Mechanisms and states of self-stress of planar trusses using graphic statics, part I: The fundamental theorem of linear algebra and the Airy stress function

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Abstract
The fundamental theorem of linear algebra establishes a duality between the statics of a pin-jointed truss structure and its kinematics. Graphic statics visualizes the forces in a truss as a reciprocal diagram that is dual to the truss geometry. In this article, we combine these two dualities to provide insights not available from a graphical or algebraic approach alone. We begin by observing that the force diagram of a statically indeterminate truss, although itself typically a kinematically loose structure, must support a self-stress state of its own. Such an “extra” self-stress state is described by the fundamental theorem of linear algebra. We show that the self-stress states of a truss are in a one-to-one correspondence with linkage-mechanism displacements of its reciprocal, and the relative centers of rotation of these mechanism displacements correspond to centers of perspective of a projection of a plane-faced three-dimensional polyhedral mesh. We prove that this polyhedral mesh is exactly the continuum Airy stress function, restricted to describe equilibrium of a truss structure. We use the Airy function to prove James Clerk Maxwell’s conjecture that a two-dimensional truss structure of arbitrary topology has a self-stress state if and only if its geometry is given by the projection of a three-dimensional plane-faced polyhedron. Although a very limited number of engineers have been aware of the relationship between trusses and a polyhedral Airy function, the authors believe that this is the first truly rigorous elucidation. We summarize the properties of this “dual duality,” which has the Airy function at its core, and conclude by showing applications to design of tensegrities, planar panelization of architectural surfaces, and optimization of trusses.

Keywords
Airy stress function, dual structures, graphic statics, Maxwell, mechanisms, projective geometry, reciprocal diagrams, reciprocal figures, self-stress, truss

Introduction
Graphic statics is a historical method of design and analysis of pin-jointed trusses which continues to have relevance to structural engineering practice.¹ In graphic statics, a diagram dual to the truss geometry represents the forces in a truss structure by the length of its lines. Called a reciprocal figure or force diagram, its lines are parallel to those in the truss (the form diagram). Closure of each polygon face in the force diagram guarantees nodal equilibrium in the form diagram.

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indeterminacy to force diagrams that, if considered as dual structures in their own right, are actually underbraced; they are kinematically loose, admitting a set of nonzero displacements which cause no deformation of the bars. Such a dual structure is a linkage mechanism. But to recover the original form diagram as a reciprocal figure, this dual structure must also allow a set of nonzero equilibrium forces. This is quite different from a typical linkage mechanism, which cannot generally be in static equilibrium.

Such an exceptional geometrically singular truss, underbraced but self-stressable, points out the inadequacy of conventional categories of statical determinacy, indeterminacy, and kinematic indeterminacy to describe the full range of structural behavior. Close examination of these exceptional truss geometries leads to a profound array of mathematical correspondences. These correspondences have been explored in Pellegrino\textsuperscript{2} and Calladine\textsuperscript{3} work on tensegrities, in Tachi\textsuperscript{4} work on origami folding and reciprocal figures, in Filippou\textsuperscript{5} work on structural frame analysis, in Kuznetsov\textsuperscript{6} work on duality in analytical mechanics of underbraced structures, in Shai and Pennock\textsuperscript{7} work on structural observation that the layout of a 2D truss that is in equilibrium (form diagram) is the projection of a 3D plane-faced polyhedron and that the 2D projection of the reciprocal polyhedron represents the forces of the original truss (force diagram).

Throughout the article, we will alternate between a graphical approach and an algebraic one in order to highlight the correspondences between these two perspectives and draw out the insights particular to each. Our final result is a “dual duality” that combines graphic statics and graphic kinematics with their algebraic versions. While this is similar to the work of Micheletti\textsuperscript{13} on the linear algebra of reciprocal diagrams or Crapo and Whiteley\textsuperscript{8} on graph theory and truss structures, the connection of this dual duality to the Airy function that we establish provides a useful perspective that leads to additional insights.

**Background**

Maxwell’s seminal 1864 paper “On reciprocal figures and diagrams of forces”\textsuperscript{36} provided the foundation of graphics statics. In that article, he made the remarkable geometrical/structural observation that the layout of a 2D truss that is in equilibrium (form diagram) is the projection of a 3D plane-faced polyhedron and that the 2D projection of the reciprocal polyhedron represents the forces of the original truss (force diagram).

Maxwell and other 19th-century engineers and natural philosophers had extensive knowledge of projective geometry and polyhedral geometry. At the time, it was considered the “modern” mathematics.\textsuperscript{14} In projective geometry, metrics and parallelism are not preserved but only intersecting lines are (Figure 1). In this highly reduced geometrical world, a limited number of things are preserved (invariant) in projective transformations. One thing that is preserved, with the proper transformations, is structural equilibrium, a fact that appears to have been first observed by Rankine.\textsuperscript{15} Projective geometry is clearly fundamental to understanding the properties of 2D projections of 3D polyhedral, but it is surprising, at least to a modern audience, the extent to which projective geometry and polyhedra connect to structural equilibrium.
One of the earliest papers to follow up Maxwell’s work on polyhedra was coauthored by the famous geometer Klein and Wieghardt. This work offered the first rigorous proof of Maxwell’s observation and was the first to connect Maxwell’s reciprocal figures to the Airy stress function, although it stopped short of proving the complete identity of the truss and continuum stress function. This important paper appears to have been largely forgotten; an exception is Kurrer.

Maxwell’s observation was forgotten and rediscovered several times over the following century, often in fields far removed from engineering. In particular, the Structural Topology research group in Montreal was formed after being alerted to it by the engineer Janos Baracs who rediscovered it independently. This group of mathematicians then began to elaborate a rigorous and highly abstract rigidity theory, largely without notice in the engineering world.

