# Geometrical Aspects of Lateral Bracing Systems: Where Should the Optimal Bracing Point Be?

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**Abstract:** This paper describes structural optimization formulations for trusses and applies them to size and geometrically optimize lateral bracing systems. The paper provides analytical solutions for simple and limit cases and uses numerical optimization for cases where analytical solutions are not available. The analysis is based on small displacements, constant or no connection costs, static loads, and elastic behavior. This work provides guidance to improve the common engineering practice of locating the bracing point at the middle or at the top of the bay/story panel. **DOI: 10.1061/(ASCE)ST.1943-541X.0000956.** © *2014 American Society of Civil Engineers.*

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## Introduction

Structural optimization has a long history of applications with buildings. Lateral bracing systems are often used to provide lateral stiffness to buildings. These may span one or several bays, single or several stories high (Fig. 1).

Topology optimization, a density-based approach to optimization, has been applied to optimal bracing system problems with various conclusions (Neves et al. 1995; Mijar et al. 1998; Allahdadian et al. 2012; Stromberg et al. 2012), including the finding that the optimal bracing point is not always at midheight. Topology optimization is a powerful technique, but the interpretation of the results and subsequent member sizing is not straightforward. An alternative approach is the *ground structure* method (Dorn et al. 1964; Ben-Tal and Bendsøe 1993; Sokół 2011). The ground structure method results in solutions with a large number of members that asymptotically converge to the theoretical optimum for problems with known (analytical) solutions (Michell 1904; Hemp 1973). The approach used in this work simultaneously optimizes truss geometry (node locations) and member sizes (Felix and Vanderplaats 1987; Hansen and Vanderplaats 1988; Lipson and Gwin 1977). The structural connectivity is fixed, and no members are added or removed. Zero cross-sectional areas are a special case which cannot be allowed because of a known discontinuity in member's stresses (Kirsch 1990). However, a very small cross-sectional area has an effect similar to removing the bar. The bracing system is modeled as an elastic truss with static loads and small displacements. The connection costs are considered to be constant or null. This indicates that the cost to connect members with different cross-sectional areas and angles is assumed not to change.

The manuscript is organized as follows: "Four Complementary Formulations" section describes the optimization formulations used and highlights the properties of each. "Formulation Equivalency" section discusses the analogies between all formulations and points out the conditions for these to be equivalent. A single brace in two and three dimensions is analytically optimized in "Single-Brace Analysis" section. Extension to multiple braces and bays with the inclusion of vertical loads are analyzed in "Multiple Bays/Stories" section. Finally, conclusions and findings are summarized in the "Conclusions" section.

## Four Complementary Formulations

The question of where is the optimal bracing point is actually a subjective one: the math and physics involved are exact, but to define an optimum, a benchmark or measure needs to be chosen. The options for the objective function (measure) are limitless, but only a few are important to the practicing engineer, and even less are used in practice. The following four objectives are explored in the present work:

- Minimize the volume;
- Minimize the load-path;
- Minimize compliance; and
- Minimize displacements.

These objectives require constraints in order for the solution to be bounded and unique. In all cases, the structural internal–external force equilibrium is enforced, either by $\mathbf{Ku} = \mathbf{f}$, or an equivalent expression. The problems considered in this paper are elastic lateral bracing systems with small deformations, and no self-weight or connection costs (or constant). Nonetheless, some of the concepts and conclusions can be extended to a wider range of options and constraints (e.g., buckling or frequencies). Each formulation has some properties, advantages, and disadvantages. A brief discussion of these will be presented in the following section.

