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STRUCTURAL OPTIMIZATION: A NEW DUAL METHOD USING MIXED VARIABLES

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SUMMARY

A new and powerful mathematical programming method is described, which is capable of solving a broad class of structural optimization problems. The method employs mixed direct/reciprocal design variables in order to get conservative, first-order approximations to the objective function and to the constraints. By this approach the primary optimization problem is replaced with a sequence of explicit subproblems. Each subproblem being convex and separable, it can be efficiently solved by using a dual formulation. An attractive feature of the new method lies in its inherent tendency to generate a sequence of steadily improving feasible designs. Examples of application to real-life aerospace structures are offered to demonstrate the power and generality of the approach presented.

1. BACKGROUND

It is now widely recognized that many optimal sizing problems can be accurately approximated by a mathematical programming problem having a simple algebraic structure: linear objective function and separable constraints.[1,2] This explicit subproblem is generated by linearizing the behaviour constraints with respect to the reciprocals of the design variables. On the other hand, it is often useful, for fabricational reasons, to link the design variables through linear inequality constraints. Therefore at each stage of the iterative optimization process, the approximate subproblem to be dealt with exhibits the following explicit form:

minimize
$$W(x) = \sum_{i=1}^{n} w_i x_i \tag{1}$$

subject to
$$c_j(x) \equiv \sum_{i=1}^{n} \frac{c_{ij}}{x_i} \leqslant \bar{c}_j \tag{2}$$

$$d_j(x) \equiv \sum_{i=1}^{n} d_{ij} x_i \leqslant \bar{d}_j \tag{3}$$

$$\underline{x}_i \leqslant x_i \leqslant \bar{x}_i \tag{4}$$

In these expressions the x_i's denote the design variables, which correspond to the transverse sizes of the structural members (bar cross-sectional areas, membrane thicknesses). The structural weight (1) is a linear objective function, because the weight coefficients w_i are prescribed parameters related to the material mass density and to geometrical quantities (bar lengths, membrane areas).

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The inequalities (2) represent high-quality explicit approximations of the behaviour constraints that impose limitations on the stresses and the displacements under static loading cases. The linear constraints (3) permit taking into consideration some technological limitations on the design variables, such as a linear progressivity rule for the thicknesses of contiguous structural members (e.g. increase in the number of layers in a laminate). Finally, the design variables are also bounded by the side constraints (4), where $\underline{x}_i$ and $\bar{x}_i$ are positive lower and upper limits that reflect fabricational or analysis validity considerations. It should be noted that these side constraints constitute a particular case of the linear constraints (3). However, they are written separately in our explicit problem statement, because the dual method approach described later can handle them more efficiently when considered apart from the general constraints (2) and (3).

The approximate constraints (2) can be obtained by virtual work considerations, in which case the c_{ij} coefficients can be interpreted as virtual strain energy densities in the structural members.[1] The c_{ij}'s can also be identified as the first derivatives of the response quantities c_j with respect to the reciprocal of the design variables:

$$c_{ij} = \frac{\partial c_j}{\partial\left(\dfrac{1}{x_i}\right)} \tag{5}$$

The approximate constraints (2) correspond therefore to first-order Taylor series expansions in terms of the intermediate variables $1/x_i$:[1]

$$c_j(x) = c_j^o + \sum_{i=1}^{n} c_{ij}^o\left(\frac{1}{x_i} - \frac{1}{x_i^o}\right) \leqslant \bar{c}_j \tag{6}$$

(the symbol ^{o} denotes quantities evaluated at the current design point x^o where the structural analysis is made). At each stage of the optimization process, the c_{ij}'s are considered as constant coefficients, and so, inequalities (2) represent explicit approximate forms of the true behaviour constraints. Note that in the case of a statically determinate structure, the c_{ij}'s are really constant coefficients, so that a single structural analysis is sufficient to get the optimal design.

Until recently we have been faced with structural optimization problems that did not have to take into account the linear constraints (3). In this case, when adopting the reciprocal variables $1/x_i$, the explicit problem amounts to minimizing a nonlinear objective function subject to linear constraints. Such a problem can be efficiently solved by using a gradient projection algorithm (primal approach). A second possibility is to take advantage of the objective function and constraints being separable and to resort to a dual approach. The dual problem consists in maximizing an auxiliary function that depends only on the Lagrangian multipliers associated with the main primal constraints (2). These Lagrangian multipliers—also called dual variables—have to remain non-negative.[1-3]

When introducing the linear constraints (3), the primal problem (1–4) can no longer be directly solved by using a primal projection method. Formulated either in terms of the direct variables x_i or in terms of the reciprocal variables $1/x_i$, it always involves nonlinear constraints. A possible strategy would be to formulate the problem in terms of the reciprocal variables and to linearize the explicit constraints (3) just as the behaviour constraints (2). The resulting approximate problem involves then linear constraints only, and it can be solved by using the same primal projection algorithm as before. However, this strategy suffers from a major drawback: the explicit constraints (3) can no longer be exactly satisfied so that the feasible subdomain might become artificially empty because of incompatible constraints. An alternative strategy is to linearize the behaviour constraints (2) with respect to the direct variables. The resulting explicit subproblem becomes then a linear programming problem, and it can be solved by the well-known SIMPLEX method. However such a conventional linearization technique converges only if the optimum of the primary problem lies at a vertex of the design space.

The research work reported in Reference 4 was devoted to dealing with the linear constraints (3) in an exact manner. To this end the separable nature of the explicit problem (1–4) was exploited by resorting to a specially devised dual method approach. However, it is shown in Reference 4 that, the primal problem (1–4) being non-convex, the dual problem happens to be non-differentiable and therefore difficult to solve by conventional maximization algorithms. One important conclusion of Reference 4 was that further research was needed to convexify the explicit constraints (3), for example by linearizing them partially with respect to reciprocal variables:

$$\tilde{d}_j(x) \equiv 2\sum_{(2)} d_{ij} x_i^o + \sum_{(1)} d_{ij} x_i - \sum_{(2)} \frac{d_{ij}(x_i^o)^2}{x_i} \tag{7}$$

where $\sum_{(2)}(\sum_{(1)})$ means 'summation over the variables for which reciprocal linearization is (not) used'. However, no criterion was suggested to select groups (1) and (2).

At this stage the concept of hybrid approximation proposed in Reference 5, and later on exploited in References 6–7, was rediscovered with the spirit of achieving convexity. The original idea set forth in Reference was to get conservative approximation to the behaviour constraints (specially buckling constraints) by resorting to mixed direct/reciprocal variables in the linearization process. Instead of employing the conventional first-order Taylor series expansions (6) in terms of direct or reciprocal variables, the following hybrid approximation was suggested:

$$\tilde{c}_j(x) \equiv c_j^o + \sum_{i=1}^{n} b_i c_{ij}^o \tag{8}$$

where

$$b_i = \begin{cases} \dfrac{1}{x_i} - \dfrac{1}{x_i^o} & \text{if } c_{ij}^o > 0 \\[2ex] -\dfrac{1}{(x_i^o)^2}(x_i - x_i^o) & \text{if } c_{ij}^o < 0 \end{cases} \tag{9}$$

According to Reference 5, 'with this hybrid approximation, a given constraint may have a linear approximation with respect to one design variable and an inverse approximation with respect to another design variable'. It was shown that this approximation 'always uses the more conservative of the two approximations considered' (direct and reciprocal).

