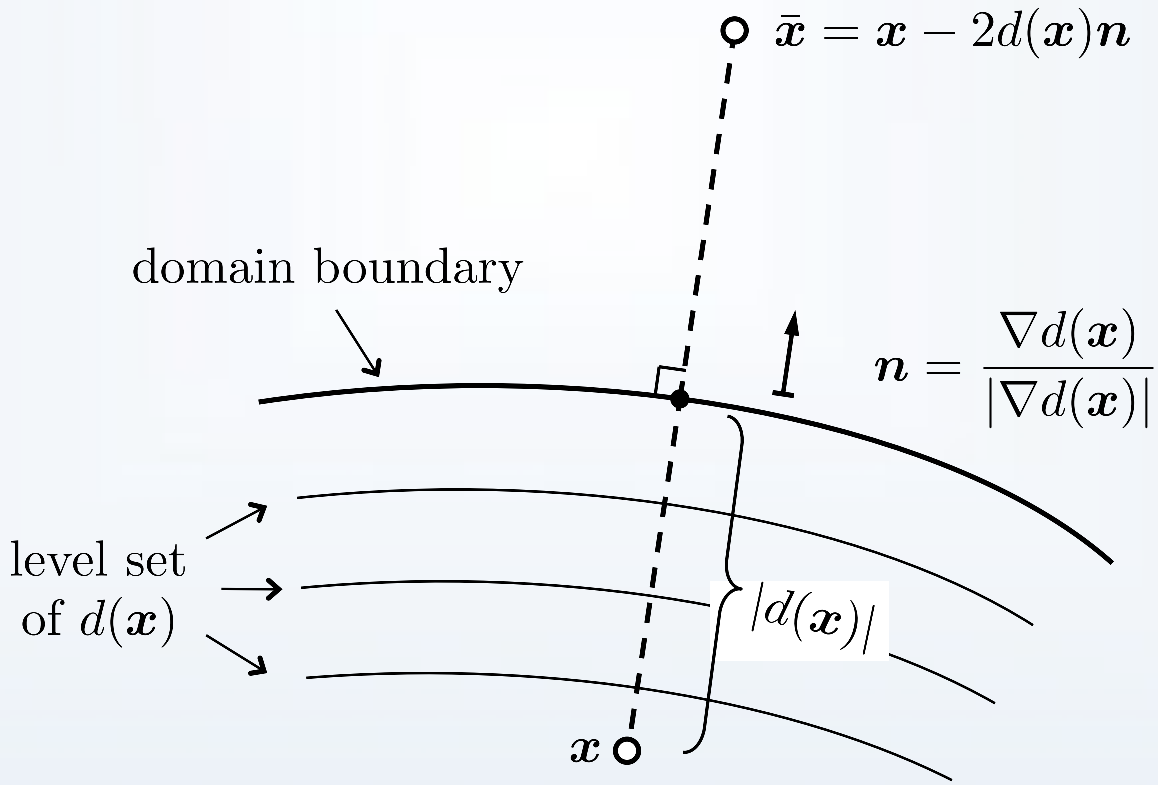


interior angles



Implicit description of geometry

- Placement and reflection of seeds can be carried out generically using an implicit description domain geometry
- In particular, the "signed distance" function, $d(\boldsymbol{x})$, can be used



Persson, Strang (2004) SIAM Rev

Construction of distance functions

- Some simple geometries and set operations:

$$\text{Circle:} \qquad d(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{x}_o| - r$$

$$\text{Half-plane } x_1 \leq a\text{:} \qquad d(\boldsymbol{x}) = x_1 - a$$

$$\text{Union:} \qquad d_{A \cup B}(\boldsymbol{x}) = \min\left(d_A(\boldsymbol{x}), d_B(\boldsymbol{x})\right)$$