### *Volume Formulation*

An intuitive formulation for a practicing engineer is to minimize the volume of structural material. Typically, the cost of a structure is proportional to its weight. Thus, minimizing the total weight of the structure minimizes its cost (when the connection and joint costs

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(a)


(b)


(c)

**Fig. 1.** Examples of single and multiple bays braced buildings: (a) John Hancock Center, Chicago (SOM, Ezra Stoller, © Esto, with permission); (b) Alcoa Building, San Francisco (SOM, © Mak Takahashi, with permission); (c) building in Presidente Riesco Ave., Santiago, Chile (image by Tomás Zegard)

are constant). A stress constraint prevents the member's cross-sectional areas from approaching zero. The *minimum volume formulation* is as follows:

$$\begin{array}{ll} \min_{\mathbf{A},\mathbf{x}} & V = \mathbf{A}^T\mathbf{L} \\ \text{s.t. } \sigma_c \le \sigma_i \le \sigma_t & \forall\ i = 1\dots n_e \\ \mathbf{A} \ge \mathbf{0} & \\ \text{with} & \mathbf{K}\mathbf{u} = \mathbf{f} \end{array} \quad (1)$$

where $\mathbf{L}$ and $\mathbf{A}$ are column vectors with the member lengths and cross-sectional areas, respectively, $\mathbf{x}$ is a vector with the joint coordinates, $\mathbf{K}$ the stiffness matrix, $\mathbf{u}$ the nodal displacements, $\mathbf{f}$ the nodal force vector, $\sigma_i$ the member stresses, and $\sigma_c$ and $\sigma_t$ the stress limits on compression and tension, respectively.

The volume formulation is arguably the most intuitive and common (Michell 1904; Hemp 1973; Achtziger 2007). The strength of this formulation is that dealing with different stress limits for compression and tension is simple and straightforward. Euler buckling constraints, for example, can also be included in the formulation with an additional stress constraint

$$-\frac{\pi^2 E}{(\kappa L_i/r_i)^2} \le \sigma_i \quad \forall\ i = 1\dots n_e \quad (2)$$

where $E$ is the modulus of elasticity, $\kappa$ is the column effective length factor$r$ is the member's radius of gyration, $r = \sqrt{I/A}$ and $I$ is the member's area moment of inertia. Because Euler's buckling criterion overestimates the buckling strength of structural members, better criteria and safety factors should be considered (AISC 2011).

## Load-Path Formulation

The *load-path formulation*, also called performance index or Michell's number (Lev 1981; Sokół 2011; Mazurek et al. 2011), has equal treatment of compression and tension members and takes into account the distance these forces have to travel through the members to the supports.

$$\begin{array}{ll} \min_{\mathbf{A},\mathbf{x}} & Z = \sum_i (N_{(t)i} - N_{(c)i})L_i = \sum_i |N_i| L_i \\ \text{s.t.} & \sum_i A_i L_i \le \bar{V} \\ \mathbf{A} \ge \mathbf{0} & \\ \text{with} & \mathbf{K}\mathbf{u} = \mathbf{f} \end{array} \quad (3)$$

where $N_i$ stands for the axial force in member $i$, and $\bar{V}$ is a prescribed limit on the volume.

The load-path formulation has the difficulty of an absolute value in the objective function (alternatively, the compression and tension loads can be split into two positive variables, thus making the objective a linear function). The biggest advantage of this formulation is that for statically determinate trusses, the axial load does not depend on the cross-sectional areas. In other words, the member sizing problem is decoupled from the geometry, reducing the design variables from $(n_d \times n_n + n_e)$ to just $(n_d \times n_n)$, where $n_d$, $n_n$, and $n_e$ are the problem's dimension, number of nodes, and number of elements, respectively. The formulation can be extended to treat compression and tension differently by introducing a parameter $\gamma = -\sigma_t/\sigma_c$ and rewriting the objective for Eq. (3) to include this penalization parameter (Sokół 2011)

$$\begin{array}{ll} \min_{\mathbf{A},\mathbf{x}} & Z^\star = \sum_i (N_{(t)i} - \gamma N_{(c)i})L_i \\ \text{s.t.} & \sum_i A_i L_i \le \bar{V} \\ \mathbf{A} \ge \mathbf{0} & \\ \text{with} & \mathbf{K}\mathbf{u} = \mathbf{f} \end{array} \quad (4)$$

Including a buckling constraint on the other hand is not straightforward.

