11th US National Congress on Computational Mechanics:

ON RESTRICTION METHODS FOR TWO-PHASE OPTIMAL SHAPE PROBLEMS

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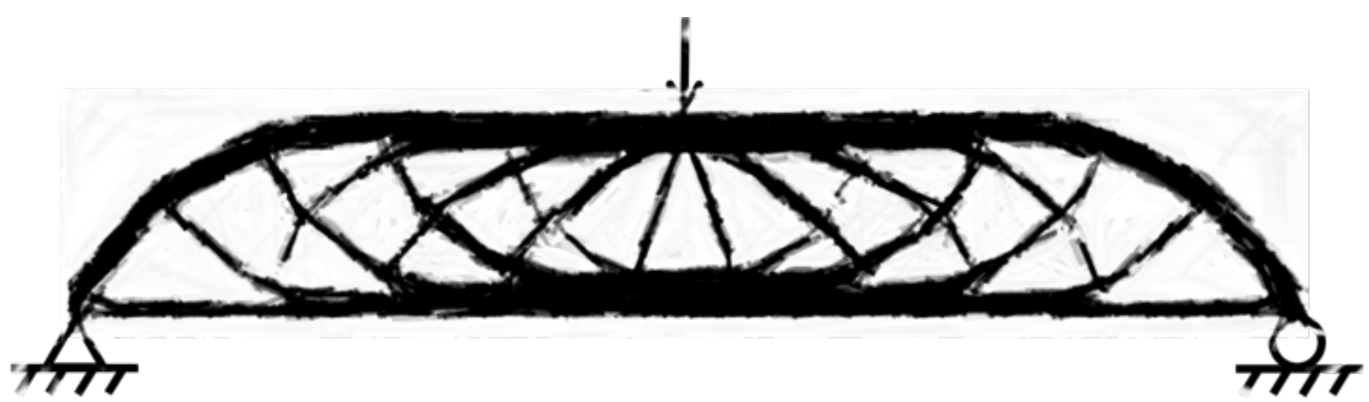
Minneapolis, Minnesota, July 25th, 2011



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- For example, we show that a consequence of the ill-posedness is that smearing of Heaviside function transforms the topology problem into the variable thickness problem

Problem statement

- The two-phase optimal shape problem is defined as:

$$\inf_{\chi \in \mathcal{A}} J(\chi, \mathbf{u}_\chi) \quad \text{where } \mathbf{u}_\chi \in \mathcal{V} \text{ solves} \quad \mathcal{B}(\mathbf{u}, \mathbf{v}; \chi) = \ell(\mathbf{v}),\ \forall \mathbf{v} \in \mathcal{V}$$

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$\mathcal{A} \subseteq L^\infty(\Omega; \{0, 1\})$ is the given space of admissible designs,

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are the bilinear and linear forms, and $\mathcal{V} = \left\{\mathbf{u} \in H^1\left(\Omega; \mathbb{R}^d\right) : \mathbf{u}|_{\Gamma_D} = \mathbf{0}\right\}$

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  - The objective function for the minimum compliance is given by

$$J(\chi, \mathbf{u}_\chi) = \ell(\mathbf{u}_\chi) + \lambda \int_\Omega \chi \mathrm{d}\mathbf{x}$$

  where λ is the volume penalty parameter

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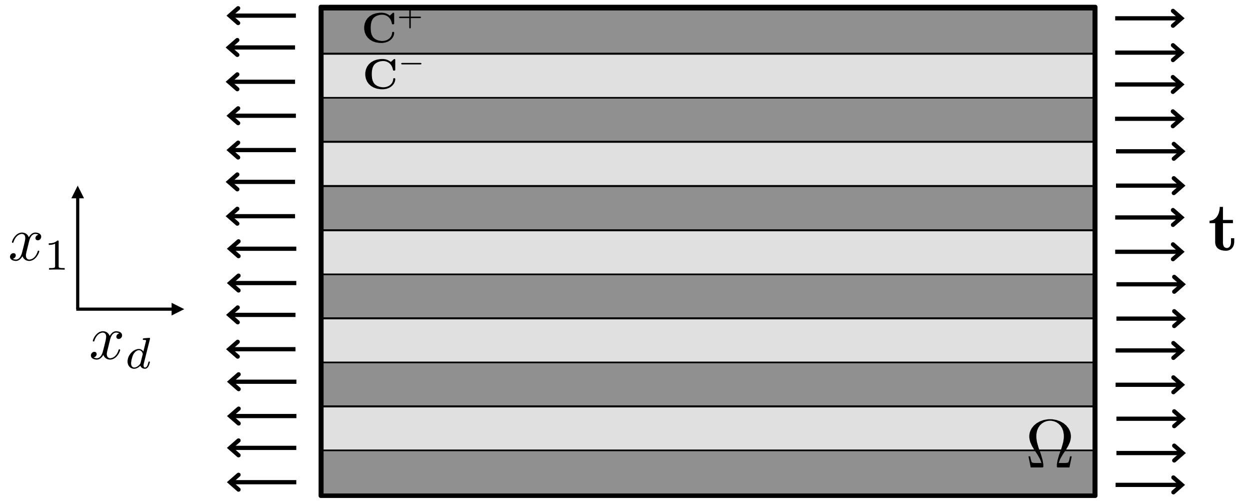