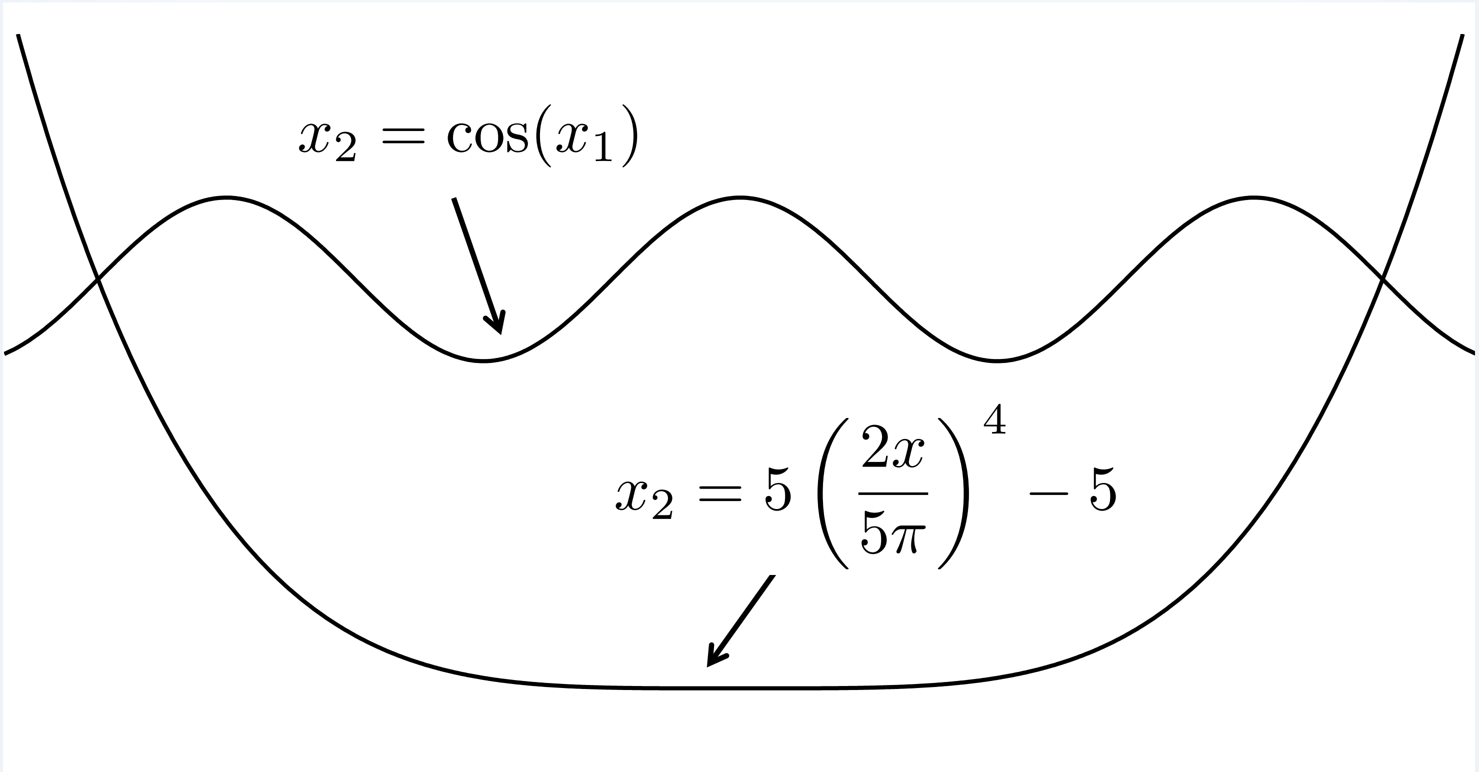
$$\text{Intersection:} \qquad d_{A \cap B}(\boldsymbol{x}) = \max\left(d_A(\boldsymbol{x}), d_B(\boldsymbol{x})\right)$$

$$\text{Difference:} \qquad d_{A \setminus B}(\boldsymbol{x}) = \max\left(d_A(\boldsymbol{x}), -d_B(\boldsymbol{x})\right)$$

Persson, Strang (2004) SIAM Rev

Implicit description of geometry

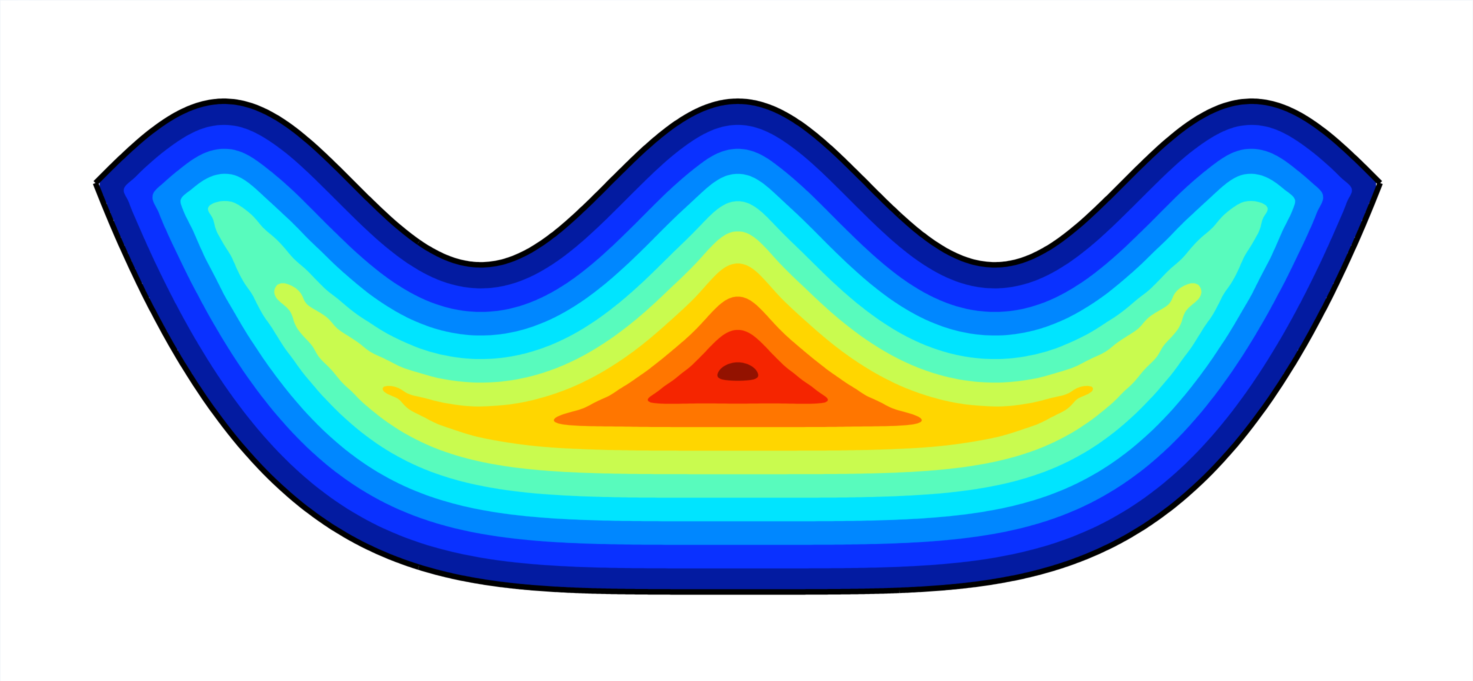
- Consider the following domain:



Domain boundary is the zero level set of $f(\boldsymbol{x})$

Implicit description of geometry

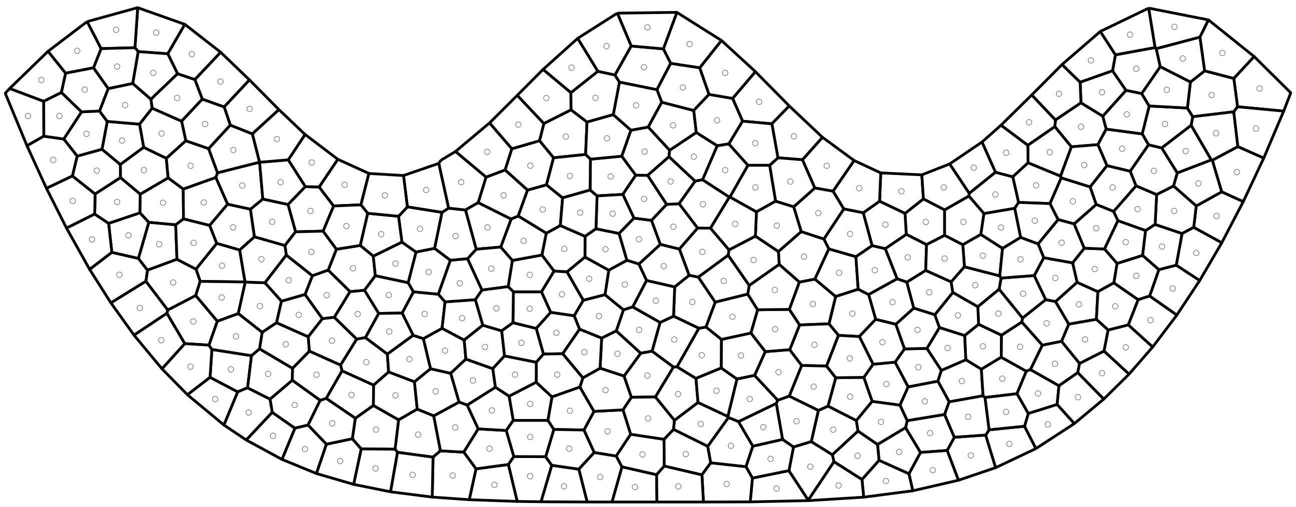
- The distance function is obtained by solving nonlinear system and basic set operations:



For each boundary: $d(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{x}_p| \times \text{sgn}(f(\boldsymbol{x}))$

Implicit description of geometry

- The final CVT mesh:



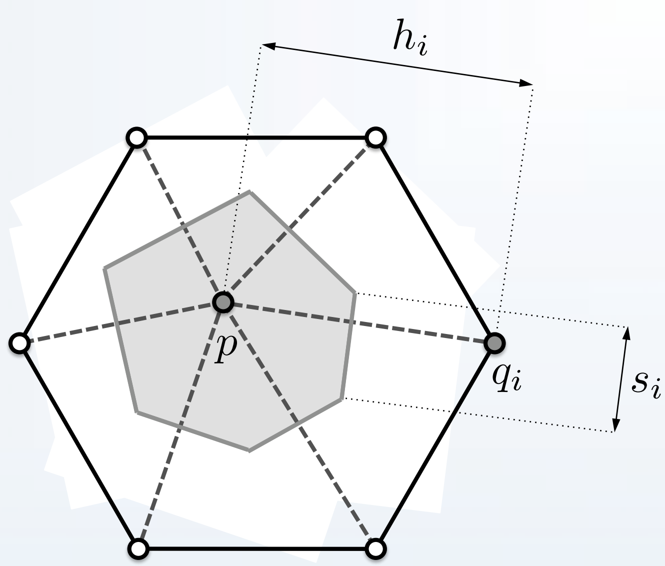
To capture the corners, seeds are reflected about both boundaries

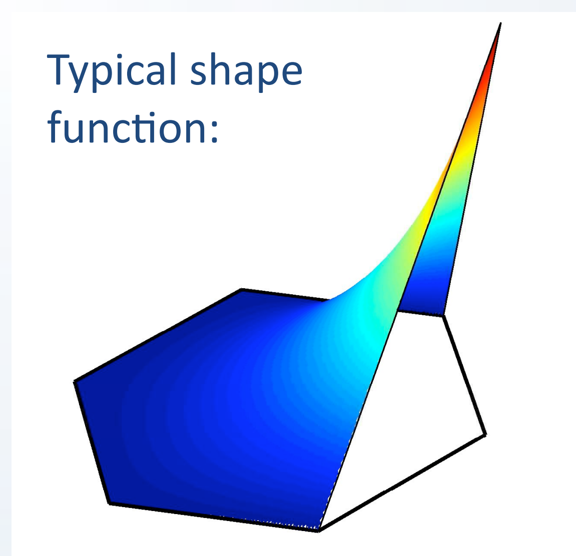
Finite element scheme

- The finite element scheme for polygonal elements is obtained based on notions of natural neighbors and natural neighbor functions

- For a convex polygon, the Laplace interpolant at an interior point is given by:

$$\phi_i(\boldsymbol{x}) = \frac{w_i(\boldsymbol{x})}{\sum_{i=1}^{n}(\boldsymbol{x})} \quad \text{where} \quad w_i(\boldsymbol{x}) = \frac{s_i(\boldsymbol{x})}{h_i(\boldsymbol{x})}$$