This “lost lore” may appear obscure and removed from practical concerns, but we will show that it is in fact quite useful. The fundamental theorem, graphic statics and kinematics, and the discrete Airy stress function provide a map for orienting oneself while exploring optimal structures. Without this map, crucial phenomena, such as the equilibrium of tensegrities, cannot be understood. With this map, superficially unrelated phenomena become related by a profound order, in which insights in one domain can propagate to others by symmetry.

**Statics and kinematics**

A given truss form diagram with \( b \) bars, \( v \) vertices (nodes), and \( f \) interior faces can be mapped to its force diagram by circling each node of the truss in fixed (e.g. counter-clockwise) order, drawing force diagram bars parallel to form diagram bars to form closed polygons. This construction leads to a reciprocal figure with

\[
\begin{align*}
  f^* &= v, \\
  v^* &= f, \\
  b^* &= b
\end{align*}
\]

(1)

where \( v^* \) is the number of vertices, \( f^* \) is the number of faces, and \( b^* \) is the number of bars in the reciprocal figure. In typical graphic statics, one treats loads applied on the outer boundary of the truss, but we must also consider self-stress states.

**Graphic statics and self-stress**

The examples in Maxwell’s paper do not distinguish between the lines of actions (lines on the form diagram) that represent truss members and those that represent applied forces. In fact, all the lines of action could be bars, and the existence of the reciprocal diagram (force diagram) represents a state of self-stress.

The diagrams in Maxwell’s paper (Figure 2) may look a little unusual to someone familiar with standard graphic statics diagrams. Bow’s notation, which became the standard method of drawing reciprocal diagrams, came a few years later. Figure 3 is a version of Figure 2 in a common form of Bow’s notation, where three of the lines of action are assumed to be external forces.

In the force diagram of Figure 3, the closed polygon \( abc \) demonstrates that the translational forces sum to zero.
The sum of moments is zero because the forces in the form diagram intersect in a point, that is, the funicular diagram shrinks to a point. By transforming internal forces into external forces or vice versa in this manner, one can easily see how graphic statics can be applied to self-stress states.

**Graphic kinematics**

The linkage-mechanism motions (displacements) of a pin-jointed plane truss can be characterized graphically by determining the location of the instant centers (ICs) of rotation of each of the bars using the Aronhold–Kennedy rule, which states that the ICs of any two bars must lie on a line through their common node (Figure 4). The rotation w about the IC yields the linkage-mechanism displacement $U_i$ of the node when multiplied by the lever arm between the IC and node, in the direction perpendicular to the line connecting the IC and the node (Figure 5). This allows graphical determination of mechanism motions: once a set of consistent ICs and the A-K lines connecting them between nodes can be drawn, all displacements can be found.

Any plane truss that admits an assignment of ICs consistent with the A-K rule has a mechanism. Such a set of consistent centers can exist regardless of the count of equations and unknowns: in Figure 6, the diagram has a mechanism even though it has equal equations and unknowns.

**Algebraic statics and kinematics**

Recall the truss kinematic equation

$$A_u U_i = V_t, \quad \text{row}_e [A_u] = \begin{bmatrix} \tau_{ij} & \tau_{ji} \end{bmatrix}$$

(2)

where $U_i$ are the translational displacements of the truss nodes in the global frame, $V_t$ is the vector of element deformations (length changes) in the local element frame, and $\tau_{ij}$ is the unit-length direction vector pointing from node $j$ to node $i$, and $\text{row}_e$ indicates the row of $A_u$ corresponding to bar $e$. Equation (2) is the matrix form of the simple geometric definition of the infinitesimal length change/deformation of the bar as

$$V_e = \tau_{ij} \cdot u_j + \tau_{ji} \cdot u_i = \tau_{ij} \cdot u_j - \tau_{ji} \cdot u_i$$

(3)

where $u_i$ and $u_j$ are the global-frame single-node displacement vectors at node $i$ and $j$.

Recall the truss equilibrium equation

$$B_n Q_i = P_t, \quad \text{row}_e [B_n] = \sum_{j \in \text{star}(i)} Q_{ij} \tau_{ij}$$

(4)

where $P_t$ are the applied nodal forces in the global frame, $Q_i$ is the vector of bar axial forces, and $\text{star}(i)$ is the set of nodes connected to node $i$ via the $n_b$ adjacent bars. Note that direction vectors $\tau_{ij}$ point toward node $i$, by the tension-positive sign convention.

We can extend this to moment-connected frame structures by introducing applied moments $P_r$, end moments $Q_e$, nodal rotations $U_r$, and element angle changes $V_r$. By introducing the linearized/infinitesimal rotation matrix

![Figure 4. Motion of bars without extension is equivalent to rotation about an instant center (IC). Connected bars must have collinear centers.](image1)

![Figure 5. Linkage-mechanism displacements of a typical single-degree-of-freedom four-bar mechanism can be obtained from one element rotations about any bar instant center. ICs of each pair of connected bars are collinear with the node connecting the bars.](image2)

![Figure 6. This truss has a mechanism even though it is statically determinate: ICx,y,z overlap to allow inner triangle motion.](image3)
\[
\mathbf{R} = \begin{bmatrix}
0 & -\tau_y & \tau_z \\
\tau_y & 0 & \tau_z \\
-\tau_z & -\tau_x & 0
\end{bmatrix}
\]

so that \( \mathbf{R}\mathbf{u}_i = \tau_i \times \mathbf{u}_i \) (5)

we can expand (2) to include the element rotational deformations (angle changes) \( \mathbf{V} \), via

\[
\begin{bmatrix}
\mathbf{A}_n \\
\mathbf{A}_n \mathbf{I}
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}_t \\
\mathbf{U}_r
\end{bmatrix} =
\begin{bmatrix}
\mathbf{V}_t \\
\mathbf{V}_r
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
\mathbf{A}_n \\
\mathbf{A}_n \mathbf{I}
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}_t \\
\mathbf{U}_r
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{L_{ij}} \mathbf{R}_{ij} \\
\frac{1}{L_{ji}} \mathbf{R}_{ji}
\end{bmatrix}
\]

with \( 1/L_{ij} \) required for \( \mathbf{A}_n \mathbf{I} \) to have the correct (angle) units.

