

A CRACK IN THE HOMOGENEOUS HALF PLANE INTERACTING WITH A CRACK AT THE INTERFACE BETWEEN THE NONHOMOGENEOUS COATING AND THE HOMOGENEOUS HALF-PLANE

E.E. Theotokoglou¹ and G.H. Paulino²

¹*Faculty of Applied Sciences, Dept. of Mechanics-Lab. of Strength of Materials
The National Technical University of Athens, Zographou Campus,
Theocaris Bld., GR-0157 73, Athens, Greece
e-mail: stathis@central.ntua.gr,*

²*Department of Civil and Environmental Engineering
University of Illinois at Urbana-Champaign, 2209 Newmark Laboratory
205 N. Mathews Ave., Urbana, IL 61801-2352, USA
e-mail: paulino@uiuc.edu*

Abstract. A fundamental solution is established for a crack in a homogeneous half-plane interacting with a crack at the interface between the homogeneous elastic half-plane and the nonhomogeneous elastic coating in which the shear modulus varies exponentially with one coordinate. The problem is solved under plane strain or generalized plane stress conditions using the Fourier integral transform method. The stress field in the homogeneous half plane is evaluated by the superposition of two states of stresses, one of which is associated with a local coordinate system in the infinite fractured plate, while the other one in the infinite half plane defined in a structural coordinate system.

1. Introduction. Turbine systems and aerospace applications require the use of structural ceramics to protect the hot sections. The thermomechanical mismatch between metal and ceramics induces high residual stresses responsible for cracking and spallation. Functionally graded materials (FGM) are composites with predetermined, continuously varying mechanical properties that reduce the residual stresses in composites (Yamanouchi et al., 1990, Holt et al., 1993, Kokoni and Choules, 1995). FGMs can be described as two-phase particulate composites where the volume fractions of their constituents differ continuously in the thickness direction (Hirano et al., 1988, Yamanouchi, 1990). Hence, the composition profile could be chosen to give the appropriate thermomechanical properties. In particular by varying the volume fractions of the constituents between zero at the interface to one hundred percent near the surface, thereby obtaining a continuous through-thickness material property variation, it is possible to obtain not only smoother stress distribution but also higher bond strength (Erdogan, 1995). In addition, their properties may be determined either experimentally or by using the method of cells (Pindera et al., 1998). Gradual variation of mechanical properties appear naturally in bones or soils and artificially in FGM. This material property graduation may be simulated by a power or an exponential law (Yamanouchi et al., 1990, Erdogan, 1995 and Bartsch et al., 2001). A number of investigators have studied crack problems in nonhomogeneous materials via different methods and procedures (Delale and Erdogan, 1988, Erdogan et al., 1991, Pindera, 1991, Erdogan, 1995, Chen and Erdogan 1996, Jin and Batra, 1996, Pei

and Asaro, 1997, Pindera et al., 1998, Shbeeb et al., 1999, Chan et al., 2001, Choi, 2001, Kim and Paulino, 2002).

A very interesting problem in layered structures such as ceramic-coated metal substrates, is the interface crack problem. It is well known that in the simplest case of two bonded elastic homogeneous materials containing an interface crack, Young's moduli and Poisson's ratios are constant and therefore a jump discontinuity exists at the interface. Thus, the stress state around the crack tip exhibits an oscillation (Rice and Sih, 1963, Erdogan, 1965). This implies wrinkling and overlapping of the crack interfaces near the crack tip. In a FGM, considering a continuous through thickness material property variation, which is bonded to a substrate and contains an interface crack, it has been shown (Delale and Erdogan, 1988), that the crack tip stress and displacements oscillations disappear. In that study, one material is assumed nonhomogeneous with its material properties varying through the thickness in a certain exponential manner and the interface crack problem is confronted by eliminating the jump discontinuity in the material properties and without considering Dundurs' parameters (Dundurs, 1969).