As previously mentioned, the aim in Reference 5 was conservativeness. However, it is remarkable that the conservative explicit approximation (8, 9) is also convex and separable. This constitutes the basis for the convex linearization method proposed in this paper. After giving the mathematical backgrounds in Section 2, the method is described in Section 3. The key idea is to employ the hybrid approximation (8, 9) not only for the behaviour constraints, but also for the objective function and for any other constraint function [e.g. linear constraints (3)], so that the resulting explicit subproblem is convex and separable. Therefore, the dual solution scheme developed in Section 4 is particularly efficient. Section 5 introduces a simple modification that can be brought to the basic method in order to make it capable of dealing with infeasible starting points. In Section 6, the new approach is applied to optimal sizing problems. Because of its generality, it is shown to further reconcilate mathematical programming and optimality criteria approaches to structural weight minimization.[8] Finally, Section 7 is concerned with various numerical applications, including real-life aerospace structures. Applications to shape optimal design problems are offered in a companion paper.[9] From all the examples treated up to now, it can be concluded that the convex linearization method usually converges within ten finite element analyses.

2. MIXED VARIABLES

Considering a differentiable function $g(x)$, *direct linearization* consists of replacing it with the first-order Taylor series expansion

$$\tilde{g}_D(x) = g(x^o) + \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i} \right)_{x^o} (x_i - x_i^o) \tag{10}$$

By this technique the function $g(x)$ is approximated as a linear function of the variables. In structural optimization another form of approximation that is often used is a linear function of the reciprocals of the design variables, which can be named *reciprocal linearization*:

$$\tilde{g}_R(x) = g(x^o) + \sum_{i} \left(\frac{\partial g}{\partial x_i} \right)_{x^o} \frac{x_i^o}{x_i} (x_i - x_i^o) \tag{11}$$

Let us now assume that for the linearization purpose the variables are arbitrarily split into two groups: group (1) continues to contain the original direct design variables, while group (2) is concerned with intermediate reciprocal variables. Performing then a first-order Taylor series expansion of the function $g(x)$, the following *mixed linearization* is obtained:

$$\tilde{g}_M(x) = g(x^o) + \sum_{(1)} \left(\frac{\partial g}{\partial x_i} \right)_{x^o} (x_i - x_i^o) + \sum_{(2)} \left(\frac{\partial g}{\partial x_i} \right)_{x^o} \frac{x_i^o}{x_i} (x_i - x_i^o) \tag{12}$$

where $\sum_{(1)} [\sum_{(2)}]$ means 'summation over the variables belonging to group (1) [(2)]'.

It is important to recognize that reciprocal linearization (11) yields a convex approximation only if all the first derivatives $(\partial g/\partial x_i)_{x^o}$ are non-negative. This feature cannot be controlled: it is an entry in the linearization process. On the other hand, mixed linearization is always capable of generating a convex approximation provided that groups (1) and (2) are appropriately chosen. Hence the idea of *convex linearization*, which is achieved when group (1) is selected as containing the variables for which $(\partial g/\partial x_i)_{x^o}$ is positive, and group (2) contains the remaining variables:

$$\tilde{g}_c(x) = g(x^o) + \sum_{+} \left(\frac{\partial g}{\partial x_i} \right)_{x^o} (x_i - x_i^o) + \sum_{-} \left(\frac{\partial g}{\partial x_i} \right)_{x^o} \frac{x_i^o}{x_i} (x_i - x_i^o) \tag{13}$$

Note that this apparently tricky linearization scheme takes advantage of the trivial fact that

$$\frac{\partial g}{\partial \left(\dfrac{1}{x_i} \right)} = -x_i^2 \frac{\partial g}{\partial x_i}$$

An attractive property of convex linearization is that it also yields the most conservative approximation among all the possible combinations of direct/reciprocal variables [i.e. selection of group (1) and group (2)]. This remarkable property is easy to prove by substracting (12) from (13) to get

$$g_c(x) - g_M(x) = \sum_{(2)+} \left(\frac{\partial g}{\partial x_i} \right)_{x^o} \frac{1}{x_i} (x_i - x_i^o)^2 - \sum_{(1)-} \left(\frac{\partial g}{\partial x_i} \right)_{x^o} \frac{1}{x_i} (x_i - x_i^o)^2 \tag{14}$$

By requiring that the x_i's are non-negative variables—a simple translation can do it—, the first

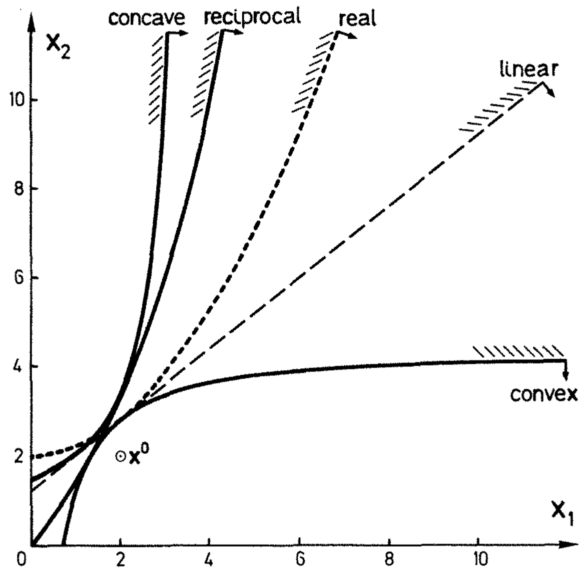

Figure 1. Linearization in mixed variables

summation in (14) will contain only positive terms from group (2), and the second, negative terms from group (1). Therefore it can be concluded that $g_c(x)$ is always greater than $g_M(x)$. In other words, convex linearization is the most conservative approximation of any mixed linearization, including the two extreme cases: direct linearization (10) and reciprocal linearization (11).

As an illustrative example, let us approximate the following constraint function at the current point $x^o = (2, 2)^T$:

$$g(x) \equiv 5x_2 - x_1^2 \leqslant 10$$

There exist four possible combinations of mixed variables, that lead to the approximate constraints:

linear:	$5x_2 - 4x_1 \leqslant 6$
reciprocal:	$-20/x_2 + 16/x_1 \leqslant 2$
concave:	$-20/x_2 - 4x_1 \leqslant -14$
convex:	$5x_2 + 16/x_1 \leqslant 22$

From Figure 1, where these four constraint surfaces are plotted, it can be intuitively verified that convex linearization yields the most conservative approximation.

In summary then, when one wants a function $g(x)$ to be approximated in a conservative way by using mixed direct/reciprocal variables, the *only* possible scheme is to employ convex linearization. The word 'only' is important, because it is the basis for the generality of the method proposed in the sequel. Indeed, this method intrinsically contains a rational scheme to select by itself the mixed variables.