We can extend the equilibrium matrix similarly. Both sets of equations are summarized in Table 1.

**Table 1.** The static and kinematic equations.

<table>
<thead>
<tr>
<th>Statics</th>
<th></th>
<th>Kinematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Global variables</td>
<td>( \mathbf{P}_t, \mathbf{P}_r ) (nodal loads)</td>
<td>Global variables</td>
</tr>
<tr>
<td>Local variables</td>
<td>( \mathbf{Q}_t, \mathbf{Q}_r ) (element forces)</td>
<td>Local variables</td>
</tr>
<tr>
<td>Truss equations</td>
<td>( \mathbf{B}_n \mathbf{Q}_t = \mathbf{P}_t )</td>
<td>Truss equations</td>
</tr>
<tr>
<td>Extension to frames</td>
<td>( \begin{bmatrix} \mathbf{B}_n &amp; \mathbf{B}_r \end{bmatrix} \begin{bmatrix} \mathbf{Q}_t \ \mathbf{Q}_r \end{bmatrix} = \begin{bmatrix} \mathbf{P}_t \ \mathbf{P}_r \end{bmatrix} )</td>
<td>Extension to frames</td>
</tr>
</tbody>
</table>

**The two dualities**

Two distinct dualities cast light on truss behavior: one graphical and the other algebraic.

**Dual structures in graphic statics**

Something that is not always obvious in graphic statics is that either diagram could represent the structure or the forces. Per Maxwell,9 “reciprocal figures are such that the properties of the first relative to the second are the same as those of the second relative to the first.” In fact, the two diagrams can be viewed as *structural duals* (Figure 7). This duality of reciprocal diagrams was explored via linear algebra and graph theory in Micheletti13 and was recently further discussed in Beghini et al.37

For a 2D structure, Calladine3 has refined the “Maxwell rigidity number” \( N \) (the number of equilibrium equations minus the number of force unknowns) to account for mechanisms and self-stress states. This can be expressed as

\[
N = 2v - b - 3 = v - f - 1 = m - s
\]

where \( m \) is the number of mechanisms and \( s \) is the number of states of self-stress. The second form of the expression is obtained using the Euler rule \( 2 = v - b + f \) for simply connected meshes.

If \( N = 0 \), the structure may be statically determinate, or it may have an equal number of mechanisms and states of self-stress. If \( N > 0 \), the truss will have mechanisms and if, \( N < 0 \), the truss will have redundant members resulting in possible states of self-stress.

Depending on the specific geometry and topology, a truss can have both states of self-stress and mechanisms, in which case \( N \) is equal to the number of mechanisms in excess of the number of states of self-stress (Figure 8).

For dual structures, we can write the extended Maxwell rule (7) for the dual as well as the original structure.

\[
\begin{bmatrix}
\mathbf{A}_n \\
\mathbf{A}_n \mathbf{I}
\end{bmatrix}
\begin{bmatrix}
\mathbf{U}_t \\
\mathbf{U}_r
\end{bmatrix} =
\begin{bmatrix}
\mathbf{V}_t \\
\mathbf{V}_r
\end{bmatrix}
\]

**Figure 7.** Graphic statics and dual structures.
We can then apply the reciprocal relations (1) and the Euler rule \(2 = v - b + f\) to show that \(N' + N = -2\). Consequently, if \(N \leq -2\), so the force diagram has equal equations and unknowns if the form diagram has two more unknowns than equations, and the force diagram has more equations than unknowns if the form diagram has three or more unknowns than equations.

But since the dual structure arose from a self-stress state of a form diagram, it must have at least one self-stress state \(s^*\) of its own to return the form diagram as its reciprocal. Then by equation (7), even an “indeterminate” dual structure must also have a mechanism \(m^*\). The case where \(N' = -1\) and \(N^* = -1\) is a boundary case: in this and only this case, both diagrams can have one self-stress state and no mechanisms.

**The static-kinematic duality in linear algebra**

Per Pellegrino,\(^2\) we can observe that \(B_m = A^T_m\); although this symmetry can also be understood as a consequence of virtual work, or of Lagrange’s analytical mechanics as in Kuznetsov,\(^6\) it is simply a geometric fact. This duality extends to the frame terms in Table 1, so \(B_{e} = A^T_{e}\) as well. This symmetry, called the static-kinematic duality, helps explain the extended Maxwell rule (7) and the peculiar properties of dual structures, once we apply the fundamental theorem of linear algebra.

**Mechanisms and self-stresses**

The fundamental theorem of linear algebra relates the four fundamental subspaces of a matrix to the four fundamental subspaces of its transpose, where the four fundamental subspaces are the row space, column space, and right and left nullspaces.\(^20\) Since the truss kinematic matrix is the transpose of the equilibrium matrix by the static-kinematic duality, we can apply it to our truss system.

**The fundamental theorem of linear algebra for truss structures**

For our truss kinematic equation (1), we can write the orthogonal decomposition

\[
U_i = U_p + U_h \quad \text{where} \quad A_h U_h = 0 \\
\text{and} \quad U_h \cdot U_p = 0
\]

so that the nullspace \(U_h\) of the kinematic matrix \(A_h\) is the set of inextensional displacements, that is, displacements that correspond to a linkage-mechanism motion of the bars. Now we know \(B_m = A^T_m\), so

\[
[A_h U_h] = U_p B_m = 0^T
\]

so that \(U_h \equiv P_i\), where \(P_i = P_c + P_r\), and \(P_c \cdot P_r = 0\)

that is, the mechanism displacements \(U_h\) are equivalent to the left nullspace of the equilibrium matrix: the mechanism-activating loads \(P_i\) (non-equilibrium loads in Kuznetsov\(^6\)).