In this work we consider for the first time the problem of a crack in the homogeneous half-plane interacting with a crack at the interface between the non-homogeneous coating and the homogeneous half-plane (Fig. 1). A very important problem, which has not been addressed yet, is that thermal barriers always include some thickness of pure ceramic material (Kokoni and Choules, 1995).

In the plane elasticity problem shown in Fig. 1, it is assumed that the cracked half-plane is homogeneous with elastic constants μ_1 , κ_1 , the coating is nonhomogeneous with elastic parameters $\mu_2(y)$, $\kappa_2(y)$, and μ_2 according to Delale and Erdogan (1988), is approximated by:

$$\mu_2(y) = \mu_1 e^{\beta y}, \quad 0 < y < h_2 \quad (1)$$

where μ_i is the shear modulus, $\kappa_i = 3 - 4\nu_i$ for plane strain and $\kappa_i = (3 - \nu_i)/(1 + \nu_i)$ for generalized plane stress, ν_i being the Poisson's ratio ($i = 1, 2$) and β is a real constant which represents the coefficient of nonhomogeneity. In previous studies (Erdogan et al., 1991, Erdogan 1995, Chen and Erdogan, 1996), it was shown that the influence of the variation in Poisson's ratio on stress intensity factors is rather insignificant and, therefore, κ may be assumed to be constant throughout the medium. In the case of nonhomogeneous ceramic coatings, for example a thin film of Gold, Silver, Lead, Zirconia or Mullite to be deposited on Nickel or steel bearing and sliding parts as dry film lubricant (Ahmed, 1987), the stiffness of coating is less than that of the substrate. In other cases involving contact problems and some thermal barrier coatings, the coating is stiffer than the substrate. Thus, the non-homogeneity constant β will range from -3 to 3 which covers nearly all cases that may arise in practice and a Poisson's ratio of 0.3 may also be considered (Chen and Erdogan, 1996, Shbeeb et al., 1999).

First, the stress and displacement fields are computed for the FGM. In the second step, the crack in the homogeneous half-plane is confronted by considering the

superposition of the solution of the infinite half-plane with a crack, with the solution of the infinite plane without a crack. Introducing the boundary conditions, all the unknowns are expressed in terms of the slopes of the crack displacement discontinuities along the cracks at the infinite half-plane and at the interface. Finally from the perturbation problem, a system of four integral equations in terms of the unknown distribution of dislocations, which solve the problem is derived. In this way we succeeded in expressing all the unknown coefficients in terms of the slopes of the crack displacement coefficients and in reducing the solution of the whole problem to the solution of a system of four singular integral equations.

2. The problem of nonhomogeneous coating. By using standard Fourier transforms for the nonhomogeneous layer 2 (Fig. 1), we have

$$u_2(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{n=1}^4 D_n(\xi) A_n(\xi) e^{\lambda_n y} e^{-ix\xi} d\xi, \quad v_2(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{n=1}^4 A_n(\xi) e^{\lambda_n y} e^{-ix\xi} d\xi \quad (2)$$

where $A_1(\xi), \dots, A_4(\xi)$ are unknown functions, $\lambda_1, \dots, \lambda_4$ are the roots of the characteristic equation, given by (Chen and Erdogan, 1990):

$$[\lambda_n^2 + \beta\lambda_n - \xi^2]^2 + \frac{3-\kappa}{\kappa+1} \beta^2 \xi^2 = 0 \quad (3)$$

and

$$D_n(\xi) = \frac{2i\xi\lambda_n + i\beta\xi(\kappa-1)}{-\xi^2(\kappa+1) + \lambda_n^2(\kappa-1) + \beta\lambda_n(\kappa-1)}; \quad n=1,2,3,4 \quad (4)$$