3. CONVEX LINEARIZATION METHOD

Considering a general mathematical programming problem

$$\text{minimize} \qquad f(x) \tag{15}$$

subject to
$$h_j(x) \geqslant 0 \tag{16}$$
$$\bar{x}_i \geqslant x_i \geqslant \underline{x}_i \tag{17}$$

the new approach presented in this paper proceeds by transforming it into a sequence of 'linearized' subproblems having a simple explicit algebraic structure. Because the method employs the convex linearization scheme described in Section 2, it is very general and easy to use: the algorithm inherently chooses itself the intermediate linearization variables. Therefore, the only input data are the initial values of the objective function and constraint functions:

$$f^o = f(x^o) \quad \text{and} \quad h_j^o = h_j(x^o) \tag{18}$$

as well as their first derivatives:

$$f_i = \left.\frac{\partial f}{\partial x_i}\right|_{x^o} \quad \text{and} \quad h_{ij} = \left.\frac{\partial h_j}{\partial x_i}\right|_{x^o} \tag{19}$$

where x^o denotes the current point, i.e. the design point where the problem is linearized.

Conventional linearization methods also benefit from these attractive properties of generality and simplicity, and it is probably the reason why they have met with considerable success in engineering design (see, for example, Reference 10). However, because such a technique replaces the primary problem with a sequence of linear programming problems, it suffers from severe limitations. It does not converge to a local minimum unless the latter occurs at a vertex of the feasible domain. Otherwise, the optimization process either converges to a non-optimal vertex or it oscillates indefinitely between two or more vertices. One way of avoiding this undesirable behaviour is to add artificial side constraints (called 'move limits') to the linear subproblem statement. These move limits must then be gradually tightened at each stage of the process by using some properly chosen update formula (see Figure 2).

Because it introduces some convex curvature in the approximate functions, the approach proposed herein does not require any control parameters such as move limits. The key idea of the method is to perform the linearization process with respect to mixed variables, either direct (x_i) or reciprocal ($1/x_i$), independently for each function involved in the problem, so that a convex and separable subproblem is generated. Separability is automatically obtained because first-order Taylor series expansions are employed, while convexity is achieved by using *ad hoc* criteria to select the mixed linearization variables (see equation 13):

objective function $f(x)$

$$\begin{array}{lll} x_i & \text{if} & f_i > 0 \\ 1/x_i & \text{if} & f_i < 0 \end{array}$$

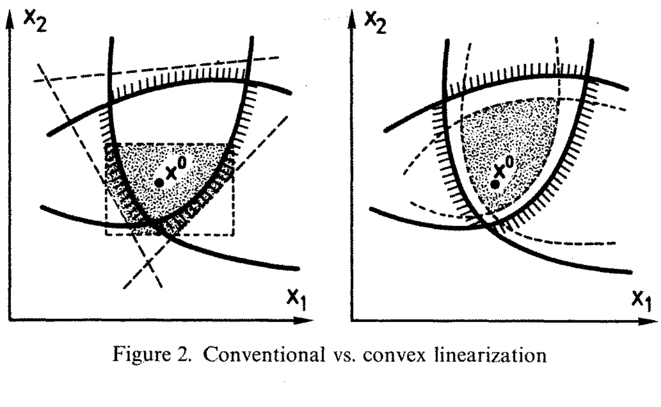

Figure 2. Conventional vs. convex linearization

constraints $h_j(x)$

$$\begin{array}{lll} x_i & \text{if} & h_{ij} < 0 \\ 1/x_i & \text{if} & h_{ij} > 0 \end{array}$$

Adopting these simple rules and normalizing the variables x_i so that they become equal to unity at the current point x^o, the following *convex*, *separable* subproblem is generated:

$$\min \quad \sum_{+} f_i x_i - \sum_{-} \frac{f_i}{x_i} \tag{20}$$

$$\text{s.t.} \quad \sum_{+} \frac{h_{ij}}{x_i} - \sum_{-} h_{ij} x_i \leqslant \bar{h}_j \tag{21}$$

$$\underline{x}_i \leqslant x_i \leqslant \bar{x}_i \tag{22}$$

where

$$\bar{h}_j = h_j^o + \sum_{+} h_{ij} - \sum_{-} h_{ij}$$

In these expressions the symbol $\sum_{+}(\sum_{-})$ means 'summation over all positive (negative) terms'. It is important to notice that, even if the main variables in the primary problem statement had been chosen as the reciprocal variables, nothing would be changed in the explicit subproblem statement.

As shown in Section 2, the first-order explicit approximations of the objective function (i.e. equation 20) and of the constraint functions (i.e. equation 21), because they result from convex linearization, are locally conservative. This means that they tend to overestimate the values of the true functions. In other words, the 'linearized' feasible domain corresponding to the explicit subproblem (20–22) is generally inside the true feasible domain corresponding to the primary problem (15–17). This property is illustrated in Figure 2. As a result, the convex linearization method has a tendency to generate a sequence of design points that 'funnel down the middle' of the feasible region. The primal philosophy, i.e. sequence of steadily improved feasible designs, is maintained.[1,3,8] This represents an attractive feature from an engineering point of view, since the designer may stop the optimization process at any stage, and still get an acceptable non-critical design, better than its initial estimate.

It is quite fascinating to realize that the convex linearization method is inherently very general. It is even capable of solving rather efficiently a linear programming problem. For all the linear problems that have been employed to validate the approach—only to validate, competition with SIMPLEX is not expected!—, as well as for many other nonlinear problems, order two convergence was surprisingly observed. As an illustration, let us consider the following linear programming problem:

$$\begin{array}{ll} \min & x_1 + 4x_2 \\ \text{s.t.} & x_2 - x_1 \geqslant 0 \\ & 3x_1 - 2x_2 \geqslant 1 \end{array}$$

As shown in Figure 3, the optimum point $x^* = (1, 1)^T$ lies at a very 'sharp angle' at the intersection of the two constraints. Nevertheless, starting from $x^o = (3, 4)^T$, the method exhibits order two convergence, furnishing the following sequence of design points: $(3, 4)^T$; $(2{\cdot}390, 2{\cdot}852)^T$; $(1{\cdot}888, 2.132)^T$; $(1{\cdot}526, 1{\cdot}660)^T$; $(1{\cdot}281, 1{\cdot}352)^T$; $(1{\cdot}127, 1{\cdot}159)^T$; $(1{\cdot}042, 1{\cdot}053)^T$; $(1{\cdot}007, 1{\cdot}009)^T$; $(1{\cdot}000, 1{\cdot}1000)^T$. Figure 3 shows the trajectory of these successive iteration points. It also represents