Finite element scheme

- By construction, the Laplace functions are non-negative, bounded and satisfy partition of unity:

$$0 \leq \phi_i(\boldsymbol{x}) \leq 1, \qquad \sum_{i=1}^{n} \phi_i(\boldsymbol{x}) = 1$$

- These functions are linearly complete, thus convergence of the Galerkin method for second order partial differential equations is ensured:

$$\sum_{i=1}^{n} \boldsymbol{x}_i \phi_i(\boldsymbol{x}) = \boldsymbol{x}$$

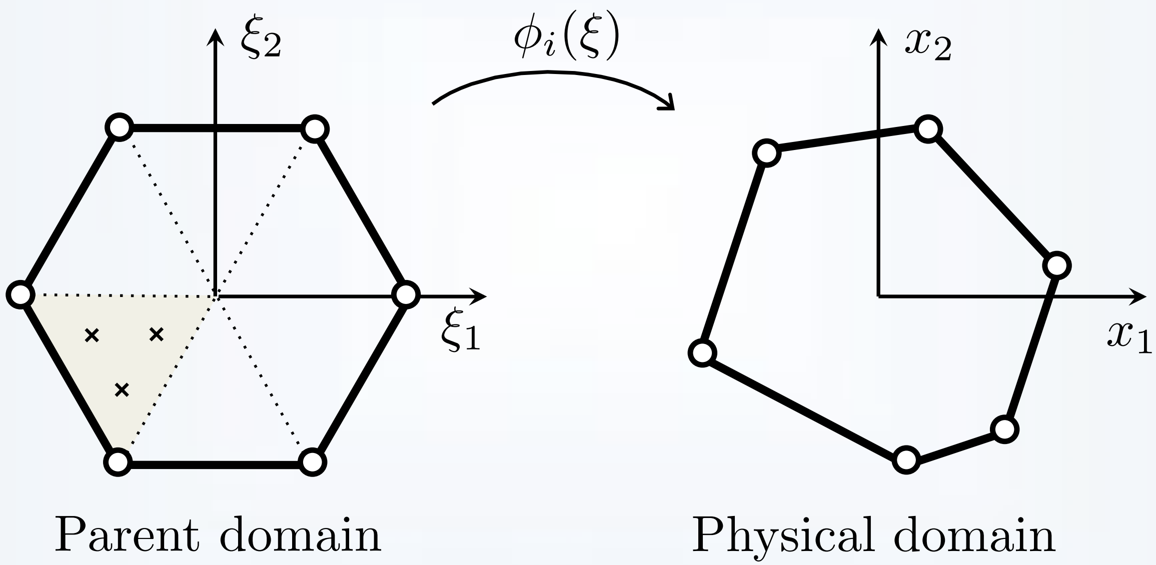
- Moreover, they are linear on the boundary of the element and satisfy the Kronecker-delta property:

$$\phi_i(\boldsymbol{x}_j) = \delta_{ij}$$

Sukumar, Tabarraei(2004) IntJ NumerMethods Eng

Finite element scheme

- The affine mapping defined by the shape functions can represent any convex polygon, and thus the Voronoi meshes can be supported by the set of reference regular n-gons.

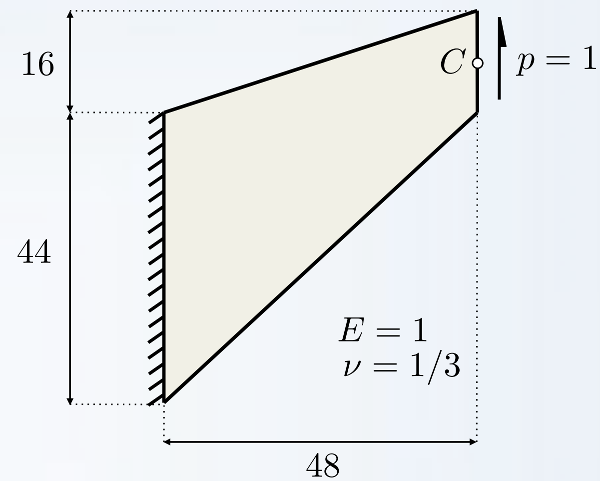


- The resulting finite elements for $n = 3$ and $n = 4$ are identical to popular constant strain triangles and bilinear quadrilateral, respectively.