- The optimal design for this problem is a rank-1 laminate, whose stiffness is precisely the $H-$limit of $\chi_n \mathbf{C}^+ + (1 - \chi_n)\, \mathbf{C}^-$

Restriction methods

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- A well-known example is the space of designs with bounded perimeter:

$$\mathcal{A} = \left\{\chi \in BV(\Omega\,\{0,1\}) : \int_\Omega |\nabla\chi|\, d\mathbf{x} \leq \overline{P}\right\}$$

Restriction of implicit function field

- Another choice (Liu et al. 2003) is to set $\mathcal{A} = H(\mathcal{F})$ where the implicit functions $\varphi \in \mathcal{F} \subseteq W^{1+\theta,2}$ satisfy:

$$\text{(R1)}: \quad \|\varphi\|_{W^{1+\theta,2}(\Omega)} \leq M$$

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- (R2) ensures that the phase boundary

$$\{\mathbf{x} \in \Omega : \varphi(\mathbf{x}) = 0\},$$

  which is where the Heaviside is discontinuous, has zero measure:
  - Without it, $\varphi_n(\mathbf{x}) = (\alpha/n^{2+\theta})\sin(nx_1)$ gives a minimizing sequence that satisfies (R1) but does not converge

Approximation of the Heaviside

- If no restrictions are placed on φ, the usual approximation of the Heaviside by

$$H_w(\varphi)(\mathbf{x}) = \begin{cases} 0, & \varphi(\mathbf{x}) < -w \\ h_w(\varphi(\mathbf{x})), & |\varphi(\mathbf{x})| \leq w \\ 1, & \varphi(\mathbf{x}) > w \end{cases}$$

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  - Note also that the conditions of optimality are the same:

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Therefore the optimal solution with such approximation will contain large "grey" regions filled by the intermediate phases

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$$J_\beta(\chi, \mathbf{u}_\chi) = J(\chi, \mathbf{u}_\chi) + \beta \int_\Omega \chi\,(1-\chi)\,\mathrm{d}\mathbf{x}$$

OR change the state equation to penalize the intermediate stiffnesses:

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  In both cases, separation of phases and thus transversality is achieved in the optimal regime.

- The condition of optimality for $\{-w < \varphi < w\}$ respectively are:

$$H_w'(\varphi)\,\{\lambda + \beta\,[1 - 2H_\omega(\varphi)] - E(\mathbf{u})\} = 0$$

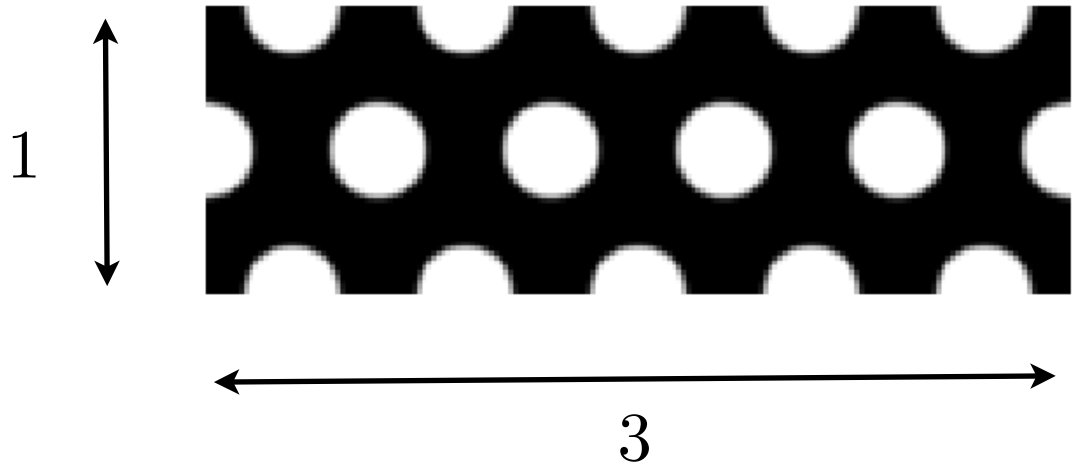
$$H_w'(\varphi)\,\left\{\lambda - p\,[H_w(\varphi)]^{p-1} E(\mathbf{u})\right\} = 0$$

Numerical investigation via an SAO algorithm

- The continuum parameters (i.e., those independent of the mesh size) are:
  - α: bound for implicit function field
  - R: radius of filtering kernel K
  - w: width of the approximate Heaviside
  - p: parameter for penalization of intermediate stiffnesses
- It is not be easy to establish an explicit relationship between ν with above parameters in general
- However the compliance problem, the transversality constant ν is directly related to α/R (which is why we set w to be fixed fraction of α/R)

Some numerical results

Initial guess:



Values of parameters used: $\alpha = 1, \quad w = 0.0375\alpha/R, \quad p = 4$

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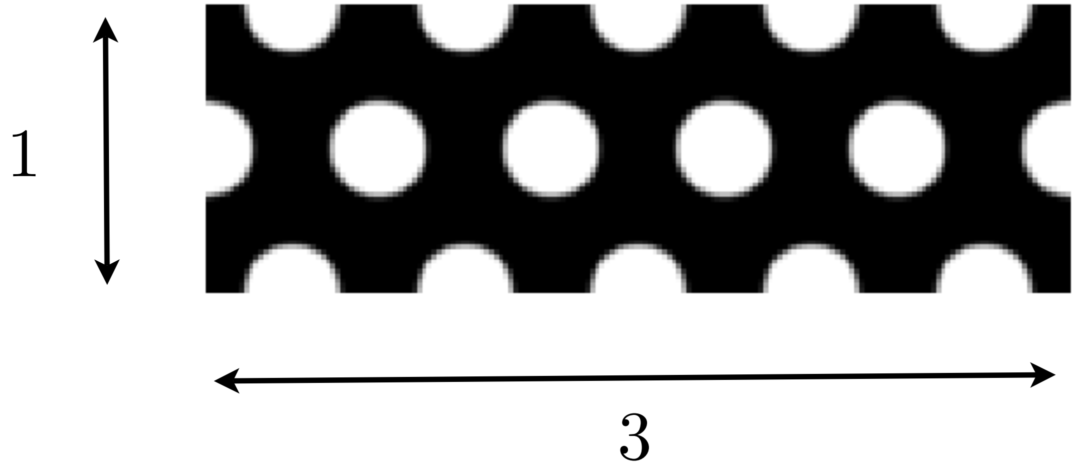

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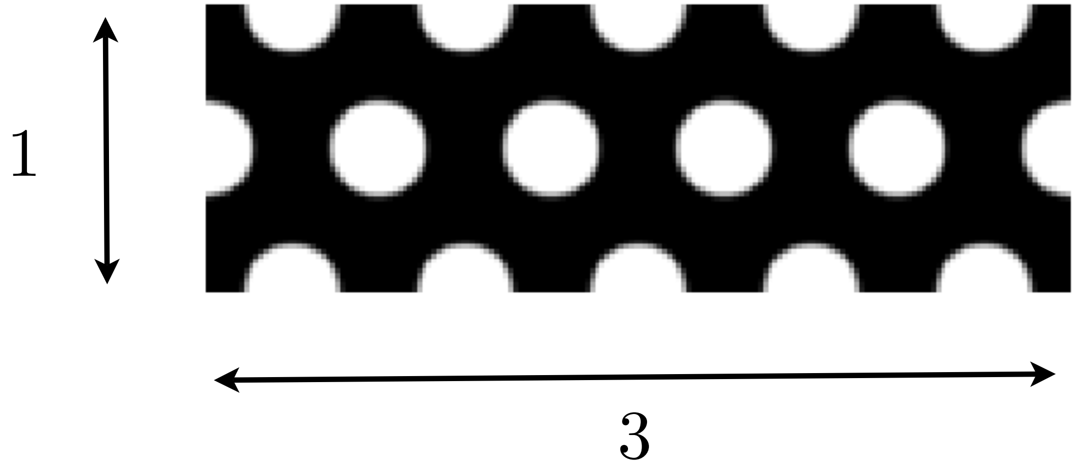

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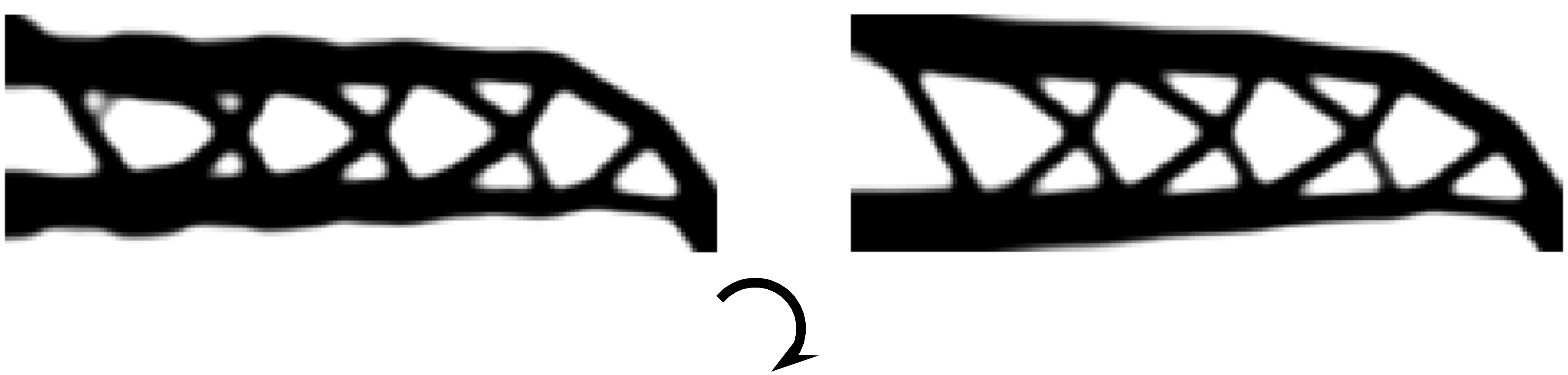