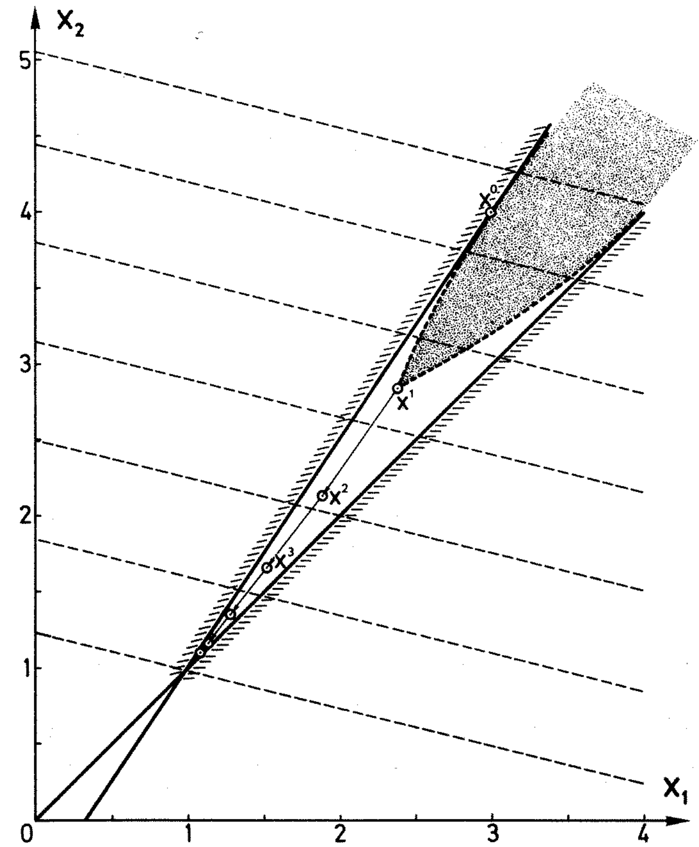

Figure 3. Application to linear programming

the first explicit subproblem:

$$\begin{aligned} \min \quad & x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + \frac{16}{x_2} \leqslant 8 \\ & \frac{27}{x_1} + 2x_2 \leqslant 17 \end{aligned}$$

In summary then, the explicit subproblem (20–22) exhibits the following remarkable properties: (a) high-quality, first-order approximation; (b) conservative feasible subdomain; (c) convexity; (d) separability.

4. DUAL SOLUTION SCHEME

Because of its properties of convexity and separability, the explicit problem (20–22) can be efficiently solved by dual methods of mathematical programming (see, for example, References 2 and 11 and Section 7.2 of Reference 12). Because the side constraints can be treated separately, the dual variables are restricted to the Lagrangian multipliers associated with the approximate behaviour constraints (21). Therefore, the Lagrangian function can be written:

$$L(x,\lambda)=\sum_{+} f_i x_i - \sum_{-} \frac{f_i}{x_i} + \sum_j \lambda_j \left(\sum_{+} \frac{h_{ij}}{x_i} - \sum_{-} h_{ij} x_i - \bar{h}_j \right) \tag{23}$$

Corresponding to the primal minimization problem (20–22) the dual maximization problem exhibits the following form:

$$\begin{aligned} \max \quad & l(\lambda)=\sum_{+} f_i x_i(\lambda) - \sum_{-} \frac{f_i}{x_i(\lambda)} + \sum_j \lambda_j \left[\sum_{+} \frac{h_{ij}}{x_i(\lambda)} - \sum_{-} h_{ij} x_i(\lambda) - \bar{h}_j \right] \\ \text{s.t.} \quad & \lambda_j \geqslant 0 \end{aligned} \tag{24}$$

where $x(\lambda)$ denotes the primal point solution of the auxiliary minimization problem (for given λ):

$$\begin{aligned} \min \quad & L(x,\lambda) \\ \text{s.t.} \quad & \underline{x}_i \leqslant x_i \leqslant \bar{x}_i \end{aligned}$$

Because the Lagrangian function is separable, this n-variable problem can be decomposed in n single variable problems:

$$\min_{\underline{x}_i \leqslant x_i \leqslant \bar{x}_i} L_i(x_i,\lambda) \tag{25}$$

The explicit statement of this minimization problem depends upon the sign of f_i:

$$\underline{f_i > 0}: L_i(x_i,\lambda) = f_i x_i + \frac{p_i}{x_i} + q_i x_i$$

$$\underline{f_i < 0}: L_i(x_i,\lambda) = -\frac{f_i}{x_i} + \frac{p_i}{x_i} + q_i x_i$$

where p_i and q_i are positive constants for given feasible λ_j:

$$p_i = \sum_{+} h_{ij}\lambda_j$$

$$q_i = -\sum_{-} h_{ij}\lambda_j$$

It turns out that each one-dimensional minimization problem (25) can be solved in closed form, yielding explicitly the primal variables x_i in terms of the dual variables λ_j:

$$\underline{f_i > 0}: x_i = \left(\frac{p_i}{f_i+q_i}\right)^{1/2} \quad \text{if} \quad \underline{x}_i^2 < \frac{p_i}{f_i+q_i} < \bar{x}_i^2 \tag{26}$$

$$x_i = \underline{x}_i \quad \text{if} \quad p_i \leqslant (f_i+q_i)\underline{x}_i^2 \tag{26a}$$

$$x_i = \bar{x}_i \quad \text{if} \quad p_i \geqslant (f_i+q_i)\bar{x}_i^2 \tag{26b}$$

$$\underline{f_i < 0}: x_i = \left(\frac{p_i - f_i}{q_i}\right)^{1/2} \quad \text{if} \quad \underline{x}_i^2 < \frac{p_i - f_i}{q_i} < \bar{x}_i^2 \tag{27}$$

$$x_i = \underline{x}_i \quad \text{if} \quad p_i \leqslant f_i + q_i \underline{x}_i^2 \tag{27a}$$

$$x_i = \bar{x}_i \quad \text{if} \quad p_i \geqslant f_i + q_i \bar{x}_i^2 \tag{27b}$$

For the sake of completeness, it is worth noticing that in the special case where $f_i = 0$, both formulae (26) or (27) can be used. The two following particular cases should also be mentioned:

$$x_i = \underline{x}_i \quad \text{if} \quad p_i = 0 \quad \text{and} \quad f_i \geqslant 0$$
$$x_i = \bar{x}_i \quad \text{if} \quad q_i = 0 \quad \text{and} \quad f_i \leqslant 0$$

Knowing $x(\lambda)$ the dual problem (24) is explicitly defined. It is a quasi-unconstrained problem and it can therefore be readily solved by using a steepest ascent algorithm, slightly modified to handle the non-negativity constraints on the dual variables. Such a gradient method requires the first derivatives of the dual function to be available. Fortunately an interesting feature of the dual formulation is that these derivatives are extremely simple to compute, because they are given by the primal constraints

$$\frac{\partial l}{\partial \lambda_j} = \sum_{+} \frac{h_{ij}}{x_i(\lambda)} - \sum_{-} h_{ij} x_i(\lambda) - \bar{h}_j \tag{28}$$

At this point it is worth pointing out that the convex linearization method can be used when some or all of the design variables, instead of varying continuously, can only take on discrete values. In such a case the dual method formulation becomes still more attractive.[13] The discrete primal variables continue to be explicitly related to the continuous dual variables. The dual function remains continuous, but it has discontinuous first derivatives. A first-order gradient projection type algorithm is under development, which is based on the DUAL 1 optimizer available in ACCESS-3[14] and SAMCEF.[15] Because the dual function gradient discontinuities can be shown to occur on specific hyperplanes in the dual space, DUAL 1 determines ascent directions by projecting the dual function gradient on the intersection of the successively encountered discontinuity planes. The DUAL 1 optimizer remains applicable to pure continuous variable problems, in which case it reduces to a special form of the conjugate gradient method.