Sukumar, Tabarraei(2004) IntJ NumerMethods Eng

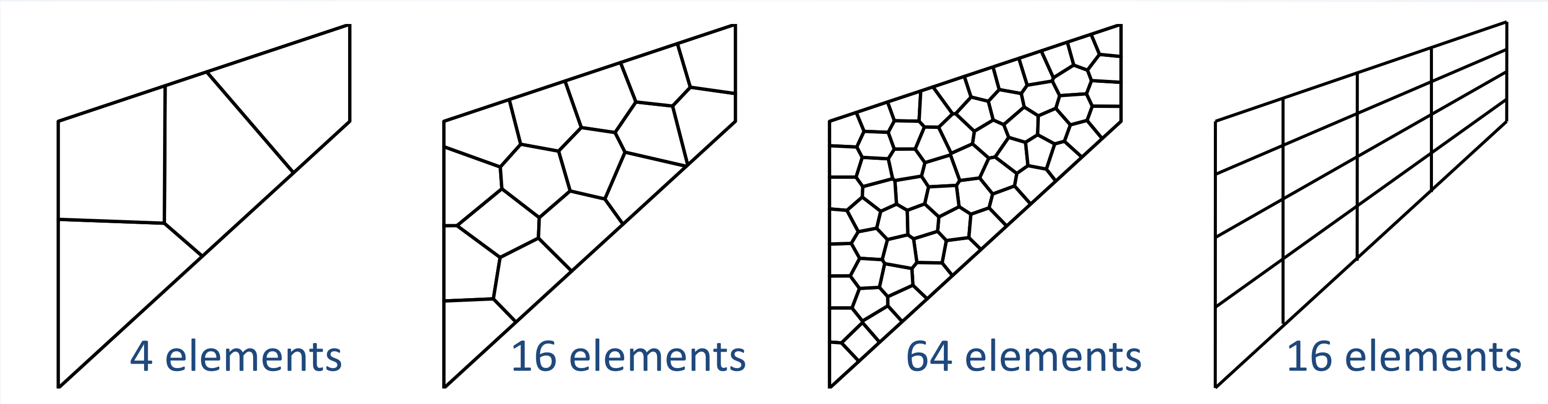
Performance of Polygonal FEM

- Consider Cook's problem consisting of a tapered swept panel subjected to uniform shear loading:



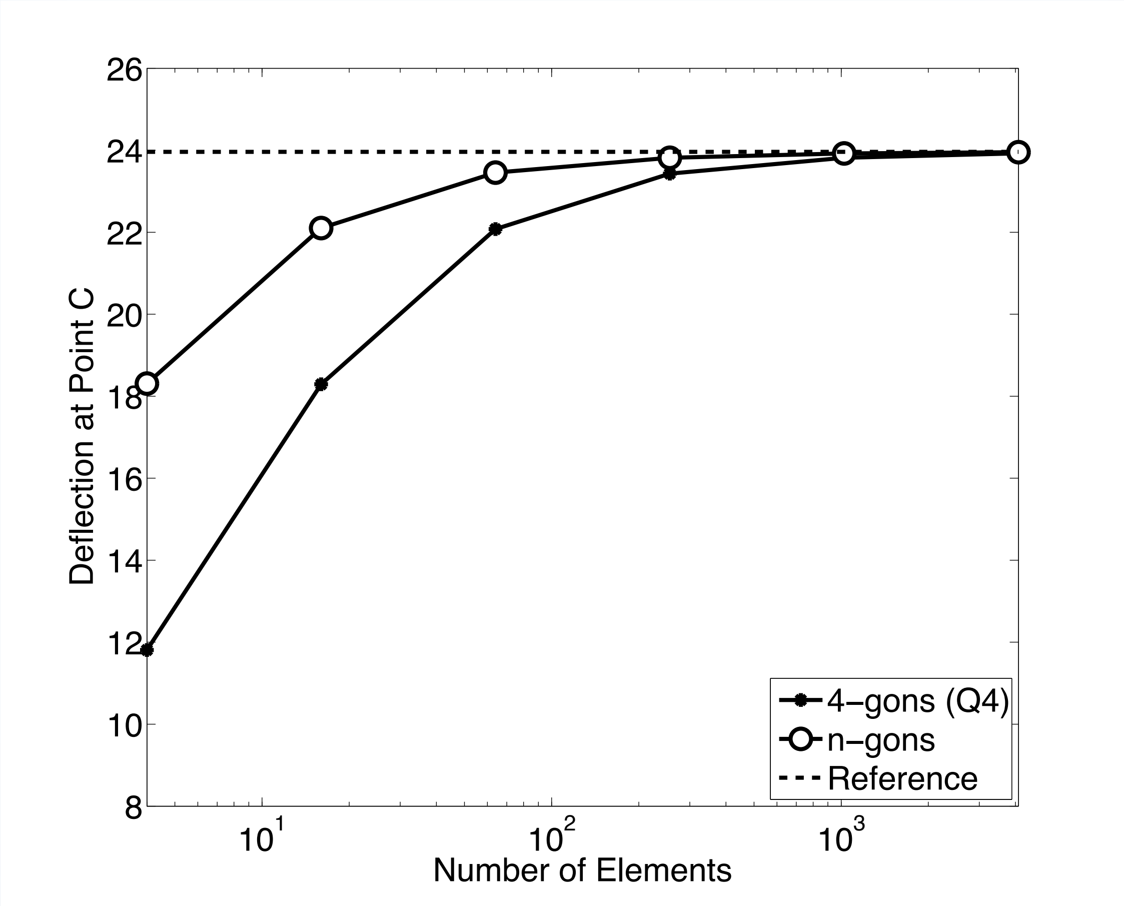
- Quantity of interest is the tip deflection at mid-depth of the panel tip (point C)

- The results using quadrilateral (progressive refinement) and polygonal (independent refinement) discretizations are compared



Performance of Polygonal FEM

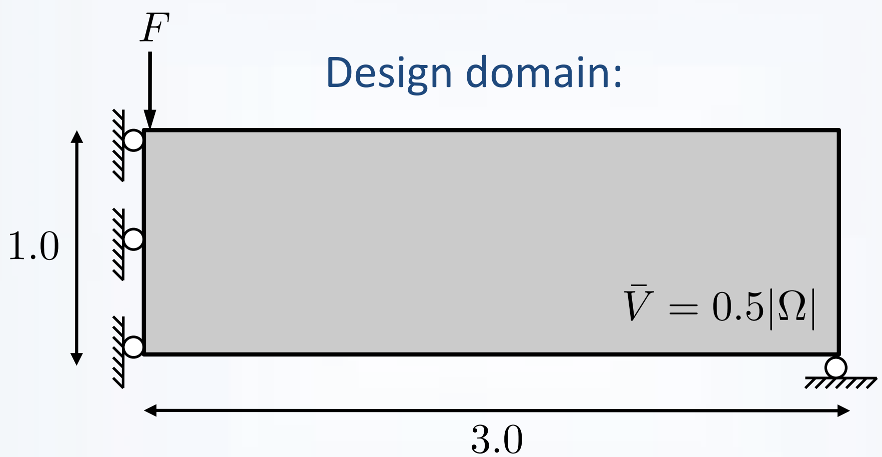
- Polygonal elements are not as stiff as the quadrilateral elements and produce more accurate results, especially with coarser meshes