We can apply an identical logic to nullspace \(Q_h\) of the equilibrium matrix \(B_m\), where \(Q_h\) are the self-stress states of the structure. Then, the self-stresses \(Q_h\) are equivalent to the left nullspace of the kinematic matrix, that is, the incompatible deformations \(V_i\).\(^4\) This should be intuitive, in that an ordinary indeterminate structure can have both self-stresses and misalignments of member lengths that cause it to not fit together \((Q_h \neq 0 \rightarrow V_i \neq 0)\), while a determinate or kinematically indeterminate structure can be fit together regardless of member misalignment \((Q_h = 0 \rightarrow V_i = 0)\). Note that, while “ordinary” structures have no mechanisms and ordinary mechanisms cannot be self-stressed, we are interested in strict generality, so we must include the exceptional cases.

We also have equivalences between the opposite components of each subspace: between the equilibrium loads \(P_i\) and the extensional displacements \(U_{pl}\), and between the compatible deformations \(V_i\) and load-balancing forces \(Q_p\). These conclusions are summarized in Table 2.

A key feature of the fundamental theorem is that the extraordinary cases we have been examining correspond to a rank-deficient matrix \(B_m\). By the fundamental theorem, each degree of rank-deficiency must introduce both a new mechanism and a new state of self-stress. In fact, Calladine’s extended Maxwell rule (7) is a corollary of the fundamental theorem. Thus, the fundamental theorem explains why reciprocal diagrams in graphic statics, required to always have an extra self-stress state, must also have an extra mechanism.

**Mechanisms and states of self-stress of dual structures**

In 2D, it is possible to understand the relationship between extra self-stress states and extra mechanisms graphically.


**Table 2.** The fundamental theorem of linear algebra for truss structures.

<table>
<thead>
<tr>
<th>Statics</th>
<th>Decompositions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Global</td>
<td>$P_i = P_c + P_r$</td>
</tr>
<tr>
<td>Local</td>
<td>$Q_i = Q_c + Q_r$</td>
</tr>
<tr>
<td>Subspaces</td>
<td></td>
</tr>
<tr>
<td>Right</td>
<td>$B_c Q_c = 0$</td>
</tr>
<tr>
<td>Left</td>
<td>$P_i^T B_c = 0$</td>
</tr>
<tr>
<td>Equivalences</td>
<td></td>
</tr>
<tr>
<td>Global</td>
<td>$U_i$ dual to $P_i$</td>
</tr>
<tr>
<td>Local</td>
<td>$Q_p$ dual to $V_i$</td>
</tr>
<tr>
<td>Kinematics</td>
<td>Decompositions</td>
</tr>
<tr>
<td>Global</td>
<td>$U_i = U_p + U_b$</td>
</tr>
<tr>
<td>Local</td>
<td>$V_i = V_p + V_b$</td>
</tr>
<tr>
<td>Subspaces</td>
<td></td>
</tr>
<tr>
<td>Right</td>
<td>$A_p U_i = 0$</td>
</tr>
<tr>
<td>Left</td>
<td>$V_i^T B_c = 0$</td>
</tr>
<tr>
<td>Equivalences</td>
<td></td>
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</tr>
<tr>
<td>Local</td>
<td>$Q_p$ dual to $V_c$</td>
</tr>
</tbody>
</table>

**Figure 9.** Mechanism displacements found from ICs (left) can be rotated $90^\circ$ into irrotational displacements (right).

First, note that it is possible to rotate the mechanism displacements $U_b$ $90^\circ$ in the plane, one node at a time, to generate a set of displacements $U_p$ that cause only bar extensions without rotations (Figure 9). Like the extensional displacements $U_p$, these *irrotational displacements* are orthogonal to $U_p$, but on a node-by-node rather than a global basis.

Second, note that such a rotated set of mechanism displacements for a reciprocal diagram generates a *local rescaling* of part of the diagram about the ICs (Figure 10). Since the displacements $U^*_i$ introduce no rotations, the bars of the rescaled force diagram are still parallel to the bars of the form diagram. Therefore, each independent mechanism of the dual, rotated to form a local rescaling, corresponds to a transformation of an independent self-stress state of the form diagram. Since the figures are reciprocal, this correspondence also holds between self-stresses of the force diagram and mechanisms of the form diagram.

Finally, note that the first self-stress state of a truss is always a *global rescaling*, altering the length of all force diagram bars but leaving their relative magnitudes unaltered. After node-by-node $90^\circ$ rotation, a global rescaling corresponds to a rigid-body rotation, rather than a mechanism. It follows that

$$s = m + 1$$

(10)

Each form diagram self-stress state save the first corresponds to a dual diagram mechanism and vice versa. This correspondence between mechanisms and self-stresses appears to have been first derived by Crapo and Whiteley. Note that equation (10) also covers the boundary case where $N=-1$ and $N^*= -1$ with $m=0$ and $m^*=0$.

**The algebra of mechanisms and states of self-stress of dual structures**

For frames, the equations of Table 1 apply. But for trusses, we have no applied moments ($P_r = 0$), and $Q_r = 0$ since all joints are pinned. Nodal rotational displacements are also not meaningful, so $U_r = 0$ in equation (6). This leaves the lower right block of equation (6) as the only nonzero block we can consider for trusses.

Applying a similar logic to equation (8), we obtain a new orthogonal decomposition of the translational displacements of the dual diagram

$$U^*_i = U_p^* + U_w^* \quad \text{where} \quad A^*_n U^*_i = 0$$

$$U^*_i \cdot U^*_w = 0$$

(11)

Clearly, any transformation of the dual diagram must be given by equation (11), else bars are no longer parallel to the form diagram.

Now note that any self-stress state of the form diagram can be scaled arbitrarily without violating homogenous equilibrium, that is, $B_a(\alpha Q_r) = 0$. It follows that such a rescaling corresponds to a deformation of the reciprocal diagram bars: $V^*_i \rightarrow (\alpha - 1)Q_i$.