However, in the pure continuous case, a second order Newton-type algorithm similar to the DUAL 2 optimizer available in ACCESS-3[14] and SAMCEF[15] has been developed to solve more efficiently the dual problem (24). It uses the gradient and the Hessian of the dual function. The components of the gradient vector are given by (28), while the elements of the Hessian matrix can be shown to be

$$\frac{\partial^2 l}{\partial \lambda_j \partial \lambda_k} = \frac{1}{2} \sum_{(+)} \frac{h_{ij} h_{ik}}{x_i^3 (f_i + p_i)} + (\cdots) \tag{29}$$

where $\sum_{(+)}$ means summation over all the terms involving positive values of the quantities f_i, h_{ij} and h_{ik}, and $(\cdots)$ denotes similar terms depending upon the signs of these quantities. It should be added that the summation in (29) is restricted to the free primal variables, i.e. the variables x_i that have not taken on their lower or upper bound value ($\underline{x}_i$ or $\bar{x}_i$). This means that the Hessian is discontinuous whenever a free primal variable becomes fixed, or conversely (see, for example, References 2, 8 and 11). Hence it is apparent that while $l(\lambda)$ is continuously differentiable in all feasible dual space, it is not, in general, twice continuously differentiable everywhere. The DUAL 2 optimizer is capable of coping with this difficulty by resorting to a specially devised and quite simple line search technique. DUAL 2 operates in a sequence of dual subspaces with gradually increasing dimension, so that the effective dimensionality of the dual problem never exceeds the number of active behaviour constraints. Because this number is relatively low for many optimization problems of practical interest, the DUAL 2 optimizer is very efficient. It is, in addition, general and reliable.

5. CONSTRAINT RELAXATION

In practical applications a difficulty that frequently occurs is that the initial design violates some of the constraints. Very often it is difficult to get a feasible design point because two or more constraints are incompatible. In the convex linearization method, this difficulty might be accute because of the conservative character of the approximate constraints. It can even lead to a breakdown of the optimization process. In fact, although conservativeness is most of the time a desirable property, it is not so when the initial starting point is seriously infeasible. In such a case it can happen that the linearized feasible subdomain be empty, so that the method can no longer be applied. To cope with this difficulty an additional relaxation variable is introduced into the explicit subproblem statement, which becomes

$$\min \quad \delta \tilde{f}(x) \tag{30}$$

$$\text{s.t.} \quad \tilde{h}_j(x) \leqslant \delta \bar{h}_j \tag{31}$$

$$\underline{x}_i \leqslant x_i \leqslant \bar{x}_i \tag{32}$$

$$\delta \geqslant 1 \tag{33}$$

where $\tilde{f}(x)$ and $\tilde{h}_j(x)$ represent the approximate functions appearing in (20) and (21). Clearly, if the relaxation variable δ hits lower bound ($\delta = 1$), nothing is changed in the problem statement, which will usually happen when the starting point x^o is feasible or nearly feasible. On the other hand, if the starting point x^o is seriously infeasible, the algorithm will find a value of δ greater than unity, which means that the linearized feasible domain will be artificially enlarged. Taking the solution of the current explicit subproblem (30–33) as a new linearization point, the next feasible subdomain will generally be non empty. The method can then be applied as initially stated in Section 3, yielding unit values of the relaxation variable δ at each subsequent iterations.

To solve the modified explicit problem (30–33), it is sufficient to increase the number of variables by one, and to add to the definitions (19):

$$\left.\begin{aligned} f_i &= \tilde{f}(x^o) \\ h_{ij} &= -\bar{h}_j \end{aligned}\right. \qquad \text{for } i = n+1$$

The dual solution scheme of Section 4 can then be employed without any transformation. Another possibility is to modify the dual algorithm by explicitly introducing the effects of the relaxation variable δ in the convex linearization method. Assuming $\tilde{f}(x^o)$ and $\bar{h}_j$ positive, the new explicit problem (30–33) will be transformed as follows after convex linearization with respect to δ:

$$\min \quad \tilde{f}(x^o)\delta + \tilde{f}(x) \tag{34}$$

$$\text{s.t.} \quad \tilde{h}_j(x) \leqslant \bar{h}_j\left(2 - \frac{1}{\delta}\right) \tag{35}$$

$$\underline{x}_i \leqslant x_i \leqslant \bar{x}_i \tag{36}$$

$$\delta \geqslant 1 \tag{37}$$

It is easily seen that δ is given in terms of the dual variables by the relation (see equation 26)

$$\delta = \left(\frac{\sum \lambda_j \bar{h}_j}{\tilde{f}(x^o)}\right)^{1/2} \quad \text{if} \quad \sum_j \lambda_j \bar{h}_j > \tilde{f}(x^o)$$

$$\delta = 1 \qquad\qquad \text{if} \quad \sum_j \lambda_j \bar{h}_j \leqslant \tilde{f}(x^o)$$

As implemented herein, relaxation is uniformly applied to all the constraints and its purpose is simply to balance the effect of conservativeness in the convex linearization method. Fc uniform relaxation being effective, it is implicitly assumed that the feasible domain correspc to the primary problem is non-empty. If it not the case, for example because two or constraints are really in conflict, then uniform relaxation is not a satisfactory technique. Cı research is directed toward other relaxation methods that would be capable of finding a mi relaxation for an infeasible problem. These methods imply introducing several addi relaxation variables δ_j, one for each constraint that the user accepts to relax.

6. APPLICATION TO OPTIMAL SIZING

Coming back to the optimal sizing problem (1–4), it can be seen that, the weight coefficier being positive, the linear objective function (1) will remain linear in the explicit subpro statement (see equation 20). Conversely, when applying convex linearization to the beha constraints, they will be mainly expressed in terms of the reciprocal variables. Indeed, struc mechanics indicates that when the component sizes are increased the stresses and displacen usually decrease. Numerical experiments support this intuitive interpretation: the first deriva of the behaviour constraints with respect to the member sizes are most of the time positive (that $h_j(x) \equiv \bar{c}_j - c_j(x)$). As a result the reciprocal variables will dominate in the convex lineariza scheme ($\sum_+$ in equation 21). On the other side, those variables for which a linear expansion is ı ($\sum_-$ in equation 21) generally will be sized at the lower bound $\underline{x}_i$, because increasing them wı raise the stresses and displacements values. Of course, this is only a global understanding. The situation is much more difficult, and it is the purpose and function of a good optimizer to find best compromise.