## Compliance Formulation

Recent works on structural optimization center around stiffness measures for the structure's performance (Bendsøe and Sigmund 2003). A typical formulation for this purpose is the *compliance formulation*: an energy measure related to maximizing the stiffness of the structure for given loads

$$
\begin{array}{lll}
\min_{\mathbf{A},\mathbf{x}} & C = \mathbf{u}^T\mathbf{K}\mathbf{u} = \mathbf{u}^T\mathbf{f} & \\
\text{s.t.} & \sum_i A_i L_i \le \bar{V} & \\
\mathbf{A} \ge \mathbf{0} & & \\
\text{with} & \mathbf{K}\mathbf{u} = \mathbf{f} &
\end{array} \tag{5}
$$

The main advantage of the compliance formulation is that the objective function is self-adjoint: when computing the sensitivity, the solution to the adjoint problem is known (Giles and Pierce 2000), making this formulation computationally attractive. Stress and buckling constraints can be implemented, but are, again, not straightforward.

### Displacement Formulation

Displacement objective functions are used typically in minimizing the maximum displacement (i.e., top story of a building or interstory drift). Considering a single displacement $\Delta = |u_j|$ (Fig. 2), the *displacement formulation* takes form:

$$
\begin{array}{lll}
\min_{\mathbf{A},\mathbf{x}} & \Delta = u_j & \\
\text{s.t.} & \sum_i A_i L_i \le \bar{V} & \\
\mathbf{A} \ge \mathbf{0} & & \\
\text{with} & \mathbf{K}\mathbf{u} = \mathbf{f} &
\end{array} \tag{6}
$$

The displacement formulation is simple and has the advantage of possessing a direct physical meaning for the engineer. It has similar characteristics to the compliance formulation, but it is not self-adjoint.

## Formulation Equivalency

The four objectives presented in Eqs. (1), (3), (5), and (6) may seem different, but under *typical* conditions, these formulations will result in the same optimal brace point location. In other words, the stiffest structure in a direction, the least compliant structure, the least weight structure, and the one with the smallest load-path all have the same optimum solution. The focus of this section is to explain when this occurs and what happens when some of the conditions are violated.

Optimal structures in material efficiency tend to be fully stressed (Michell 1904; Lev 1981; Topping 1983). The proof is intuitive with the formulation in Eq. (1): for a structure that is not fully stressed, a reduction of the cross-sectional areas will decrease the objective without violating the constraints. With no displacement, buckling, or symmetry constraints (manufacturing constraints), the optimal design is often fully stressed. This statement is not true for the case of multiple loading conditions and is not considered in the present work. The fully stressed condition leads to the stress–ratio method [that relates to Michell's solutions (Lev 1981)], where the cross-sectional areas are updated at each iteration as

$$
A_{i(\text{new})} = \max\left(\frac{\sigma_c}{\sigma_i}, \frac{\sigma_t}{\sigma_i}\right) A_i \tag{7}
$$

Optimal structures have a tendency to be statically determinate, but as (Schmidt 1962) correctly concluded, *a statically indeterminate form could sometimes give a lighter structure than a statically determinate one*, although this situation is only valid with multiple loading scenarios.

### Load-Path to Volume

The connection between volume and load-path was pointed out by Michell (Lev 1981). If the structure is fully stressed, then there exist limit stress values $\sigma_c$ and $\sigma_t$ such that

$$
N_{(c)i} = \sigma_c A_i \qquad N_{(t)i} = \sigma_t A_i \tag{8}
$$

With these, the objective in Eq. (4) becomes

$$
Z^{\star} = \sum_i \left[\sigma_t A_i|_{(t)} - \left(-\frac{\sigma_t}{\sigma_c}\right)\sigma_c A_i|_{(c)}\right] L_i = \sigma_t \sum A_i L_i = \sigma_t \mathbf{A}^T\mathbf{L} \tag{9}
$$

and Eq. (3) is a subcase of the previous. Therefore, if the structure is fully stressed, minimizing the performance index is equivalent to the formulation in Eq. (3) (multiplied by a constant).