Using the strain-displacement relations and the constitutive equations (Chen and Erdogan, 1995), the stress field is finally given by

$$\begin{aligned} \sigma_{2,yy}(x, y) &= \frac{\mu_1}{(\kappa-1)\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{n=1}^4 G_n(\xi) A_n(\xi) e^{\lambda_n y} e^{-ix\xi} d\xi; \quad y > 0 \\ \sigma_{2,xy}(x, y) &= \frac{\mu_1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{n=1}^4 H_n(\xi) A_n(\xi) e^{\lambda_n y} e^{-ix\xi} d\xi; \quad y > 0 \end{aligned} \quad (5)$$

where

$$G_n(\xi) = \lambda_n(\kappa+1) - i\xi D_n(3-\kappa), \quad H_n(\xi) = \lambda_n D_n - i\xi; \quad n=1,2,3,4 \quad (6)$$

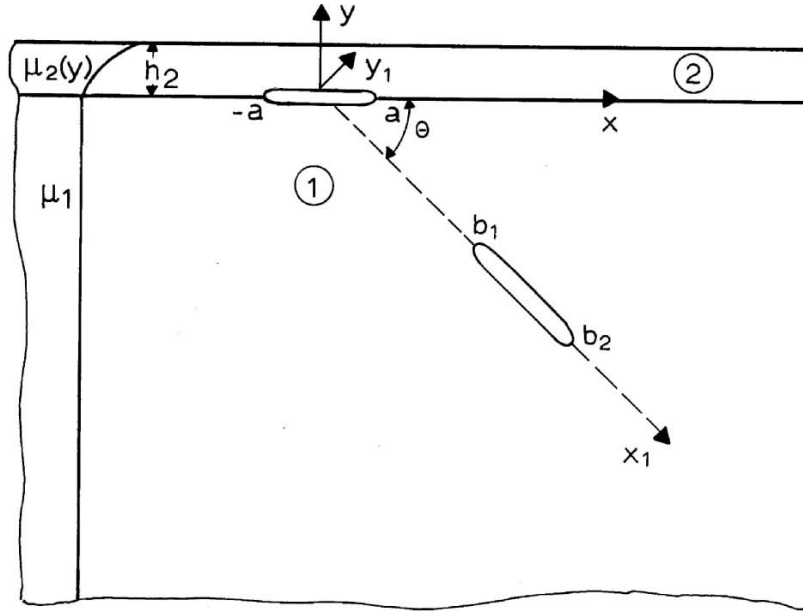


Figure 1: A crack in the homogeneous half-plane interacting with a crack at the interface.

3. A crack in the homogeneous half-plane. The stress and the displacement fields in the cracked homogeneous half-plane, considering the superposition principle, are given by

$$\begin{aligned} u_1(x, y) &= u_1^{(1)}(x, y) + u_1^{(2)}(x, y), \quad v_1(x, y) = v_1^{(1)}(x, y) + v_1^{(2)}(x, y) \\ \sigma_{1ij}(x, y) &= \sigma_{1ij}^{(1)}(x, y) + \sigma_{1ij}^{(2)}(x, y); \quad i, j = x, y \end{aligned} \quad (7)$$

where the superscript (1) refers to the field components *in an infinite plane with a crack* and the superscript (2) to those in the *half-plane without the crack*.

Taking into consideration the Fourier transform for the displacements $u_1^{(2)}(x, y)$ and $v_1^{(2)}(x, y)$ along the x -coordinate, and the regularity condition, $\lim_{y \rightarrow -\infty} u_1^{(2)} = \lim_{y \rightarrow -\infty} v_1^{(2)} = 0$, it is finally obtained

$$\begin{aligned} u_1^{(2)}(x, y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (R_1(\xi) + yR_2(\xi)) e^{|\xi|y} e^{-ix\xi} d\xi, \quad -\infty < y < 0 \\ v_1^{(2)}(x, y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Q_1(\xi) + yQ_2(\xi)) e^{|\xi|y} e^{-ix\xi} d\xi, \quad -\infty < y < 0 \end{aligned} \quad (8)$$

where

$$R_1(\xi) = -\frac{i\xi}{|\xi|} Q_1(\xi) - \frac{iK}{\xi} Q_2(\xi), \quad R_2(\xi) = -i\frac{\xi}{|\xi|} Q_2(\xi); \quad -\infty < y \leq 0 \quad (9)$$

Using the strain-displacement relations, and the constitutive equations, (Chen and Erdogan, 1995), the stress field $\sigma_{ij}^{(2)}(i, j = x, y)$ for the region, $y < 0$, may be obtained.