These considerations permit justifying *a posteriori* the now well-established approximaı concepts approach, where the behaviour constraints are linearized with respect to the reciprı variables.[1,2,16] It is worth recalling that the approximation concepts approach, when combi with dual methods, has proven to provide a highly efficient structural synthesis capability usually generates a nearly optimal design within less than ten finite element analyses. In additi this method has led to a perspective where optimality criteria techniques are seen to reside wit the general framework of a mathematical programming approach to structural optimization.[1,2] Finally, the method has been successfully extended to deal with pure discrete and mi continuous-discrete variable problems (e.g. available standard gauge sizes; number of plies laminated composite structures).[13]

It is apparent that the convex linearization method proposed in this paper can be viewed as enhanced approximation concepts approach, keeping all its attractive properties. It can also interpreted as a further generalization of optimality criteria techniques. By writing down t Kuhn–Tucker conditions for the explicit subproblem (20–22), the following generalized optimal criterion is obtained:

$$x_i^2 = \frac{\sum_+ h_{ij}\lambda_j}{f_i - \sum_- h_{ij}\lambda_j} \qquad \text{if} \quad f_i < 0$$
(3

and

$$x_i^2 = \frac{f_i - \sum_+ h_{ij}\lambda_j}{\sum_- h_{ij}\lambda_j} \qquad \text{if} \quad f_i < 0$$
(3

These explicit redesign relations give the sizing variables x_i in terms or the Lagrangian multipliers λ_j. They must of course be employed in conjunction with the side constraints, which have always to be enforced (see equations 26 and 27). To help fix ideas let us consider Berke's optimality criteria technique for the minimum weight design of a truss subject to a single displacement constraint.[18] In this technique the bar cross-sectional areas are subdivided into a group of active (or free) variables ($i = 1, \tilde{n}$) and a group of passive (or fixed) variables ($i > \tilde{n}$). With this formulation the explicit problem depends only on the $\tilde{n}$ active design variables [see problem (1–4)]:

$$\min \quad \sum_{i=1}^{\tilde{n}} w_i x_i \tag{40}$$

$$\text{s.t.} \quad \sum_{i=1}^{\tilde{n}} \frac{c_i}{x_i} \leqslant \bar{c} - c^o \tag{41}$$

where c^o denotes the contribution of the passive members to the constrained displacement. The stationarity conditions of the Lagrangian function

$$L(x_i, \lambda) = \sum_i w_i x_i + \lambda\left(\sum_i \frac{c_i}{x_i} - \bar{c} + c^o\right) \tag{42}$$

provide the solution to the problem:

$$\begin{cases} x_i^2 = \lambda \dfrac{c_i}{w_i} & i = 1, \tilde{n} \qquad (43) \\ x_i = \underline{x}_i & i > \tilde{n} \qquad (44) \end{cases}$$

The value of the positive Lagrangian multiplier λ is obtained by substituting (43) into the constraint condition (41).

In (equation (43)) the coefficients c_i have to be positive for the square root to be extracted, which yields a means to distinguish between the active and passive variables. Note that the contribution c^o of the passive variables to the constrained displacement is negative, so that they must obviously take on the minimum possible cross-section. The condition that an active variable is characterized by $c_i > 0$ is the same as the one used in convex linearization to select direct and reciprocal variables. In fact, the explicit problem (40, 41) that furnishes Berke's optimality criterion (43) is identical to the convex subproblem (20, 21) considered in this paper. Also, the redesign relations (38), when restricted to the particular case under consideration, are equivalent to (43) (with $f_i = w_i > 0$ and $h_{i1} = c_i$). The idea of active/passive design variables is thus one way of achieving convexity.

Berke[18] has also proposed a correct optimality criterion for the case of multiple displacement constraints. However, he noticed that, conversely to the previous case, it is no longer possible to achieve a closed form solution to the explicit problem. Also it becomes difficult to select the active and passive design variable groups and to detect the strictly critical displacement constraints. From this time all the optimality criteria school has attempted for many years to derive explicit redesign relations that could solve at least approximately any structural optimization problem (see, for example, References 19 and 20). The key in fact, lies in dual methods. Since the dual maximization problem (24) is quasi-unconstrained and explicit, its exact solution can be generated at a low computational cost, which is comparable to that required by the recursive techniques of conventional optimality criteria approaches. The dual algorithms can handle a large number of inequality constraints. They are inherently capable of identifying the active behaviour constraints through the non-negativity constraints on the Lagrangian multipliers. They also automatically sort out the active and passive design variable groups by using the explicit relationships between primal and dual variables.

It is fascinating to observe that convex linearization and dual formulation form really two strategies made for each other. Convex linearization yields a convex, separable subproblem, and dual methods need it. A dual solution scheme can only generate an exact solution to the explicit subproblem, and convex linearization provides a conservative approximation.

The convex linearization method proposed in this paper further generalizes the approximation concepts approach to structural optimization—as well as the now obsolete optimality criteria techniques—while keeping all its attractive features. It should again be emphasized that the method benefits from a much broader generality. At each iteration the optimizer requires as only entries the initial values and first derivatives of the functions describing the mathematical programming problem to be dealt with. As a result of such a generality new potentials can now be envisaged. Other types of objective function than the structural weight could be considered, such as: minimize stress concentration; maximize fundamental natural frequency, etc. A broader class of constraints can be conceived: explicit constraints on the design variables such as the linear constraints (3); stress flow constraints, etc. Finally, new kinds of design variables can be addressed, such as geometrical variables to deal with shape optimal design.[9,21]

In the last section, examples of application to optimal sizing of real-life aerospace structures will be offered, which demonstrate the efficiency of the convex linearization method. To be convinced of its generality, Reference 9 can be consulted, where difficult shape optimization problems are successfully treated.

7. NUMERICAL EXPERIMENTS

The convex linearization method has been first experimented on some simple problems, such as the 2-bar and 10-bar trusses classical in the structural optimization literature, by adding linear inequality constraints on the bar cross-sections. The results are not reported herein, because they are not very significant, no comparison with other methods being available. In this paper three examples are offered. The first one has been elaborated on the famous 10-bar truss problem. The other two are concerned with real-life aerospace structures.

10-bar truss

The first example has been specially devised to make the classical 10-bar truss problem difficult to solve by conventional methods (see Figure 4). The displacements at nodes 4 and 5 are limited

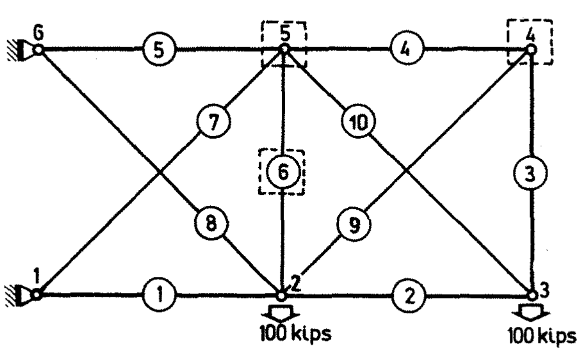

Figure 4. 10 bar-truss

Table I. Iteration history for 10-bar truss example

Iteration	Weight	u_4	u_5	f_6
0	8393	1·898	0·8372	−40125·0
1	7290	1·635	0·7389	− 478·0
2	4856	1·980	0·8458	− 675·0
3	4221	1·968	0·972	− 2366·0
4	4095	1·994	0·9987	− 2458·0
5	4067	1·998	0·9992	− 2491·0
6	4058	1·999	0·9998	− 2498·0
7	4053	1·999	0·9999	− 2499·0
8	4050	2·000	1·000	− 2500·0

to 2 in. and 1 in., respectively. Instead of assigning a maximum allowable stress limit in the critical member 6, the stress flow (i.e. the force) in member 6 is limited to 2500 lb.

Stress flow constraints are difficult to deal with. For a statically determinate truss, the bar forces are constant, and no change in the design can modify them. In the statically indeterminate case under consideration, stress flows are not affected by a scaling of the design variables, so that, if they were the only imposed constraints, the minimum weight design should be zero. In the design space, this means that a stress flow constraint is represented by a restraint surface that passes through the axes origin.