### Compliance to Load-Path

The compliance problem in Eq. (5) with a single load $P$, independent of the variables and displacements, can have the objective simplified as

$$
C = \mathbf{u}^T\mathbf{f} = P\Delta \tag{10}
$$

where $\Delta$ is the displacement in the direction of the load $P$. For a truss with a single point load, the principle of work and energy (Baker 1992) states that

$$
P\Delta = \sum_i \frac{N_i^2 L_i}{A_i E} = \sum_i \frac{|N_i|^2 L_i}{A_i E} \tag{11}
$$

Including the fully stressed assumption with $|N_i|/A_i = \bar{\sigma}$, results in

$$
C = \sum_i \frac{|N_i|^2 L_i}{A_i E} = \frac{\bar{\sigma}}{E} \sum_i |N_i| L_i \tag{12}
$$

thus making it equivalent to the formulation in Eq. (3) (multiplied by a constant).

### Displacement to Compliance

If the displacement problem from Eq. (6) has a single constant load $P$ (independent of design variables and displacements), and the displacement $\Delta$ being minimized is in the direction of the force $P$, then

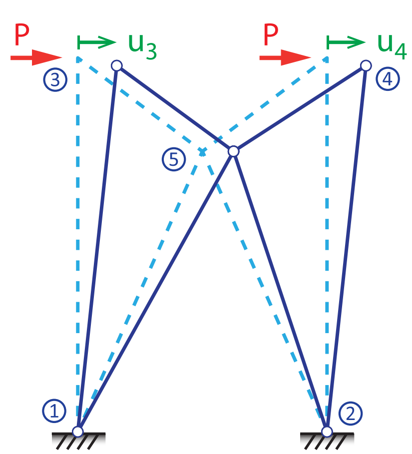


**Fig. 2.** Displacements of a lateral bracing system due to a load $P$; because of symmetry, $u_3 = u_4 = \Delta$

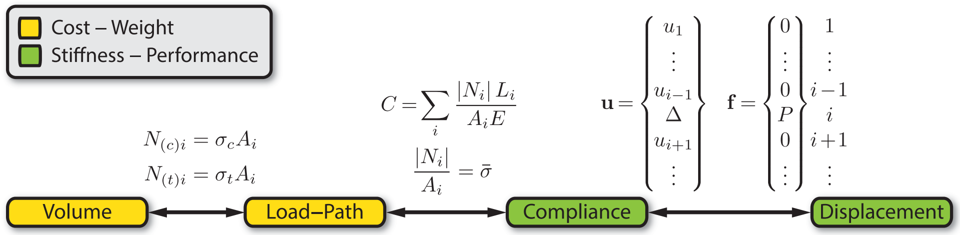


**Fig. 3.** Equivalency requirements between formulations

$$\min \Delta = \min P\Delta = \min \mathbf{u}^T \mathbf{f} \tag{13}$$

leading to the formulation in Eq. (5). If the objective function in Eq. (6) is a linear combination of several displacements of the truss, then the equivalency is preserved *if and only if* the loads in the displacement directions are in the same ratio as the coefficients in the linear combination. If the aforementioned condition is not met, then the optimal design, which minimizes some displacement $\Delta$, is not fully stressed.

The relationship and requirements for equivalency between formulations are then summarized in Fig. 3. The requirements for equivalency shed light on when these connections may be broken. In particular, previous studies have already shown cases where the minimum weight and fully stressed design are not equivalent (Kicher 1966; Razani 1965).

## Single-Brace Analysis

The following analytical derivations for two-dimensional braces extend the conclusions derived by (Stromberg et al. 2012), where the optimal bracing point was found to be at $x = 0.75H$ for single-bay multiple-story braces optimized for compliance. In addition, Stromberg has an application of these optimal braces in a real design (Stromberg et al. 2012). The findings in this section form the basis for more general bracing rules in the following sections. The bracing point location in Fig. 4 will be optimized using formulations discussed earlier.

Using the symmetry condition, only half of the brace needs to be analyzed. Consideration of different values for $\sigma_t$ and $\sigma_c$ is not important because the direction of lateral loads in buildings is uncertain, therefore a single value $\bar{\sigma}$ is used to limit positive and negative stresses.

