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# A simple Galerkin boundary element method for three-dimensional crack problems in functionally graded materials

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**Abstract.** This paper presents a Galerkin boundary element method for solving crack problems governed by potential theory in nonhomogeneous media. In the simple boundary element method, the nonhomogeneous problem is reduced to a homogeneous problem using variable transformation. Cracks in heat conduction problem in functionally graded materials are investigated. The thermal conductivity varies parabolically in one or more coordinates. A three dimensional boundary element implementation using the Galerkin approach is presented. A numerical example demonstrates the efficiency of the method. The result of the test example is in agreement with finite element simulation results.

## Introduction

The boundary element method (BEM) is an efficient solution method because only boundary discretization is necessary for numerical implementation [1]. For crack propagating problems, remeshing the evolving geometry is much simpler with boundary element analysis than with finite element analysis. Fracture geometries arise in important technological applications in which the governing relation is based on potential theory. For instance, modeling of subsurface flow must often contend with either fractures in the rock or soil systems having embedded thin layers of different permeability.

In this paper, heat transfer problems in FGMs have been chosen for the numerical implementation, although the techniques can be extended to other types of problems which are governed by potential theory. Recently, a simple three dimensional (3D) BEM has been proposed by Sutradhar and Paulino [2] where nonhomogeneous problems are transformed to known homogeneous problems for a class of variations (quadratic, exponential and trigonometric) of thermal conductivity. The material property can have a functional variation in either one, two or three dimensions. They also extended the simple BEM to transient problems [3]. In the present work, the simple BEM is further extended to crack problems by following the approach by Paulino and Sutradhar [4]. Only parabolic material variation has been considered in this paper although exponential and trigonometric material variations can be dealt with in similar fashion [4]. This paper demonstrates that by simple change in the treatment of the

boundary conditions of an existing homogeneous Laplace equation code, the solution for cracks in nonhomogeneous media with a quadratic material variation can be obtained.

The remainder of this paper is organized as follows. The simple boundary element method is presented in Section 2. Section 3 details the BEM formulation and the fracture algorithm. The treatment of the boundary conditions is described in Section 4. Section 5 demonstrates, by means of simple example calculations, that the methodology works and is efficient, and Section 6 contains some closing remarks.

## The Simple Boundary Element Method

The governing differential equation for a potential function $\phi$ defined on a region $\Omega$ bounded by a surface $\Sigma$, with an outward normal $\mathbf{n}$, can be written as

$$\nabla \bullet (k(x,y,z)\nabla\phi) = 0 \tag{1}$$

where $k(x,y,z)$ is a position dependent material function. Equation (1) is the field equation for a wide range of problems in physics and engineering such as heat transfer, fluid flow motion, flow in porous media, electrostatics and magnetostatics. The boundary conditions of the problem can be of the following types:

$$\phi = \overline{\phi} \quad \text{on } \Sigma_1 \quad (\textit{Dirichlet type}) \quad \text{and} \quad q = -k(x,y,z)\frac{\partial\overline{\phi}}{\partial\mathbf{n}} = \overline{q} \quad \text{on } \Sigma_2 \quad (\textit{Neumann type}) \tag{2}$$

where $\Sigma = \Sigma_1 + \Sigma_2$ for a well-posed problem. The boundary value problem is a Neumann problem if the flux is known on the whole boundary, and the problem is a Dirichlet problem if the potential is known on the whole boundary. Mixed boundary conditions are also frequently encountered: flux is prescribed over some portion of the boundary and potential is prescribed over the complementary portion of the boundary.

By defining the variable

$$v(x,y,z) = \sqrt{k(x,y,z)}\,\phi(x,y,z), \tag{3}$$

Eq. (1) can be rewritten as

$$\nabla^2 v + k'(x,y,z)v = 0 \tag{4}$$

where

$$k' = \frac{\nabla k \bullet \nabla k}{4k^2} - \frac{\nabla^2 k}{2k}. \tag{5}$$

If $k'(x,y,z) = 0$, then Eq. (4) becomes the standard Laplace equation, i.e. $\nabla^2 v = 0$. If $k$ varies only with $z$, then one obtains

$$k(z) = k_0(c_1 + c_2 z)^2, \tag{6}$$

where $c_1$ and $c_2$ are arbitrary constants, and $k_0$ is simply a reference value for $k$. This variation can be readily extended to more dimensions. For instance, the general expression for material property variation in three dimensions are given below,

$$k(x,y,z) = k_0(d_1 + d_2 x + d_3 y + d_4 z + d_5 xy + d_6 yz + d_7 zx + d_8 xyz)^2. \tag{7}$$

Here $d_i (i = 1..8)$ are arbitrary constants. Thus for quadratic variation of the thermal conductivity, by using the variable tranformation, the nonhomogeneous problem is transformed into the standard Laplace problem. By appropriate treatment of the boundary condition due to the variable transformation, a standard code for Laplace equation can be used to solve the problem. As a result, the BEM implementation becomes very simple and avoids dealing with hypersingular kernels associated with the FGM Green's function [5].