The initial design ($a^o = 20\,\text{in.}^2$ for each bar) is seriously infeasible so that the first explicit subproblem does not admit any solution. Therefore, the relaxation scheme discussed in Section 5 is employed in the convex linearization method. After this first difficult iteration the optimization process becomes normal, each subsequent feasible subdomain being non-empty. The iteration history data are given in Table I. Note that except the initial design, all other designs are feasible.

Composite plate

The second example is especially interesting, because it is at the origin of the development of the convex linearization method presented in this paper. The problem consists in the weight minimization of the composite plate representated in Figure 5 (part of a floor for the Airbus plane).

The tensile solicitation is applied by imposing prescribed displacements on one side of the plate. The other boundary conditions depicted in Figure 5 result from symmetry considerations. The structure is made up of 0°, + 45°, − 45° and 90° high strength graphite–epoxy laminates.

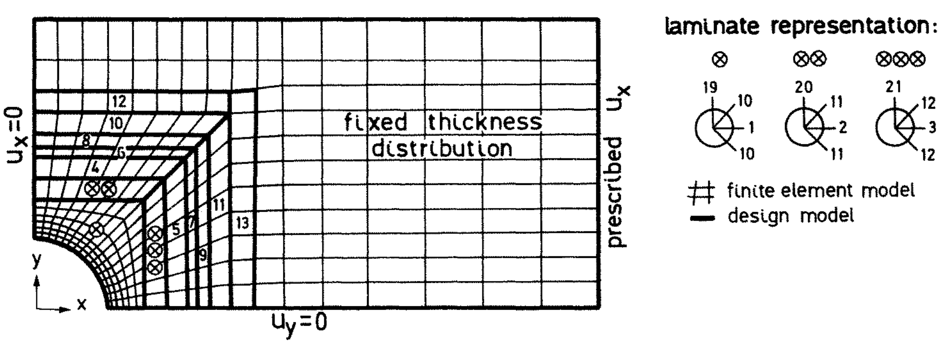

Figure 5. Composite plate

The laminates are represented by stacking four orthotropic membrane elements in each quandrangular region shown in Figure 5. The layer thicknesses in each basis direction (0°, ±45°, 90°) are the design variables. The finite element model involves 4 × 288 linear isoparametric elements, and 946 degrees-of-freedom. After design variable linking according to the subdivision in regions of Figure 5, it remains 39 independent design variables.

The behaviour constraints correspond to strength requirements based on the TSAI-AZZI failure criterion (with different tensile and compressive allowable stresses). This criterion is applied in the most critical element in each of the 39 linking regions. For fabricational reasons 27 linear inequality constraints are assigned to the design variables. Typical linear constraints are as follows (see Figure 5):

$$\begin{aligned}
0 &\leqslant -t_1 - 2t_{10} + 9t_{19} \leqslant 300 \\
0 &\leqslant -t_2 - 2t_{11} + 9t_{20} \leqslant 300 \\
0 &\leqslant -t_3 - 2t_{12} + 9t_{21} \leqslant 300 \\
0 &\leqslant t_1 + 2t_{10} + t_{19} \\
&\quad -t_2 - 2t_{11} - t_{20} \leqslant 0{\cdot}325 \\
0 &\leqslant t_1 + 2t_{10} + t_{19} \\
&\quad -t_3 - 2t_{12} - t_{21} \leqslant 0{\cdot}325
\end{aligned} \tag{45}$$

These constraints are linearly dependent and very sparse, which complicate the solution of the optimization problem. Finally, lower and upper bounds are assigned to the design variables.

The problem was first treated without the linear constraints (45) by using the DUAL 2 and DUAL 1 optimizers, respectively, in the pure continuous and in the pure discrete cases. No difficulty occurred during the optimization process, that generated the optimal design in less than ten finite element analyses (see Reference 4). However, when the linear constraints (45) are introduced, it becomes impossible to solve the problem by using the specially devised dual algorithm mentioned in Section 1 to handle linear constraints. The essential difficulty is that, after the first structural analysis, the explicit subproblem (see equations 1–4) leads to first-order discontinuities in the dual space (non-convexity). Two alternative strategies were then imagined with only limited success,[4] and finally the idea of the convex linearization method arose.

When resorting to the convex linearization method, the constraint relaxation technique described in Section 5 must be activated, because the first explicit subproblem is infeasible. This explains the increase in weight after the first iteration, as indicated in Figure 6, which plots the iteration history data.

For comparison, Figure 6 also represents the results generated by the recursive linear programming approach, obtained by linearizing the behaviour constraints with respect to the direct sizing variables, and by using the Simplex algorithm. This alternative optimization strategy is quite successful in the present case, because the problem involves 39 design variables and, at the optimum, 39 constraints are active: 1 TSAI-AZZI stress constraint, 6 linear constraints, and 32 side constraints. However, it is important to recognize that this Simplex method is not, in general, a valid approach, because it implies the optimum lying at a vertex of the design space. The final optimized design can be found in Reference 4.

Engine mount structure

The last example is concerned with a real-life application of optimization techniques to the European launcher Ariane 4. Four strap-on liquid boosters will be attached to a future version

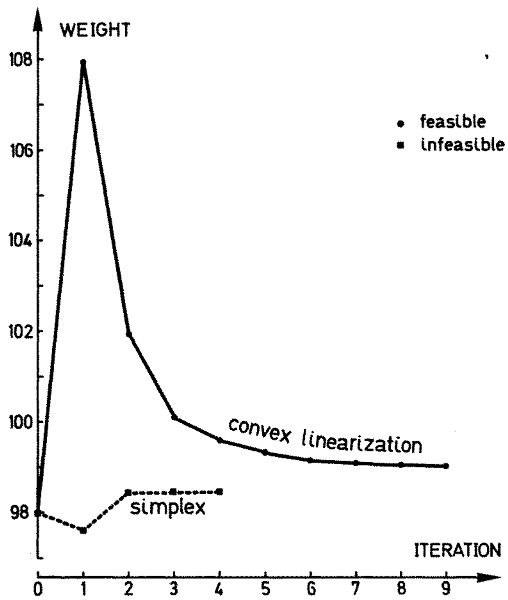

Figure 6. Iteration history for composite plate

of the launcher in order to double the thrust. The Belgian company Sabca is to design and build the three main structures of the booster: the forward skirt, the intertank skirt and the engine mount structure (see Figure 7). From the beginning the interest of resorting to the optimization capabilities of the SAMCEF finite element system was recognized at various levels of the company. The main reason was the fundamental importance of obtaining a light weight structure: 1 kg gained on the booster permits increasing the payload by 0·14 kg. It was not possible to achieve this goal by conventional design techniques because of unusual specifications (stiffness requirements; stress flow limitations).

This application has led to many ups and downs, especially because the specifications were initially in conflict, so that no feasible design could be obtained. Due to the lack of space, attention is focused in this paper only on the engine mount structure.

The finite element model shown in Figure 8 involves 4883 degrees-of-freedom and 1008 finite elements (second-degree displacement field). The objective is to minimize the structural weight subject to the following behaviour constraints: (a) stiffness requirements at the bold-bearing joint, as well as at the point where the engine load is introduced, in order to take into account dynamics aspects; (b) limitation of the normal stress flow in the upper ring, in order to diffuse the load transmitted to the upper flange joint; (c) maximum allowable Von Mises stresses under four loading conditions; (d) in addition, local stiffness requirements must be taken into consideration at various critical points (e.g. where equipments are supported). Note that all the stiffness constraints consist in fact in assigning upper limits to influence coefficients.