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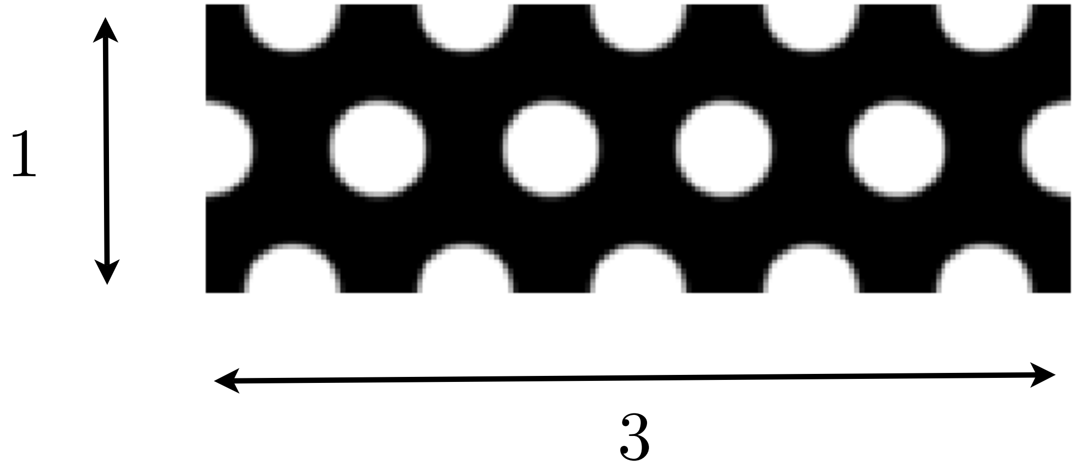

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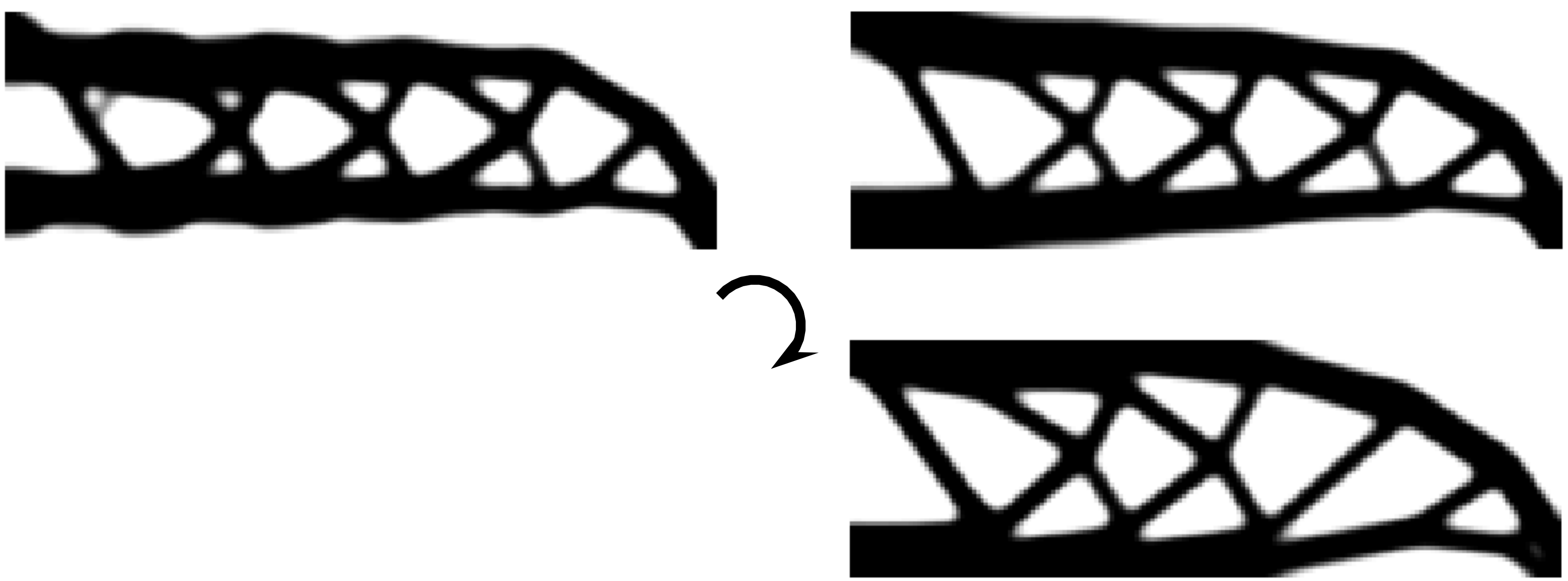