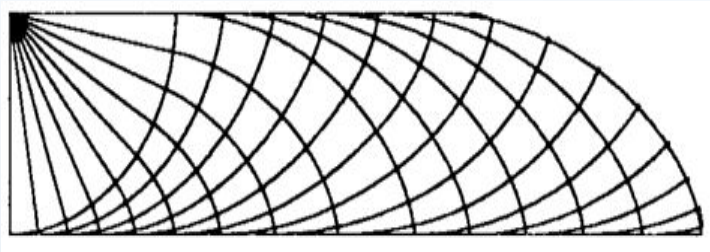
Topology Optimization Results

- The results are presented for minimum compliance problem based on SIMP formulation and MMA as the solver

Design domain:



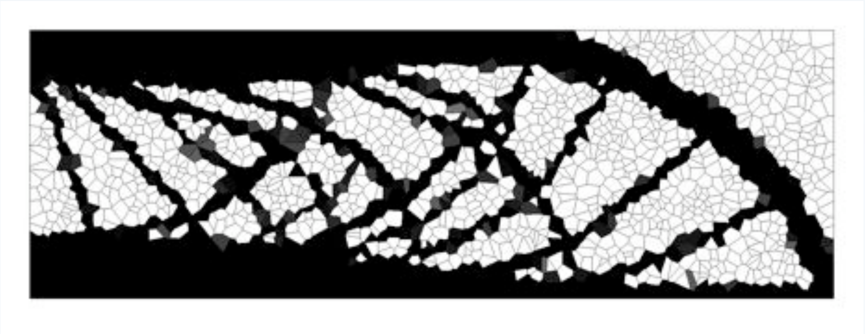
Michell-type solution:



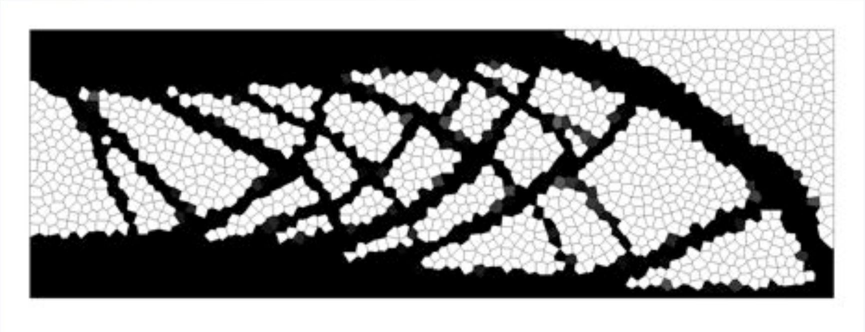
Lewinski, Zhou, Rozvany (1994) IntJ MechSci

Topology Optimization Results

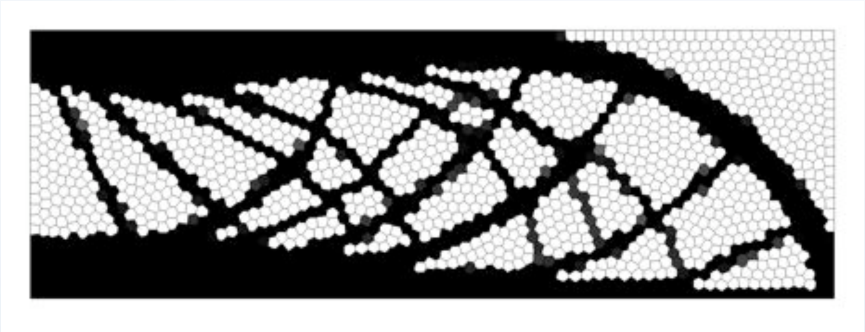
- Results based on different levels of mesh regularity



Random mesh
Final compliance = 188.19



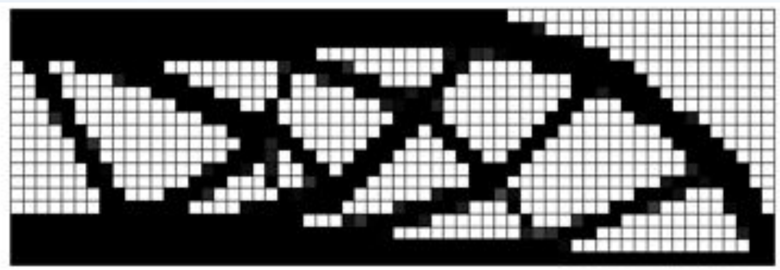
Quasi-random mesh
Final compliance = 188.25