The compliance of the structure is

$$C = \frac{4P^2}{EB^2}\left(\frac{L_1^3}{A_1} + \frac{L_2^3}{A_2} + \frac{L_3}{A_3}y^2\right) \tag{14}$$

where $B$ denotes the bay width (Fig. 4). The volume for the half-structure is

$$V = \sum_{i=1}^{3} A_i L_i = A_1 L_1 + A_2 L_2 + A_3 H \tag{15}$$

where $H$ denotes the bay height (Fig. 4). The derivatives of the member lengths $L$ with respect to variable $y$ are:

$$\frac{dL_1}{dy} = \frac{y}{L_1} \qquad \frac{dL_2}{dy} = \frac{y-H}{L_2} \qquad \frac{dL_3}{dy} = 0 \tag{16}$$

and derivatives of the axial loads with respect to the variable $y$ are

$$\frac{dN_1}{dy} = \left(\frac{-2P}{B}\right)\left(\frac{y}{L_1}\right) \qquad \frac{dN_2}{dy} = \left(\frac{2P}{B}\right)\left(\frac{y-H}{L_2}\right) \qquad \frac{dN_3}{dy} = \frac{2P}{B} \tag{17}$$

where $N$ is the axial load on the member, $P$ is the load at the top, $B$ and $H$ are the width and height of the brace, respectively, and $x$ and $y$ locate the bracing point.

The three-dimensional case is symmetric with respect to the $x_1x_3$ and $x_2x_3$ planes (lateral forces in a building are often considered in one direction at a time); therefore, only one quarter of the brace needs to be solved: corner columns participate in both loading directions. Fig. 5 illustrates the bracing system loaded in the

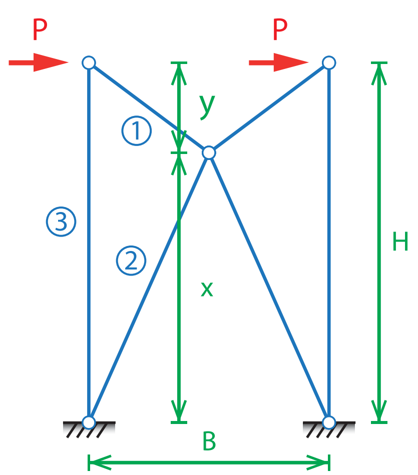


**Fig. 4.** Two-dimensional lateral bracing system

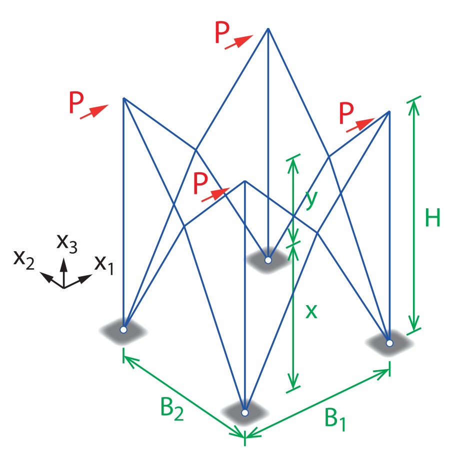


**Fig. 5.** Three-dimensional lateral bracing system

plane $x_1x_3$. If the base is square $B_1 = B_2$, the resulting optimal braces in both planes are the same. This can be interpreted as the *cost* of the diagonals being twice that of the two-dimensional case: a cost (or multiplicity) variable $\alpha = 1$ for the two-dimensional case and $\alpha = 2$ for the three-dimensional case will be used.

### Minimum Volume Optimal

The Lagrangian for the minimum volume objective is:

$$\begin{aligned}\mathcal{L} = \alpha A_1 L_1 + \alpha A_2 L_2 + A_3 H + \lambda_{11}(-A_1\bar{\sigma} - N_1) \\ + \lambda_{12}(-A_1\bar{\sigma} + N_1) + \cdots \\ \lambda_{21}(-A_2\bar{\sigma} - N_2) + \lambda_{22}(-A_2\bar{\sigma} + N_2) + \cdots \\ \lambda_{31}(-A_3\bar{\sigma} - N_3) + \lambda_{32}(-A_3\bar{\sigma} + N_3)\end{aligned} \tag{18}$$

which has a single feasible optimum at

$$\begin{aligned}\lambda_{11} = \alpha L_1/\bar{\sigma} \qquad \lambda_{12} = 0 \qquad \lambda_{21} = 0 \\ \lambda_{22} = \alpha L_2/\bar{\sigma} \qquad \lambda_{31} = 0 \qquad \lambda_{32} = H/\bar{\sigma}\end{aligned} \tag{19}$$