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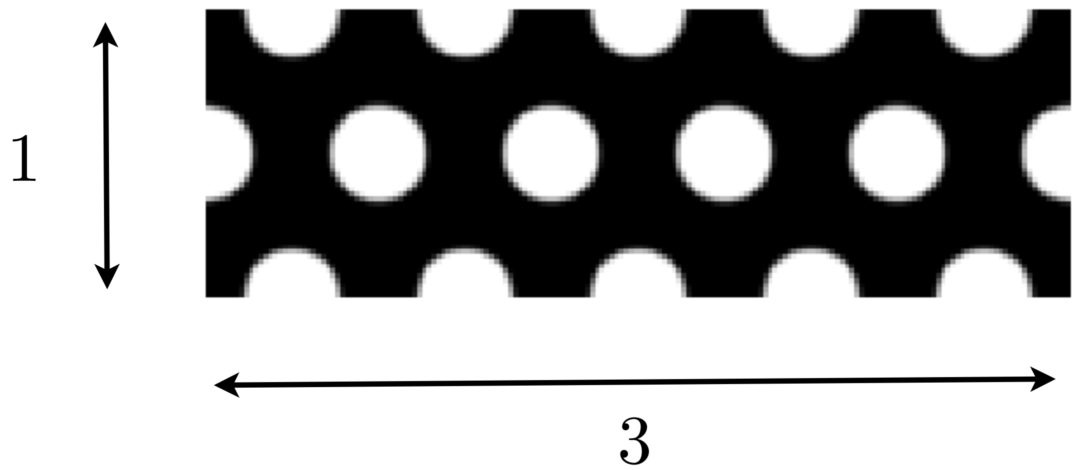

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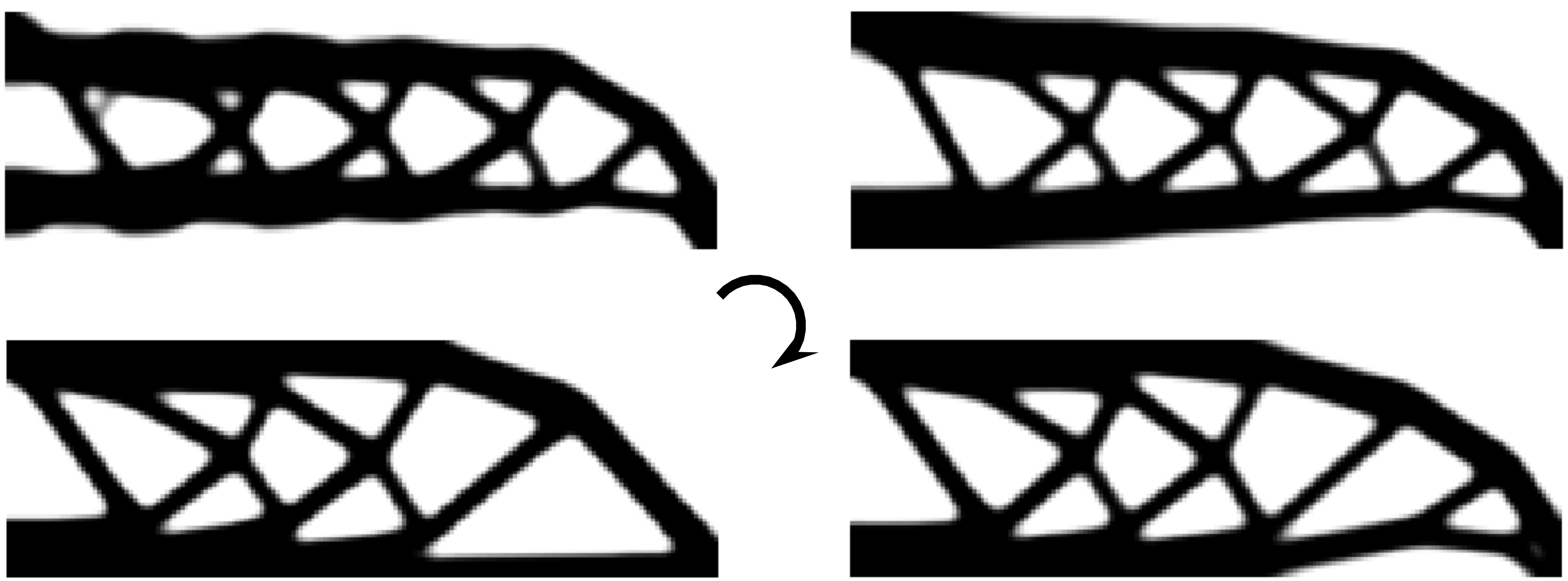
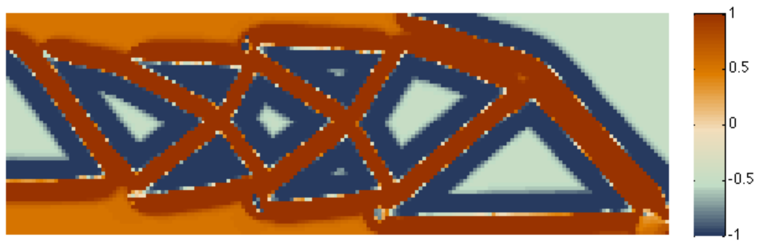