## Formulation

### Boundary Integral Equations

The numerical methods employed in the current work use Galerkin techniques for the boundary integral equations (BIE) and the hypersingular boundary integral equations (HBIE). The Galerkin boundary element method has emerged as a powerful numerical method in computational mechanics in recent years. In Galerkin formulation standard continuous $C^0$ elements can be used for evaluation of the hypersingular integrals unlike with the collocation BEM where the smoothness requirement is higher such that Hermite or Overhauser elements are required. The collocation BIE is defined as

$$\mathcal{B}(P) \equiv \phi(P) + \int_\Sigma \left(\frac{\partial}{\partial \mathbf{n}} G(P,Q)\right) \phi(Q) dQ - \int_\Sigma G(P,Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q) dQ \tag{8}$$

and thus for an exact solution $\mathcal{B}(P) \equiv 0$.

The HBIE for the Laplace equation $\nabla^2 \phi = 0$ is an expression for the surface flux $\partial\phi/\partial\mathbf{n} = \nabla\phi \bullet \mathbf{n}$, usually written in the form

$$\frac{\partial \phi}{\partial \mathbf{N}}(P) + \int_\Sigma \phi(Q) \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P,Q)\, \mathrm{d}Q - \int_\Sigma \frac{\partial G}{\partial \mathbf{N}}(P,Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q)\, \mathrm{d}Q = 0\ . \tag{9}$$

Here $\mathbf{n} = \mathbf{n}(Q)$, $\mathbf{N} = \mathbf{N}(P)$ denote the unit outward normal on the boundary surface $\Sigma$, and $P$(source) and $Q$(field) are points on $\Sigma$. The fundamental solution $G(P,Q)$ is usually taken as the point source potential

$$G(P,Q) = \frac{1}{4\pi r}\ , \tag{10}$$

where $R = Q - P$ and $r = \|R\|$ is the distance between $P$ and $Q$. The hypersingular kernel is therefore given by

$$\frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P,Q) = \frac{1}{4\pi} \left( \frac{\mathbf{n} \bullet \mathbf{N}}{r^3} - 3 \frac{(\mathbf{n} \bullet R)(\mathbf{N} \bullet R)}{r^5} \right)\ . \tag{11}$$

The limit to the boundary approach is considered where the limit is taken with the source point $P$ approaching the boundary from *outside* the domain, and as a consequence the 'free term' $\partial\phi(P)/\partial\mathbf{N}$ from Eq. (9) is not present. Thus, Eq. (9) takes the form

$$\mathcal{F}(P) \equiv \int_\Sigma \phi(Q) \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P,Q)\, \mathrm{d}Q - \int_\Sigma \frac{\partial G}{\partial \mathbf{N}}(P,Q) \frac{\partial \phi}{\partial \mathbf{n}}(Q)\, \mathrm{d}Q = 0 \tag{12}$$

with the free term automatically incorporated in the *exterior limit* evaluation of the integral in this equation.

Following standard practice, the boundary potential and flux are approximated in terms of values at element nodes $Q_j$ and shape functions $\psi_j(Q)$, *i.e.*,

$$\phi(Q) = \sum_j \phi(Q_j)\psi_j(Q) \quad \text{and} \quad \frac{\partial \phi}{\partial \mathbf{n}}(Q) = \sum_j \frac{\partial \phi}{\partial \mathbf{n}}(Q_j)\psi_j(Q). \tag{13}$$

In a Galerkin approximation, these shape functions are employed as weighting functions for enforcing the integral equations, and Eq. (9) takes the form

$$\int_\Sigma \psi_k(P)\mathcal{F}(P)\, \mathrm{d}P = 0 \quad \text{and} \quad \int_\Sigma \psi_k(P)\mathcal{B}(P) dP = 0. \tag{14}$$

As a result the Galerkin technique possesses the important *local support* property. This technique is especially suitable to treat corners [6].