By using a simplified finite element model (half-cylinder), the optimization program has shown that the normal stress flow limitation and one of the stiffness requirements were incompatible. After a while the responsible companies decided to reconsider these specifications and new stiffness constraints were imposed, which made it possible to obtain a feasible design. Several runs were then performed, with more and more accurate definitions of the fabricational constraints (design variable linking; lower and upper bounds on the thicknesses). For the last optimization

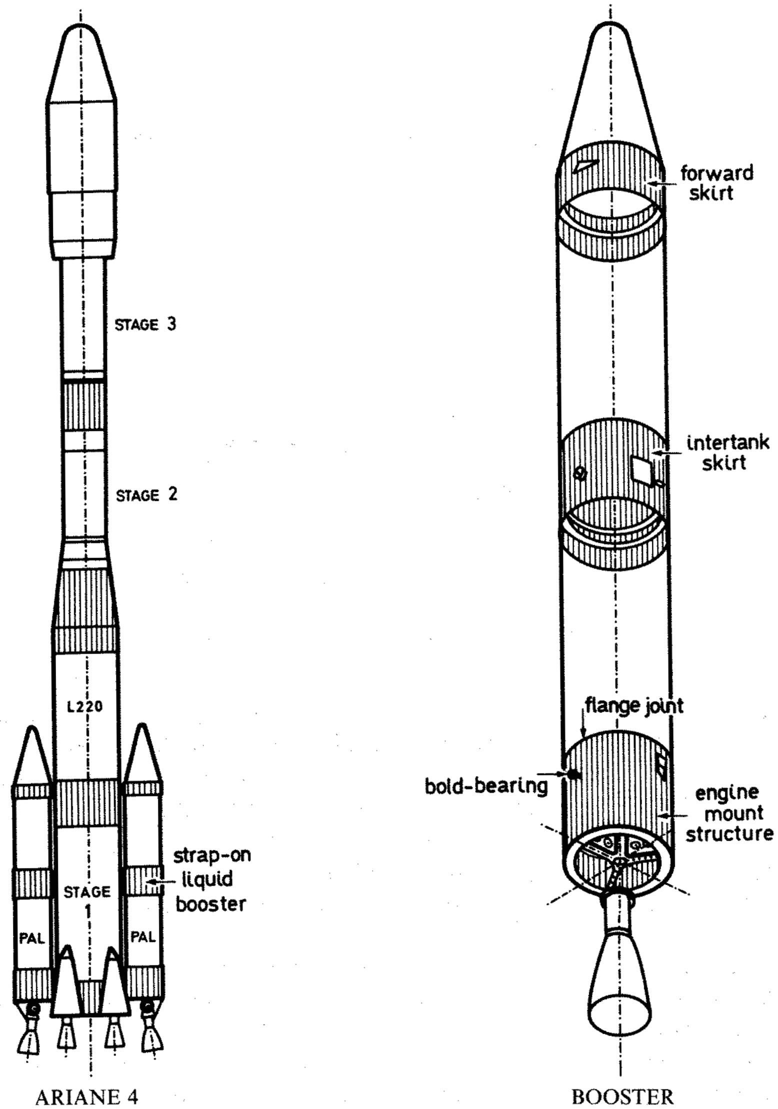

Figure 7.

process, which involves 35 design variables and 20 behaviour constraints, convergence is achieved within 7 finite element analyses.

Finally, it is worth mentioning that, when the fabricational process was started, it was found that the optimization results could not be used as such because of technological requirements that did not appear at first. The thickness distributions had to be modified. After analysing the new design, the stress flow constraints in the upper ring were seen to be seriously violated at the bold bearing joint level. Therefore, it was decided to perform an ultimate optimization run with an appropriate design variable linking. This final problem involves 62 design variables and 7 active behaviour constraints out of 40; 6 variables reach their upper bound, and 1 its lower bound.

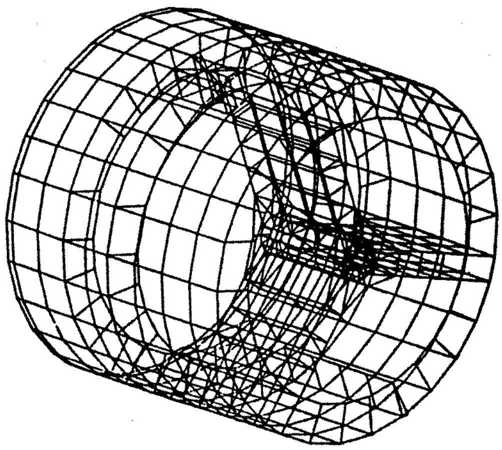

Figure 8. Engine mount structure (finite element model)

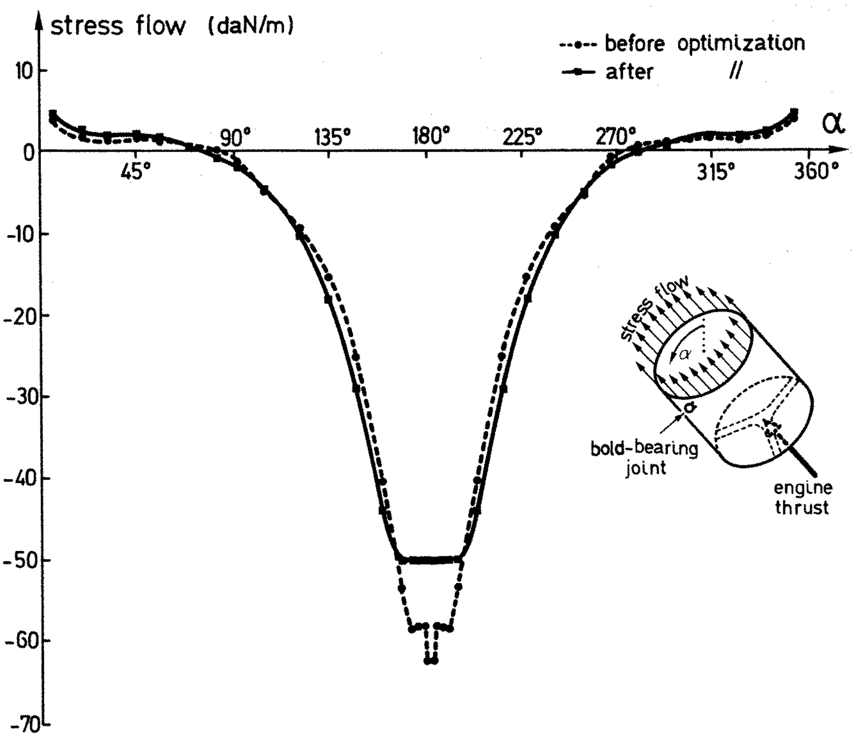

Figure 9. Stress flow transmitted to flange joint

As shown in Figure 9, the stress flow along the upper ring was properly cut to its limiting value. This additional run required 4 more structural analyses, each analysis demanding about 1 hr CPU on a VAX 11/780 computer. The optimal design is now being built by Sabca.

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