Some numerical results

$H_w(\varphi)$

η



$R = 0.075$

$\varphi = K_R \star \eta$

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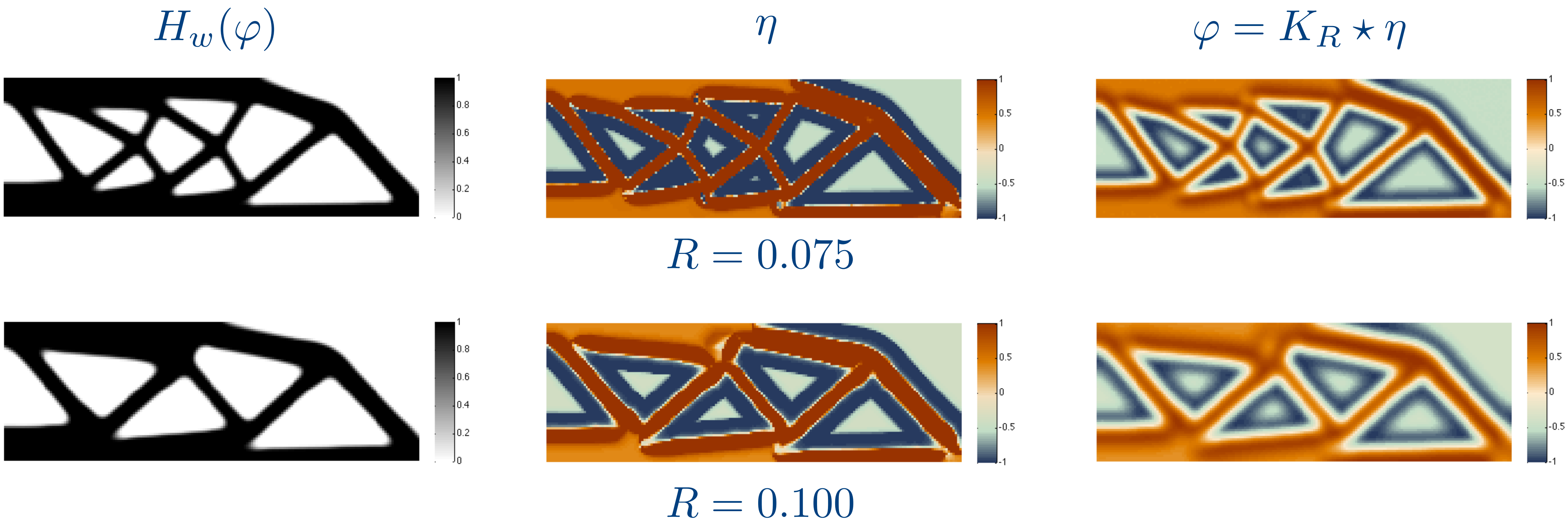

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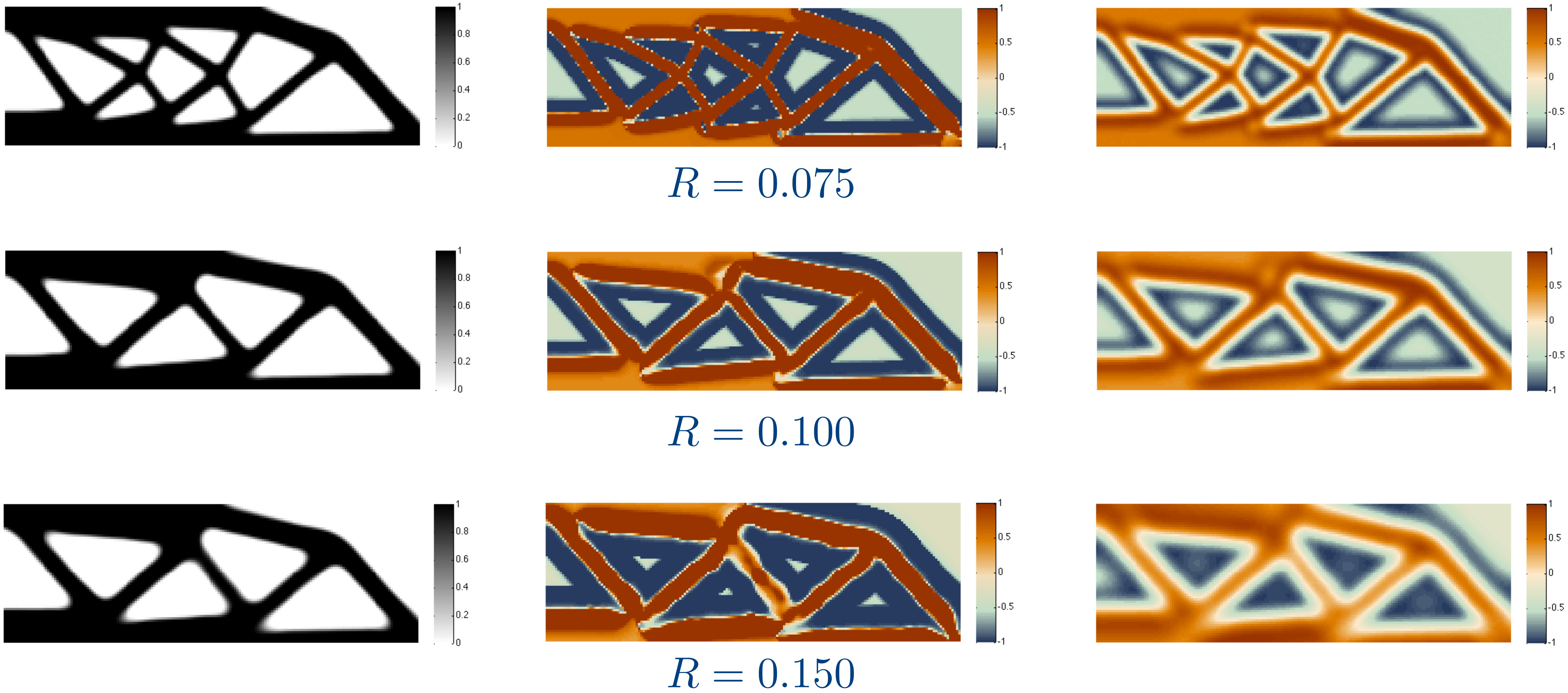