Combining these two observations, we have that

$$A^*_n U^*_i = V^*_i$$

where

$$A^*_n U^*_i = 0 \quad \text{and} \quad V^*_i \rightarrow (\alpha - 1)Q_i$$

(12)
Therefore, each self-stress state of the form diagram must correspond to an irrotational displacement $U^{i^*}$ of the force diagram. The algebra is thus consistent with what we have observed geometrically in section “Mechanisms and states of self-stress of dual structures.”

It is important to note a subtle point. For a typical (geometrically nonsingular) form diagram with two or more equations than unknowns, $m = 0$, so there is only one self-stress state $s^*$ of the force diagram. It follows that only one of the mechanisms of force diagram is the extra mechanism required for this self-stress state to exist. The rest are the ordinary mechanisms expected for an underbraced truss. It is this extra mechanism that has unusual geometric properties relating to Maxwell’s conjecture.

The geometry of mechanisms and states of self-stress of dual structures

The unexpected mechanism of the force diagram given by Figure 10 can be combined with some observations from projective geometry, in particular, Desargues’ Theorem: Two triangles are in perspective with respect to an axis of homology if and only if they are perspective with respect to a center of homology.

If one observes a graphical representation of Desargues’ Theorem, one can see that the triangles that are in perspective along with the lines from the center of homology (the point where the perspective lines connecting the triangles meet) form the A-K lines of a mechanism with centers of rotation at points on the axis of homology (the line where the planes containing the triangles intersect) and at the centers of homology (see Figure 11).

We thus observe that the conditions of IC alignment that allow this normally rigid figure to have an unexpected mechanism are the same as the geometric conditions that it be the projection of a plane-faced polyhedron. While this observation is specific to a Desargues configuration, the correspondence between relative centers (not absolute centers) of an extra mechanism of a truss and projective perspectives of a plane-faced 3D appears to be general. We have not been able to prove it directly, but the connection is suggestive. If proven, it may offer insight into such exotic mechanisms in mechanical engineering applications.

Proof of Maxwell’s conjecture via the Airy stress function

We have already observed that a structure may have “extra” mechanisms and self-stresses if its geometry is such that it has a rank-deficient equilibrium matrix. One naturally asks what exactly needs to be special about the structural geometry for the matrix to be rank-deficient. This condition turns out to be exactly Maxwell’s observation that a 2D truss must be a projection of a 3D plane-faced polyhedral mesh to have a self-stress state.\footnote{Several}
rigorous proofs of Maxwell’s conjecture have been presented; \(^{16}\) we will prove it on a novel mechanical basis.

**Theorem (Polyhedral Stress Function).** A 2D truss structure with arbitrary topology has a self-stress state if and only if its geometry corresponds to the orthogonal projection of a plane-faced polyhedral mesh. This mesh is a special case of the ordinary continuum Airy stress function, restricted to describe the equilibrium state of a truss. The jump in stress function derivatives perpendicular to the truss bars gives the force in the bars. Holes in the mesh induce special restrictions corresponding to the classical Cesaro integral conditions for a multiply connected surface, which can be understood as conditions of rigid-body moment equilibrium for each hole.

We prove this from the Airy stress function first and then show the converse. In this section, the Einstein summation convention is applied, unless otherwise noted.

**The Airy stress function for a truss is a plane-faced polyhedral mesh**

The Airy stress function \( \psi \) is a continuous scalar function that identically satisfies the continuum equations of equilibrium of a 2D body in the absence of body forces

\[
N_{ij} = e_{ik}e_{jl}\psi_{kl} \Rightarrow N_{ij,j} = 0 \tag{13}
\]

where \( N_{ij} = (\sigma_{ij})/t \) is the membrane stress in a plate with plane stress \( \sigma_{ij} \) and constant thickness \( t. \)\(^{21}\) Here, \( e_{12} = 1 = -e_{21}, e_{11} = 0 = e_{22} \) is the Levi-Civita symbol/alternator, which is equivalent to a 90° rotation matrix in 2D. Then, \( N_{ij} = e_{ik}e_{jl}\psi_{kl} \) vanishes due to the summation of \( \psi_{ij} = \psi_{ji} \) with \( e_{ij} = -e_{ji}. \) Equation (13) holds in the interior of the body; at the boundary, applied forces may be required.

The Airy function can be regarded as a kind of tensor potential function for a self-stress state, and in fact, it has been shown to have certain analogies to the vector potential of a magnetic field.\(^{22}\) Like any potential, the value of an integral of an Airy function is path-independent. Critically, the stress function is complete: all self-stresses can be expressed via equation (13).

We wish to show that equation (13) can describe the force equilibrium of a self-stressed truss. It is necessary to first assume that the truss is inscribed in the interior of a planar continuum body. This allows us to use standard calculus techniques to reduce the continuum Airy function to a discrete truss stress function. This continuum “background” can then be discarded; while necessary to establish the identity between the continuum and discrete stress function, it is only an ansatz. Once established, the discrete result can be used independently, as has been done historically, for example, in Maxwell,\(^{23}\) Pellegrino,\(^{24}\) or Hegedus.\(^{25}\)

Following Csonka,\(^{38}\) we express the truss forces \( P_i \) as the integrals of \( N_i \) over a section cut along a curve \( r(s) \) through the planar body from point \( r_1 \) to \( r_2 \), parameterized in terms of arc-length \( s \), where \( n \) is the normal vector to the cut (Figure 2). Then

\[
P_i = \int_{r_1}^{r_2} N_i n_j ds = \int_{r_1}^{r_2} e_{ik}e_{jl}\psi_{kl} n_j ds
\]

\[
= \int_{r_1}^{r_2} e_{ik}e_{jl} (e_{ji} n_j) ds
\]

\[
= \int_{r_1}^{r_2} e_{ik} (\tau_{ij} n_j) ds
\]

\[
= \int_{r_1}^{r_2} e_{ik} (\frac{\partial^2 \psi}{\partial x_i \partial s}) ds
\]

\[
= \int_{r_1}^{r_2} e_{ik} (\frac{\partial^2 \psi}{\partial x_i}) ds
\]