**Fracture algorithm**

The Dual BEM approach is used for the crack formulation. The displacement discontinuity approach is not suitable for treating cracks using the simple BEM approach [4]. Consider a body of arbitrary shape B which contains a crack, as shown in Figure 1. The boundary Γ of

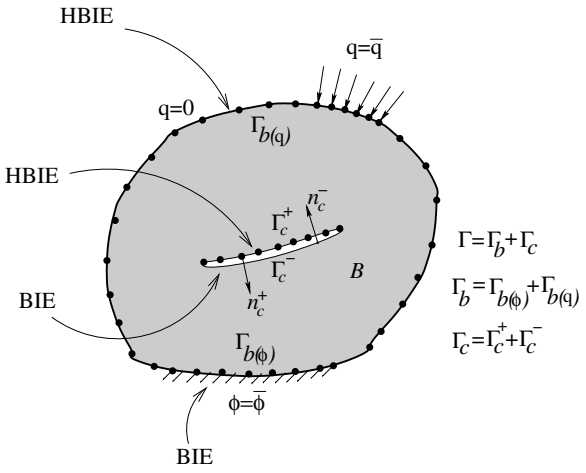



Fig. 1: Configuration of the fracture scheme using the dual BEM approach.

the body B is composed of non-crack boundary $\Gamma_b$ and the crack surface $\Gamma_c$. The portion of the boundary $\Gamma_b$ with prescribed potential is denoted by $\Gamma_{b(\phi)}$, and the portion with prescribed flux boundary is denoted by $\Gamma_{b(\mathbf{q})}$. The crack surface $\Gamma_c$ consists of two coincident surfaces $\Gamma_c^+$ and $\Gamma_c^-$, where $\Gamma_c^+$ and $\Gamma_c^-$ represent the upper and lower crack surfaces respectively. In the dual BEM, the HBIE is used in one crack surface and the BIE is used in the other crack surface. The outward normals to the crack surfaces, designated by $n_c^+$ and $n_c^-$ are oriented in opposite directions and at any point on the crack $n_c^- = -n_c^+$. As a consequence, only one side of the crack surface needs to be discretized. We write, in a $3 \times 3$ block matrix form, a dual equation Galerkin approximation for a fracture geometry. Specifically, the first block row will represent the outer, or non-crack, boundary equations, and the equation for a particular node, as per the usual Galerkin procedure, is chosen according to the prescribed boundary data. In accordance with the dual BEM approach, the second and third rows will denote, respectively, the hypersingular and standard equations written on the crack surface. With these definitions, the equations take the form

$$\begin{pmatrix} h_{11} & h_{12} & -h_{12} \\ h_{21} & h_{22} & -h_{22} \\ h_{31} & h_{32} & -h_{32} \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & g_{12} \\ g_{21} & g_{22} & g_{22} \\ g_{31} & g_{32} & g_{32} \end{pmatrix} \begin{pmatrix} \hat{\Omega}_1 \\ \Phi_2^{\mathbf{n}} \\ \Phi_3^{\mathbf{n}} \end{pmatrix} \tag{15}$$

The vector of unknowns on the non-crack boundary can be a mixture of potential and flux, and is therefore denoted by $\Omega_1$. The corresponding vector of prescribed boundary values is indicated by $\hat{\Omega}_1$. On the fracture, $\Phi$ represents the vector of unknown potential values, $\Phi^{\mathbf{n}}$ the specified flux, and the subscripts $\{2, 3\}$ label the two sides of the crack. The matrix $\mathbf{H}$ on the left therefore multiplies the vector of unknowns, and the right hand side consists of known quantities.

### Treatment of boundary conditions

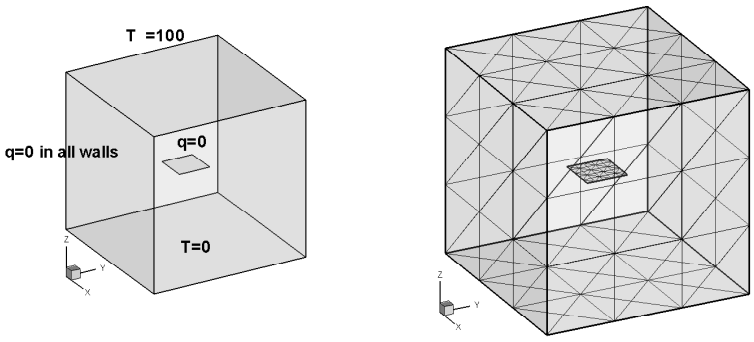
In order to solve the boundary value problem based on the modified variable $v$, the boundary conditions of the original problem have to be incorporated in the modified boundary value problem. Thus for the modified problem, the Dirichlet and the Neumann boundary conditions given by Eq. (2), change as follows:

$$v = \sqrt{k}\,\overline{\phi} \quad \text{on } \Sigma_1 \quad \text{and} \quad \frac{\partial v}{\partial \mathbf{n}} = \frac{1}{2k}\frac{\partial k}{\partial \mathbf{n}} v - \frac{\overline{q}}{\sqrt{k}} \quad \text{on } \Sigma_2 \tag{16}$$

Notice that the Dirichlet boundary condition of the original problem is affected by the factor $\sqrt{k}$. Moreover, the Neumann boundary condition of the original problem changes to a mixed boundary condition or Robin boundary condition. This later modification is the only major change on the boundary value problem.

### Numerical example

A test example is presented in order to verify the present formulation. A square crack with zero surface flux is embedded inside a unit cube ($L = 1$) with prescribed constant temperature on two sides. The problem of interest and corresponding BEM mesh is shown in Figure 2(a) and Figure 2(b), respectively. The top surface of the cube at $[z = 1]$ is maintained at a temperature of $T = 100$ units while the temperature in the bottom at $[z = 0]$ is zero. The remaining four faces are insulated (zero normal flux).




(a) (b)

Fig. 2: (a) Geometry and boundary condition of the problem with a square crack inside a cube. (b) The BEM mesh consists of 108 quadratic triangular elements on the boundary of the cube, and 100 quadratic triangular elements on the crack surface

The quadratic variation of the thermal conductivity $k(x, y, z)$ is defined as

$$k(x, y, z) = k(z) = 5(1 + \beta z)^2, \tag{17}$$

in which $\beta$ is the nonhomogeneity parameter. Here the solution of the problem is verified using the commercially available software ABAQUS [7] using the user-defined subroutine UMATHT [2]. The FEM mesh consists of 1000 20-noded brick elements (quadratic). The temperature profiles on the upper and lower crack surfaces along the center of the square crack for different values of $\beta$ are plotted and compared with the FEM solution in Figure 3. The results are in good agreement including the case when the nonhomogeneity parameter is relatively high ($\beta = 10$).

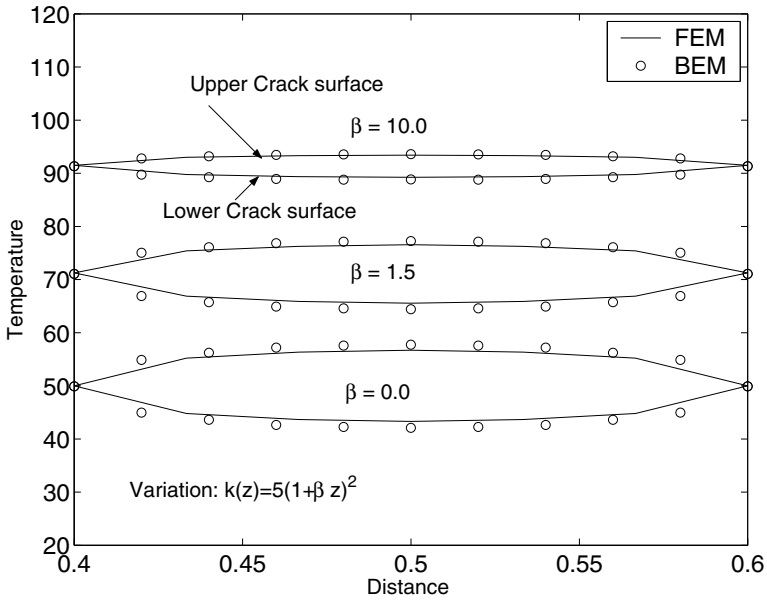



Fig. 3: Temperature profile of the upper and lower crack surface along the center of the crack

## Conclusion

This paper presents a simple boundary element method to solve cracks problems governed by potential theory in nonhomogeneous media. For quadratic material variation, by simple modification of the boundary conditions, a standard code for Laplace equation for homogeneous media is used to solve the crack problem. The dual boundary element method has been chosen for the crack formulation. A numerical example is presented to verify the proposed formulation.

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