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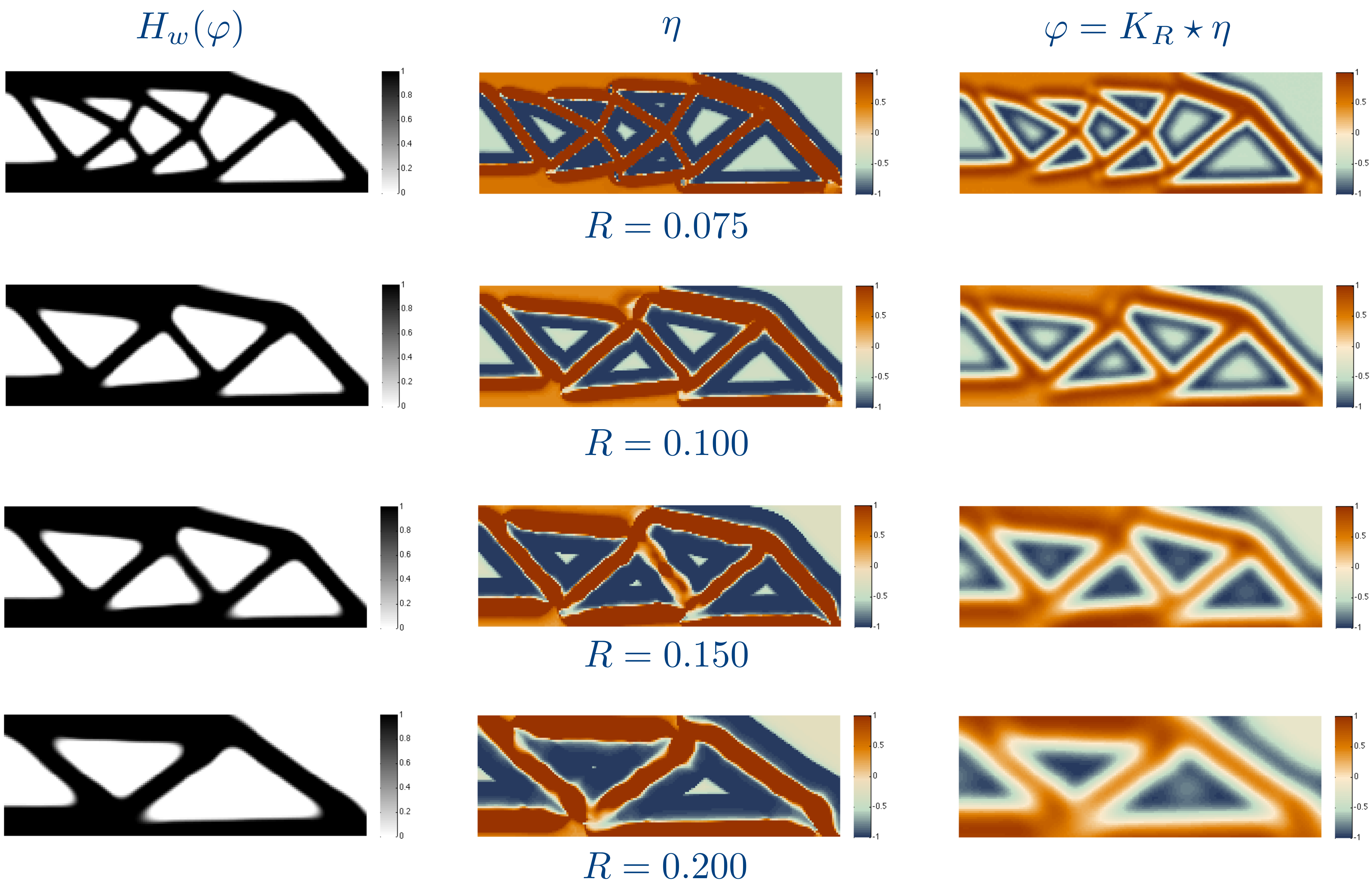
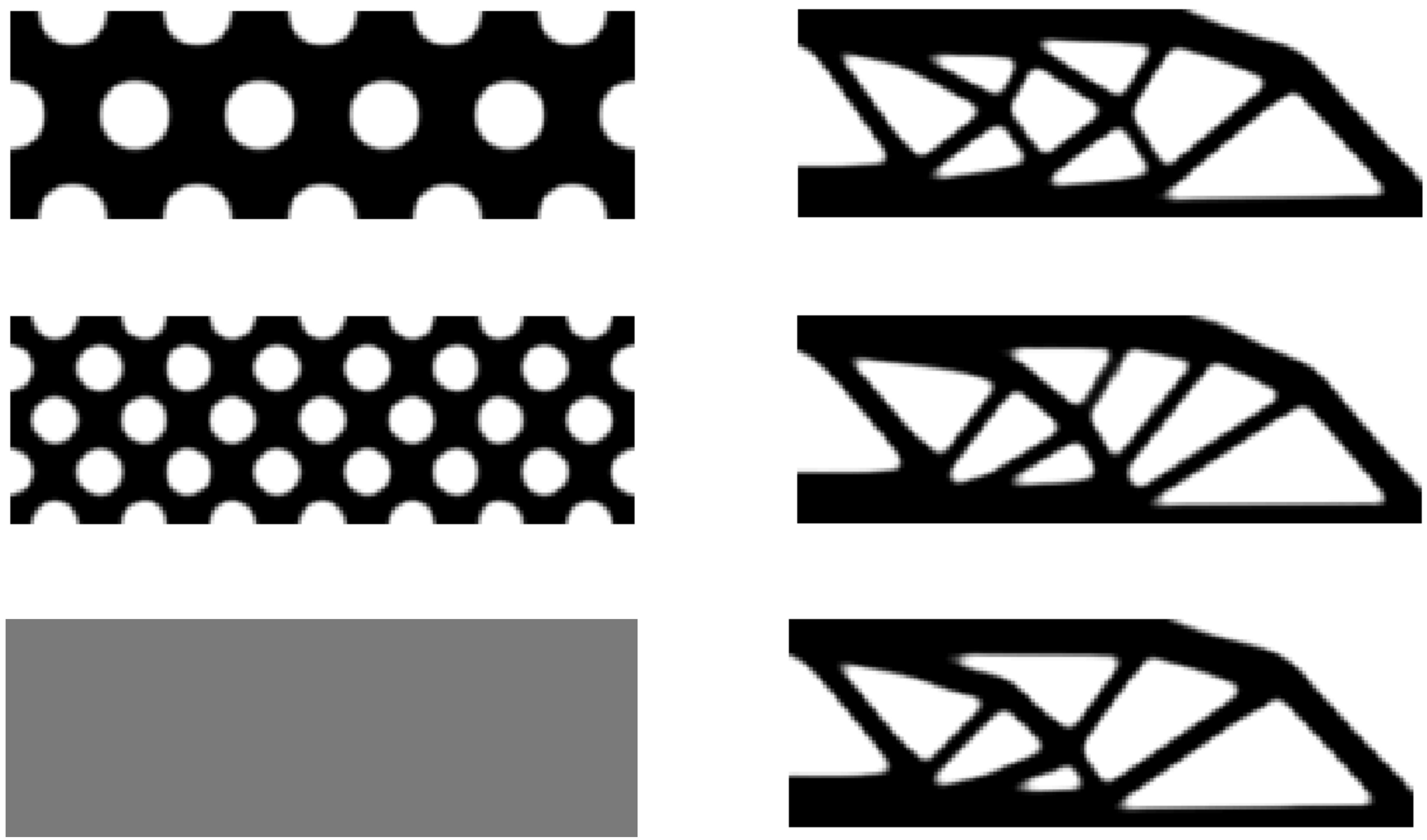