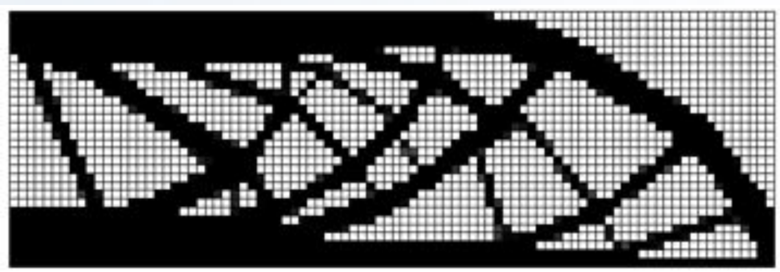
CVT mesh
Final compliance = 187.30

Topology Optimization Results

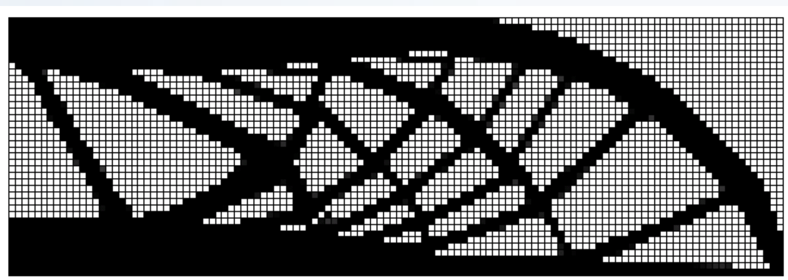
Q8 mesh:



1200 elements, 3761 nodes

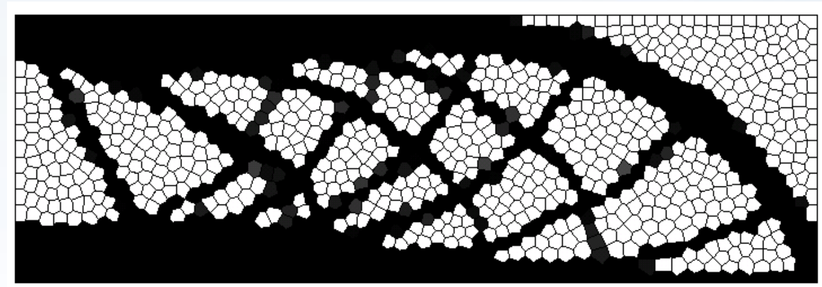


2700 elements, 8341 nodes

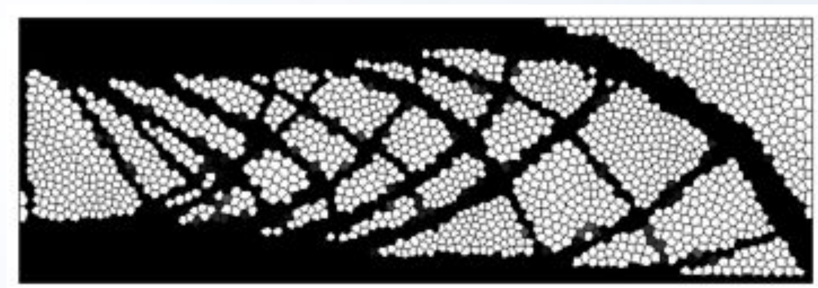


4800 elements, 14721 nodes

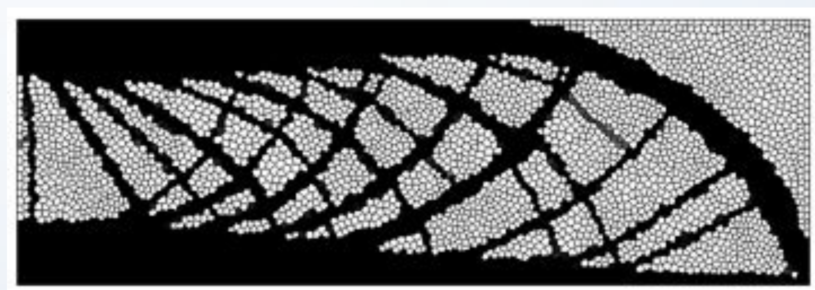
Voronoi mesh:



1336 elements, 2674 nodes



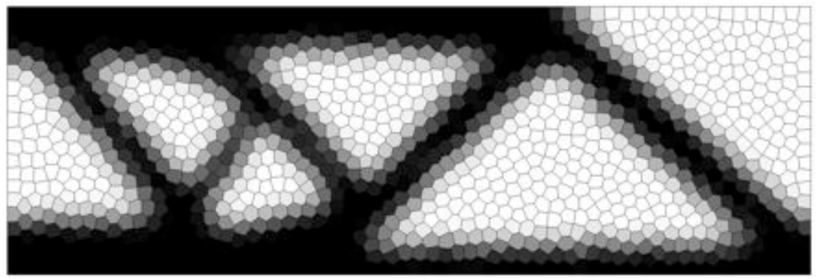
2869 elements, 5740 nodes



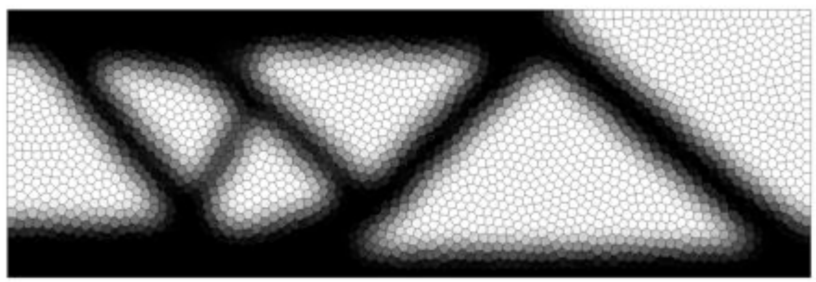
5062 elements, 10126 nodes

Topology Optimization Results

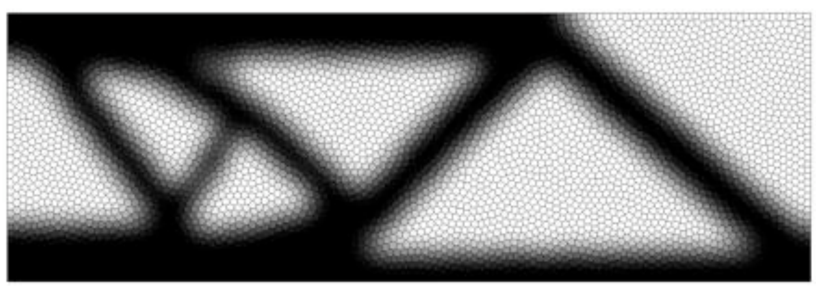
- Results using projection method:



1200 elements
Final compliance = 260.56



2700 elements
Final compliance = 257.43



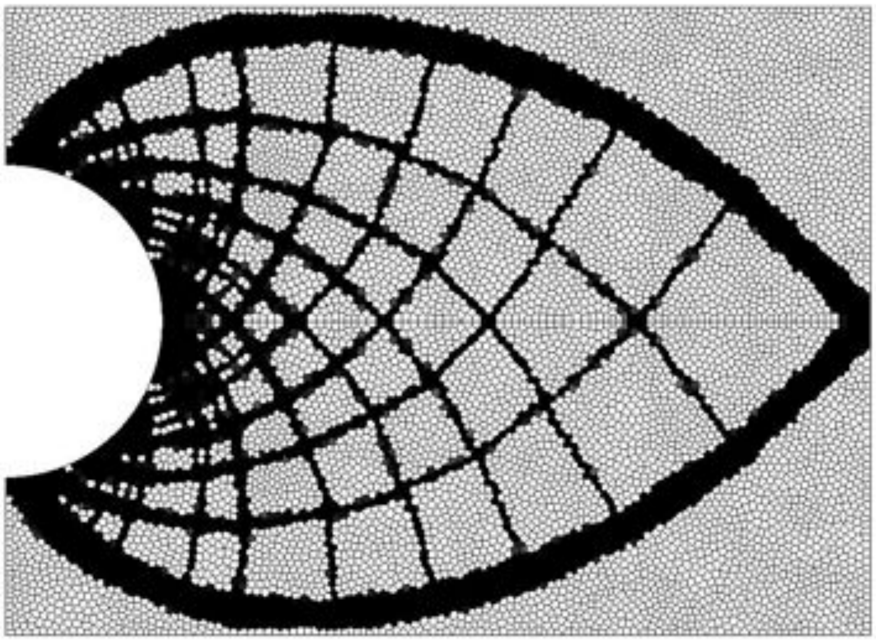
4800 elements
Final compliance = 257.99

Guest, Prevost, Belytschko(2004). IntJ NumerMethods Eng

Topology Optimization Results

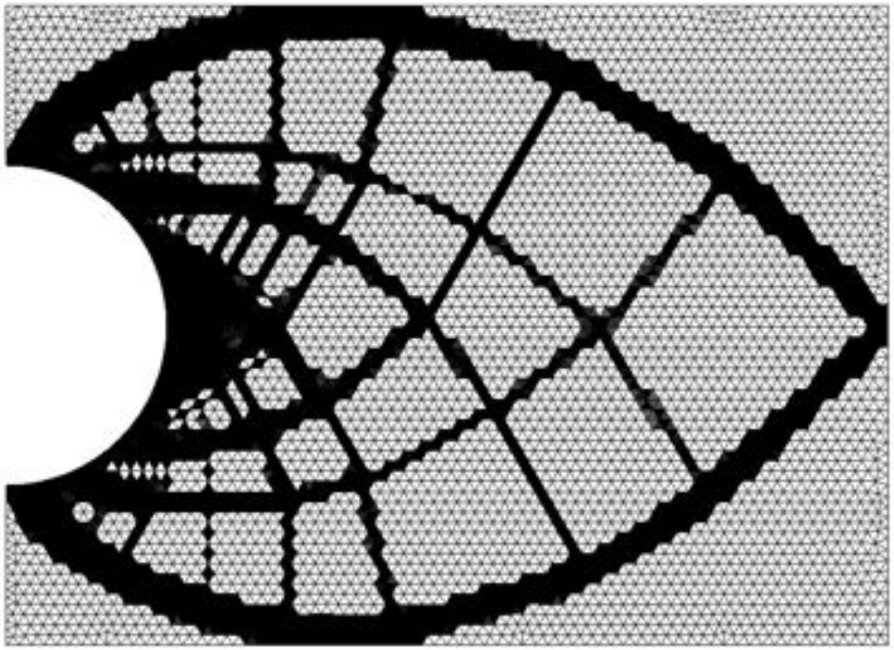
- Michell cantilever beam with circular support

Voronoi mesh:



Number of nodes = 19973
Number of elements = 10000

T6 mesh:



Number of nodes = 17441
Number of elements = 8584

Concluding remarks

- Solutions of discrete topology optimization problems with fixed mesh representation include a form of mesh dependency that stems from the geometric features of the spatial discretization

- To address this problem, we employ fully unstructured meshes to reduce the influence of the simplex geometry on topology optimization solutions

- To this end, we propose a meshing algorithm for generating arbitrary polygonal discretization that enjoy higher levels of directional isotropy and good finite element quality