The displacement field in the cracked infinite plane, according to Fourier transform, is given by

$$u_1^{(1)}(x_1, y_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U_{32}(y_1, \xi) e^{-ix_1\xi} d\xi, \quad v_1^{(1)}(x_1, y_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V_{32}(y_1, \xi) e^{-ix_1\xi} d\xi, \quad y_1 < 0 \quad (10)$$

taking into account that $\lim_{y_1 \rightarrow +\infty} u_1^{(1)} = \lim_{y_1 \rightarrow +\infty} v_1^{(1)} = 0$, we get

$$U_{32}(y_1, \xi) = (R_{41}(\xi) + y_1 R_{42}(\xi)) e^{|\xi|y_1}, \quad V_{32}(y_1, \xi) = (Q_{41}(\xi) + y_1 Q_{42}(\xi)) e^{|\xi|y_1}, \quad y_1 < 0 \quad (11)$$

where

$$R_{41}(\xi) = -\frac{i\xi}{|\xi|} Q_{41}(\xi) - \frac{iK}{\xi} Q_{42}(\xi), \quad R_{42}(\xi) = -\frac{i\xi}{|\xi|} Q_{42}(\xi); \quad y_1 < 0 \quad (12)$$

We introduce the following slopes of the crack displacement discontinuity along the crack (b_1, b_2) ,

$$f_3(x_1) = \frac{\partial}{\partial x_1} (v_1^{(1)}(x_1, 0^+) - v_1^{(1)}(x_1, 0^-)), \quad f_4(x_1) = \frac{\partial}{\partial x_1} (u_1^{(1)}(x_1, 0^+) - u_1^{(1)}(x_1, 0^-)); \quad b_1 < x_1 < b_2 \quad (13)$$

with the properties:

$$f_n(x_1) = 0, \quad 0 < x_1 < b_1, \quad x_1 > b_2, \quad \int_{b_1}^{b_2} f_n(x_1) dx_1 = 0; \quad n = 3, 4 \quad (14)$$

From the application of the inverse Fourier transform, it is finally obtained that:

$$\begin{Bmatrix} Q_{41}(\xi) \\ Q_{42}(\xi) \end{Bmatrix} = \begin{bmatrix} \frac{1}{2i\xi} & -\frac{(\kappa-1)}{(\kappa+1)} \frac{1}{2|\xi|} \\ -\frac{|\xi|}{i\xi(\kappa+1)} & \frac{1}{\kappa+1} \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} \quad (15)$$

with

$$A(\xi) = \frac{1}{\sqrt{2\pi}} \int_{b_1}^{b_2} f_3(t) e^{i\xi t} dt, \quad B(\xi) = \frac{1}{\sqrt{2\pi}} \int_{b_1}^{b_2} f_4(t) e^{i\xi t} dt \quad (16)$$

4. Boundary conditions along the coating and along the interface between the coating and the homogeneous half-plane. From the boundary condition along $y = h_2$ (Fig. 1),

$$\sigma_{2yy}(x, h_2) = 0, \quad \sigma_{2xy}(x, h_2) = 0; \quad -\infty < x < \infty \quad (17)$$

and taking into consideration relations (5), we have:

$$A_3(\xi) = e^{(\lambda_1 - \lambda_3)h_2} \frac{(H_1 G_4 - H_4 G_1)}{H_4 G_3 - H_3 G_4} A_1(\xi) + e^{(\lambda_2 - \lambda_3)h_2} \frac{(H_2 G_4 - H_4 G_2)}{H_4 G_3 - H_3 G_4} A_2(\xi) = G_{31}(\xi) A_1(\xi) + G_{32}(\xi) A_2(\xi) \quad (18)$$

$$A_4(\xi) = e^{(\lambda_1 - \lambda_4)h_2} \frac{(G_3 H_1 - G_1 H_3)}{H_3 G_4 - G_3 A_4} A_1(\xi) + e^{(\lambda_2 - \lambda_4)h_2} \frac{(G_3 H_2 - G_2 H_3)}{H_3 G_4 - G_3 A_4} A_2(\xi) = G_{41}(\xi) A_1(\xi) + G_{42}(\xi) A_2(\xi)$$

and from the boundary conditions along $y = 0$,

$$\sigma_{2yy}(x, 0^+) = \sigma_{1yy}(x, 0^-), \quad \sigma_{2xy}(x, 0^+) = \sigma_{1xy}(x, 0^-); \quad -\infty < x < \infty \quad (19)$$

A_1 and A_2 are expressed in terms of Q_1 , Q_2 , f_3 and f_2 . We introduce:

$$f_1(x) = \frac{\partial}{\partial x} (v_2(x, 0^+) - v_1(x, 0^-)), \quad f_2(x) = \frac{\partial}{\partial x} (u_2(x, 0^+) - u_1(x, 0^-)); \quad -a < x < a \quad (20)$$

which are the slopes of the crack displacement discontinuities along the interface crack $(-a, a)$, with the following properties

$$f_n(x) = 0, \quad |x| > a, \quad \int_{-a}^a f_n(x) dx = 0; \quad n = 1, 2 \quad (21)$$

Application of the inverse Fourier transform to equations (20), and taking into consideration relations (2), (7), (8), (10) and (11), we get $Q_1(\xi)$ and $Q_2(\xi)$ in terms of unknown distribution of dislocations f_1 , f_2 , f_3 and f_4 .

5. Conclusions. General procedure for the solution of a crack in the homogeneous half plane interacting with a crack at the interface between the nonhomogeneous coating and the homogeneous half-plane, is given. From our analysis, all the unknown coefficients are expressed in terms of the slopes of the crack displacement discontinuities, along the crack at the homogeneous half-plane and along the crack at the interface between the coating and the homogeneous half-plane.

The proposed procedure needs neither the inverse of a coefficients matrix (Shbeeb et al., 1999) which may be create serious numerical problems nor the stiffness matrix procedure (Pindera, 1991, Choi, 2001).

Taking into consideration the four boundary conditions arising from the perturbation problem, namely

$$\begin{aligned} \sigma_{1,y_1}(x_1, 0^-) = p_1(x_1), \quad \sigma_{1,x_1}(x_1, 0^-) = p_2(x_1); \quad b_1 < x_1 < b_2 \\ \sigma_{2,yy}(x, 0^-) = p_3(x), \quad \sigma_{2,xy}(x, 0^-) = p_4(x); \quad -a < x < a \end{aligned} \quad (22)$$

where p_1 , p_2 , p_3 and p_4 are the traction forces on the crack surfaces, σ_{2iy} ($i = x, y$) are given by (5) and σ_{1iy} ($i = x_1, y_1$) by (7) after an appropriate coordinate transformation, the system (22) of four integral equations that solves the general problem is derived.

From the numerical solution of (22), the stress and the displacement fields in the coating and the substrate and the stress intensity factors at the crack tips may be calculated with respect to the coefficient of inhomogeneity β (equation 1) and to the material constants.