where we have used the fact that \( e_{ij} \) is equivalent to a 90° rotation to rotate the normal \( n \) to the cut into the tangent \( \tau \) to the cut. By equation (14), the forces are

\[
P_x = \int_{r_1}^{r_2} \frac{\partial^2 \psi}{\partial x} ds = -\int_{r_1}^{r_2} \frac{\partial \psi}{\partial y} \mid_{r_1} ds
\]

\[
= -\frac{\partial \psi}{\partial y} \mid_{r_1} + \frac{\partial \psi}{\partial y} \mid_{r_2}
\]

\[
P_y = \int_{r_1}^{r_2} \frac{\partial^2 \psi}{\partial y} ds = \frac{\partial \psi}{\partial x} \mid_{r_1} - \frac{\partial \psi}{\partial x} \mid_{r_2}
\]
so each force is given by the jump in the Airy function derivative in the direction perpendicular to the direction of the force.

For equation (15) to correspond to forces in a truss with straight bars, we must satisfy two sets of conditions. Without loss of generality, place the truss bar so that it is aligned with the z-axis. For a section cut r(s) lying entirely on one side of the truss bar (Figure 13, left), we must have

\[ P_x = 0 \Rightarrow \psi_y \bigg|_{r_i} - \psi_y \bigg|_{r_j} \]

\[ P_y = 0 \Rightarrow \psi_x \bigg|_{r_i} - \psi_x \bigg|_{r_j} \]  \hspace{1cm} (16)

Since the integrals (15) do not depend on the shape of the path between \( r_i \) and \( r_j \), equation (16) implies that the Airy stress surface \( \psi \) must be planar on either side of the truss bar.

For a section cut that crosses the truss bar (Figure 13, right), we must have

\[ P_y = 0, \quad P_x = Q \text{ constant} \Rightarrow Q = -\frac{\partial \psi}{\partial y} \bigg|_{r_i} + \frac{\partial \psi}{\partial y} \bigg|_{r_j} \]  \hspace{1cm} (17)

\( P_y = 0 \) is already satisfied by planarity of each face plus continuity of \( \psi \) across the bar, so equation (17) shows that the force \( Q \) in the bar is given by the jump in first derivative across the bar, in the direction orthogonal to the bar in the x-y plane.

It is easy to verify that the stress function leads to identical satisfaction of equilibrium at a node \( i \) by integrating the divergence of the stress over a region with an arbitrary boundary enclosing the node and applying Gauss's theorem.

Taken together, equations (16) and (17) prove that the Airy stress function for a truss must be composed of planar faces meeting at folds whose derivative jumps give forces in the bars. This “discrete stress function” appears to have its roots in Maxwell’s comments\(^2\) and has been explored by Klein and Weighardt\(^1\), Fraternali and Carpentieri\(^2\), Pellegrino\(^4\), and Hegedus\(^5\). To the authors’ knowledge, this is the first time it has been rigorously derived as the restriction of the continuum stress function to truss equilibrium, rather than presented as a separate but analogous construction.

It should be noted that engineering approaches to the plastic design of concrete structures (e.g. the strut and tie method) have also relied on a link between continuum and discrete forces. While this connection has been fairly rigorously developed using the lower bound theorem of plastic design\(^6\), the Airy function appears not to have been used. Our claim is only to have made the connection of the continuum and discrete Airy functions explicit. While the connection between stress fields and forces may be intuitively obvious, it is nonetheless instructive to rigorously step from one to the other, since it makes it clear that the discrete Airy function is not some separate ad hoc construction, but a simple special case of the continuum function.

\[ \text{A self-stressed truss has an associated polyhedral mesh} \]

It is not strictly necessary to prove the converse of the above argument. We know the stress function is complete, so all truss self-stress states can be described with it; nothing is left out. Since we have simply restricted the complete continuum function to describe truss equilibrium, we already know that we have a complete description of truss equilibrium.

Nonetheless, proving the converse can help one understand the connection between polyhedra and truss equilibrium more clearly. We briefly summarize the argument, and refer the reader to Tachi\(^4\), Borcea and Streinu\(^1\), or Klein and Weighardt\(^1\) for details.

For a collection of planes \( z=\alpha x+\beta y+\gamma \) to form a continuous mesh (i.e. be compatible), we require

\[ \alpha_i x_i + \beta_i y_i + \gamma = \alpha_j x_j + \beta_j y_j + \gamma \]

\[ \alpha_i x_i + \beta_i y_i + \gamma = \alpha_j x_j + \beta_j y_j + \gamma \]  \hspace{1cm} (18)

for any edge with nodes \( i,j \) shared by adjacent faces \( e,f \). This is trivial only for three planes; for more than three, nontrivial compatibility conditions must be met for all the planes to intersect in one point. Subtracting the two equations (18) and moving terms to one side, we obtain

\[ (\alpha - \alpha_f) (x_i - x_j) + (\beta - \beta_f) (y_i - y_j) = 0 \]  \hspace{1cm} (19)

which can be satisfied via

\[ \begin{bmatrix} \alpha - \alpha_f \\ \beta - \beta_f \end{bmatrix} = q_{ij} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_i - x_j \\ y_i - y_j \end{bmatrix} \]  \hspace{1cm} (20)

**Figure 13.** To obtain truss equilibrium, we must have zero force along a section cut on either side of the bar (left). We must also have a constant force in the bar direction for a section cut crossing the bar (right).
with \( q_\theta \) an arbitrary scalar and \( R_{90} \) a 90° rotation matrix (identical to \( e_{ij} \)).

For a simply connected mesh, it is sufficient that each cycle of planes about a node be compatible. Recovering the \( \gamma \)-values of the planes from equations (20) and (18) leads to a sum that, after setting \( q_\theta = Q_{ij} / L_{ij} \), can be identified as the homogeneous equilibrium equations \( B_{ij}Q_{ij} = 0 \) for a node. The additional compatibility conditions for non-simply connected meshes reduce to rigid-body moment equilibrium of each hole in the mesh. Therefore, given a set of forces \( B_{ij}Q_{ij} = 0 \), the resulting polyhedron is unique up to the choice of one initial plane \( (\alpha, \beta, \gamma) \) required to remove rigid motions.