Some numerical results

Initial guess | Final solution



Concluding remarks

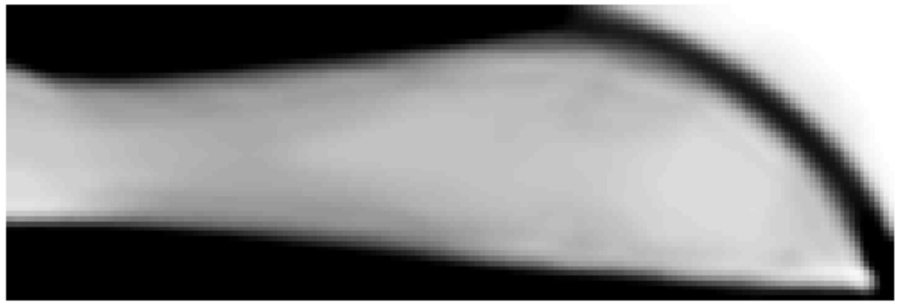
- The nature of the continuum optimal shape problem has implications for the numerical formulations and algorithms
- In addition to smoothness, a uniform “transversality” condition must be imposed on the implicit function field
- Within the restriction framework, the Ersatz material model (filling the voids with compliant material $\mathbf{C}^-$) can be justified

QUESTIONS?

Approximation of the Heaviside

- This fact can be illustrated numerically:

Initial φ



Final solution

- With transversality condition (R2) imposed, however, we can prove that as $w \to 0$, the optimal solution $\chi_w^* = H_w(\varphi^*)$ converge to solution of problem with $\mathcal{A} = H(\mathcal{F})$