References

- Ahmed, N.A.G. (1987). *Ion Plating Technology, Developments and Applications*, John Wiley and Sons.
- Bartsch, M., Baufeld, B., Mull, K. and Sick, C. (2001). Proceedings Materials week (CD-ROM), Munich Germany.
- Chan, Y.-S., Paulino G.H. and Fannjiang, A.C. (2001). The crack problem for nonhomogeneous materials under uniplane shear loading – A displacement based formulation, *International Journal of Solids and Structures*, **38**, 2989-3005.
- Chen P.F. and Erdogan F. (1996). The interface crack problem for a nonhomogeneous coating bonded to a homogeneous substrate, *Journal of Mechanics and Physics of Solids*, **44**, 771-787.
- Choi, H. J. (2001). The problem for bonded half-planes containing a crack at an arbitrary angle to the graded interfacial zone, *International Journal of Solids and Structures*, **38**, 6559-6588.
- Delale F. and Erdogan F. (1988a). On the mechanical modeling of the interfacial region in bonded half-planes, *ASME Journal of Applied Mechanics*, **55**, 317-324.
- Delale F. and Erdogan F. (1988b). Interface crack in a nonhomogeneous elastic medium, *International Journal of Engng. Science*, **26**, 559-568.
- Dundurs, J. (1969). Edge bonded dissimilar orthogonal elastic wedges under normal and shear loading, *ASME Journal of Applied Mechanics*, **36**, 650-652.
- Erdogan, F. (1965). Stress Distributions in Bonded Dissimilar Materials with cracks, *ASME Journal of Applied Mechanics*, **32**, 418-427.
- Erdogan F. (1995). Fracture Mechanics of Functionally Graded Materials, *Composites Engineering* **5**, 753-770.
- Erdogan, F., Kaya A.C. and Joseph, P.F. (1991). The crack problem in bonded nonhomogeneous materials, *Journal of Applied Mechanics*, **58**, 410-418.
- Hirano, T., Yamada, M., Teraki, M., Niino, M. and Kumakawa, A. (1988). A Study on functionally gradient material design system for a thrust chamber. Proceedings 16th International Symposium on Space Technology and Science, Sapporo, Japan.
- Holt, B.J., Koizumi, M., Hirai T. and Munir, A.Z. (1993) (Editors) *Functionally gradient materials*. In Ceramic Trans, Vol. 34, *The American Ceramic Society*, Westerville, OH.

- Jin, Z.-H. and Batra, R. (1996). Some basic fracture mechanics concepts in functionally graded materials, *Journal of Mechanics and Physics of Solids*, **44**, 1221-1238.
- Kim, J.H., Paulino, G.H. (2002). Finite element evaluation of mixed mode stress intensity factors in functionally graded materials, *International Journal for Numerical Methods in Engineering*, **53**, 1903-1935.
- Kokoni, K. and Choules BD. (1995). Surface thermal fracture of functionally graded ceramic coatings: Effect of architecture and materials, *Composites Engineering* **7**, 865-877.
- Pei, G. and Asaro, R.J. (1997). Cracks in functionally graded materials, *International Journal of Solids and Structures*, **34**, 1-17.
- Pindera M.-J. (1991). Local/global stiffness matrix formulation for composite materials and structures, *Composites Engineering*, **1**, 69-83.
- Pindera, M.-J., Aboudi, J., Arnold, S.M. (1998). Thermomechanical analysis of functionally graded coatings with different microstructural scales, *Journal of American Ceramic Society*, **81**, 1525-1536.
- Rice, J.R. and Sih, G.C. (1963). Plane problems of cracks in dissimilar media, *ASME Journal of Applied Mechanics*, 418-423.
- Shbeeb, N.I., Binienda, W.K. and Kreider, K.L. (1999). Analysis of the driving forces for multiple cracks in an infinite nonhomogeneous plate, Part I: theoretical analysis, *Journal of Applied Mechanics*, **66**, 492-500.
- Yamanouchi, M., Koizumi, M., Hirai T. and Shiota I. (Editors) (1990). *FGM-90*. Proceedings 1st Int. Symp. Functionally Gradient Materials, FGM Forum, Sendai, Japan.