Given our connection to the Airy function, we can also note that the moment equilibrium conditions are analogous to the Cesaro integral conditions on the continuum Airy function for non-simply connected bodies. Finally, note that loads on the boundary can be exchanged for an extended funicular polygon structure (Figures 2 and 3), extending the description to structures loaded on the outer boundary and completing the connection to the stress function. This completes the proof.

**Reciprocal diagrams and Airy functions**

We have shown that Maxwell’s plane-faced polyhedral are Airy stress functions for 2D truss structures, and that the force diagram of a truss can be regarded as a structure with its own mechanisms and states of self-stress. Therefore, the reciprocal diagram must have its own Airy function describing its own self-stress (Figure 14).

Examining equation (18), we can see that the roles of form diagram coordinates \( x_i, y_i \) and Airy function slopes \( \alpha_i, \beta_i \) are completely interchangeable. Minding the 90° rotation in equation (20), we have that \( (x'_i, y'_i) = (-\beta_i, \alpha_i) \);
the slopes of the Airy function planes of the form diagram give the coordinates of the force diagram. The resulting force diagram bars are parallel to the form diagram bars, as required.

Since the relation is reciprocal, we also could take the force diagram as the structure, in which case $(x_i, y_i) = (\alpha_i^*, \gamma_i^*)$, with $z_i^* = \alpha_i^* x_i^* + \beta_i^* y_i^* + \gamma_i^*$ the reciprocal Airy function. These two Airy functions thus form a pair of reciprocal polyhedra. The geometrical description of one of the reciprocal polyhedra (Airy stress function) has all the information needed to create the other reciprocal polyhedron and both 2D diagrams.

A numerical example

Figure 15 has been adapted from Maxwell. The projection of a polyhedron with eight faces (octahedron). For this diagram, $N = 6 - 8 - 1 = -3$, and the diagram has three states of self-stress and no mechanisms (the crossing lines are not considered to be connected if there is not a node). The first self-stress state allows for the existence of the reciprocal diagram. The subsequent two states of self-stress correspond to mechanisms in the dual truss, as per equation (10).

Figure 5(d) is a possible reciprocal diagram for Figure 5(a). It has 8 nodes and 12 bars, and it can be shown that it is the projection of a polyhedron with six quadrilateral faces (a distorted cube) that is reciprocal to the octahedron of Figure 5(a). Figure 5(d) has $N^* = 8 - 6 - 1 = 1$. Figure 5(d), considered as a truss, has two mechanisms ($m^* = 2$) and one state of self-stress ($s^* = 1$).

Figure 5(d) will always have at least one mechanism, but since Figure 5(d) is reciprocal to Figure 5(a), it is the

Figure 15. Reciprocal figures with $N = -3$, $m = 0$, $s = 3$ and $N^* = 1$, $m^* = 2$, and $s^* = 1$: (a) projection of original polyhedron with nodes labeled, (b) top of original polyhedron with nodes and faces labeled, (c) bottom of original polyhedron with nodes and faces labeled, and (d) projection of reciprocal polyhedron with nodes, faces, and relative instant center coordinates labeled.
projection of a polyhedron and has a state of self-stress along with an additional mechanism \((m^* = 2, s^* = 1)\). As stated in section “The geometry of mechanisms and states of self-stress of dual structures,” it appears that the relative centers of rotation of this extra mechanism correspond to the centers of homology of the Airy function face planes; although the authors have not proven this in general, it has held for every example we have examined.

We can examine the ordinary and extra mechanism numerically using the kinematic matrix \(A\). We form this matrix and compute its nullspace in MATLAB. The existence of the extra mechanism is very sensitive to the position of the coordinates, vanishing if they are perturbed by even a few digits of precision, so by perturbing the coordinates, we can isolate the ordinary from the extra mechanism. The displaced shapes are plotted in Figure 16.

While the ordinary mechanism behaves as one would expect, with all bars moving save for the fixed top bar, the extra mechanism has the peculiar property that several bars besides the fixed bar fail to move during the mechanism motion. The alignment of the ICs along the lines of homology \(X, Y, Z\) of Figure 15d appears to lock these bars out of the motion.

Table 3 includes the coordinates for the vertices for the reciprocal polyhedra and the corresponding Airy stress functions.

### Applications

We show several applications of these ideas.

#### Design of 2D tensegrity structures

One application of these geometrical relationships is for the design of 2D tensegrity structures. One can create an arbitrary plane-faced polyhedron, and any projection of that polyhedron represents a potential self-stressed tensegrity structure (Figure 17). The truss on the right in Figure 17 is similar to one discussed in Pellegrino.²

By beginning with the Airy polyhedron, we can generate a whole family of 2D tensegrities through 3D rotations and/or projective transformations of the polyhedron, followed by projection to the plane.

#### Planar panelization of architectural surfaces

Consider the polyhedron in Figure 18(a) and (b). Each face of the polyhedron is a plane that can be described as \(z = \alpha x + \beta y + \gamma\), where \(z\) is the value of the Airy stress function. The vertices of the reciprocal polyhedra are \((\beta, \alpha, \gamma)\). Conversely, the intersections of the reciprocal stress functions’ planes define the coordinates of the vertices of the original form diagram Airy polyhedron, as noted in section “Reciprocal diagrams and airy functions.” It should be noted that all the “ridges” of the original polyhedron map into the force diagram with the same sign (tension or compression) and all the “valleys” map into the opposite sign. The coordinate data of this example are given in Table 4.

The diagrams in Figure 8(a) and (b) represent a truss with a large number of members connected at some of the nodes. Since the form diagram is triangulated, it is composed of a variety of irregular quadrangles, pentagons, hexagons, and an octagon. Therefore, the reciprocal polyhedron (the Airy function of the reciprocal) has planar faces with a wide variety of polygon types.

This leads to a design method for a problem that is common in architecture: how to create a domed surface such as a roof with planar faces. First, one creates a triangulated, self-stressed cable net for our original form diagram. The number of bars attached to each node should match the desired number of polygon sides for each face of the dome.
per equation (1). From the axial forces, a reciprocal diagram can be created with the desired faces. The nodal coordinates of the reciprocal define the slopes of a polyhedron that can then be scaled to create a plane-faced architectural surface where the projection of the edges aligns with the reciprocal diagram bars. This method of planarization was used in the design of the topologically complex grid shell roof of the Amsterdam Maritime Museum by Chris Williams.

**Offsets as design and optimization degrees-of-freedom**

In the previous example, we could have adjusted the forces in the statically indeterminate cable net to alter the reciprocal. We have already shown that an alteration of the reciprocal corresponds to an irrotational motion $U_i$ of the reciprocal nodes. We can also connect these irrotational displacements to the linear algebra of frames.
The displacements $U_i$ are the pure-extensional or irrotational displacements, since they do not produce any rotations of the bars. Geometrically, these correspond to consistent offsets of the mesh edges: modulo block-wise division by $L_{ij}$, $A_{rt}U_i = 0$ from equations (11) and (6) is identical to the offset equation in Pottmann et al.\textsuperscript{28} Per section “Mechanisms and states of self-stress of dual structures,” any consistent offset of the dual transforms an existing force diagram into a new self-stress of a fixed form diagram, and any consistent offset of the form diagram retains the same self-stress forces in a fixed dual diagram. The irrotational displacements/offsets therefore generate a formal geometry of self-stresses and their transformations, in Klein’s sense. The irrotational displacements of the force diagram are thus exactly the design degrees-of-freedom of the structure.

This insight removes the focus on locating design degrees-of-freedom in local “free edges” in Van Mele and Block:\textsuperscript{29} the most general design degrees-of-freedom are global consistent offsets. These offsets can be computed generally and automatically from the nullspace of the infinitesimal rotation matrix $A_{rt}$, for example, via the singular value decomposition. Once computed, a basis of this nullspace can be used to drive interactive computational design of truss structures. Although not proven, it seems likely that “free edges” are one particular basis of the general nullspace of $A_{rt}$ and its associated consistent offsets. The space itself is fully general; any basis that spans it can be chosen, although an orthonormal basis is typically preferred in computational applications.

Consistent offsets also provide a means of extending our dualities to 3D. In section “Mechanisms and states of self-stress of dual structures,” we relied on a 90° rotation to connect the mechanism displacements of the dual to its offsets/irrotational displacements. This identity breaks down in 3D, since there is no longer any canonical out-of-plane axis to rotate about. But equation (23) is unchanged in 3D, so we can always find $U_i$ numerically.

One should note that the extension to 3D we are discussing is of the parallel-bar reciprocal type, as explored in Tachi,\textsuperscript{4} Sauer,\textsuperscript{30} or Wallner and Pottmann.\textsuperscript{31} In this type, dubbed “Cremona” reciprocals in Baracs et al.,\textsuperscript{18} a mesh of bars in space forms a force diagram reciprocal to a form diagram mesh where each bar is in the reciprocal is parallel to a bar in the form diagram. This is fundamentally different from the Rankine-type 3D polyhedral reciprocal\textsuperscript{15} discussed in Akbarzadeh et al.\textsuperscript{32} The Cremona-type 3D reciprocal allows easy exploration of self-stresses (or form changes) via offsets of the force diagram (or form diagram), since each such offset keeps the bars parallel and thus preserves the reciprocal duality of the two diagrams. Since the linear algebra of equilibrium and mechanism motion is unchanged in 3D, all the dualities of the fundamental theorem of linear algebra apply to these 3D reciprocals exactly as for 2D reciprocals. A full description of this type of 3D truss reciprocal in terms of a corresponding 3D discrete Airy function remains an interesting open problem.

Finally, we note that, in 2D, each edge offset of the form diagram induces a face offset of the corresponding Airy stress function (“parallel redrawing” in Crapo and Whiteley). We know that every 2D self-stress corresponds to an Airy function, so the transformations of reciprocal diagrams are in one-to-one correspondence with the transformations of polyhedral Airy functions.

### Table 4. Coordinate data for Figure 10.

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**Conclusion**

We have shown that

1. Any self-stressed 2D truss must be the projection of a plane-faced polyhedron;
2. This polyhedron is exactly the continuum Airy stress function specialized to describe truss equilibrium;
3. The slopes of the Airy function planes correspond to the coordinates of the reciprocal diagram nodes;
4. The reciprocal diagram of a truss has its own self-stress state and thus its own reciprocal Airy function;
5. Each self-stress state of a truss saves the first corresponds to a mechanism in its reciprocal and vice versa;
6. Outside the \( N=-1, N^*=−1 \) boundary case, one mechanism of the reciprocal appears to be an extra mechanism with relative ICs lying on the lines of homology of the projected Airy function planes;

Of these results, only “2” is original to this article; the rest are collected from results scattered across disparate literature and assembled into a single map, with the stress function at its center. This map of all the relevant dualities for truss structures helps orient oneself theoretically, and leads directly to a general method for interactive graphic statics as well as a practical method for designing plane-faced architectural surfaces.

Note that the topology of the truss is completely irrelevant in this presentation. Faces of the truss can be any polygon whatsoever; they are not restricted to triangles as in Whiteley,\textsuperscript{33} Fraternali and Carpentieri,\textsuperscript{26} De Goes et al.,\textsuperscript{34} Pellegrino,\textsuperscript{24} or Hegedus.\textsuperscript{25} We can have any degree in Whiteley,\textsuperscript{33} Fraternali and Carpentieri,\textsuperscript{26} De Goes polygon whatsoever; they are not restricted to triangles as in this presentation. Faces of the truss can be any faced architectural surfaces.

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