Therefore, the optimal bracing point is located at

$$x = \frac{2\alpha + 1}{4\alpha} H \qquad y = \frac{2\alpha - 1}{4\alpha} H \tag{20}$$

### Load-Path Optimal

The Lagrangian for the minimum load-path objective (introducing fictitious equivalent forces in the three-dimensional case to enforce symmetry, i.e., taking $\alpha = 2$) is

$$\mathcal{L} = \alpha \frac{2PL_1}{B} L_1 + \alpha \frac{2PL_2}{B} L_2 + \frac{2Py}{B} H \tag{21}$$

with no constraint because the structure is statically determinate. The optimal bracing point is located at

$$x = \frac{2\alpha + 1}{4\alpha} H \qquad y = \frac{2\alpha - 1}{4\alpha} H \tag{22}$$

### Compliance Optimal

The Lagrangian for the minimum compliance objective (introducing fictitious equivalent forces in the three-dimensional case to enforce symmetry, i.e., taking $\alpha = 2$) is

$$\mathcal{L} = \frac{4P^2}{EB^2}\left(\frac{L_1^3}{A_1} + \frac{L_2^3}{A_2} + \frac{L_3}{A_3}y^2\right) + \lambda(\alpha A_1 L_1 + \alpha A_2 L_2 + A_3 H - \bar{V}) \tag{23}$$

The optimum is found for

$$\lambda = \frac{4P^2 y^2}{EB^2 A_3^2} \tag{24}$$

and the optimal bracing point is at

$$x = \frac{2\sqrt{\alpha} + 1}{4\sqrt{\alpha}} H \qquad y = \frac{2\sqrt{\alpha} - 1}{4\sqrt{\alpha}} H \tag{25}$$

### Displacement Optimal

The Lagrangian for the minimum displacement objective (introducing fictitious equivalent forces in the three-dimensional case to enforce symmetry, i.e., taking $\alpha = 2$) is

$$\mathcal{L} = \Sigma \frac{N_i^2 L_i}{A_i E} + \lambda(\alpha A_1 L_1 + \alpha A_2 L_2 + A_3 H - \bar{V} \tag{26}$$

The stress values in the members as a function of $\lambda$ are:

$$\sigma_1 = \sqrt{PE\lambda} \qquad \sigma_2 = \sigma_3 = \sqrt{\alpha PE\lambda} \tag{27}$$

and the optimal bracing point is found at:

$$x = \frac{2\sqrt{\alpha} + 1}{4\sqrt{\alpha}} H \qquad y = \frac{2\sqrt{\alpha} - 1}{4\sqrt{\alpha}} H \tag{28}$$

### Result Summary

Results obtained with the four objectives discussed are compared in Table 1.

The methods can be grouped (or characterized) by objectives: weight/cost and stiffness/performance. In two-dimensional braces, all four formulations result in the same solution. In the three-dimensional case, however, optimizing for weight/cost or for stiffness/performance results in different bracing points. It is important to note that the member stresses will not be constant for the 3D performance-optimized case ($\sigma_2 = \sigma_3 = \sqrt{\alpha}\sigma_1$): the optimal design and fully stressed condition don't match (Kicher 1966; Razani 1965). In other words, given the symmetry constraints in two axes, the fully stressed condition is broken, and thus, the equivalency between all formulations cannot be achieved (Fig. 3).

The decrease in the objective function for the optimal bracing point (as in Table 1), compared to the midpoint brace ($x = 0.5H$), depend on the aspect ratio of the brace. As expected, the improvement in the compliance objective (displacement decrease) with a volume constraint, is the square of the improvement in the volume objective with stress constraints, for a single two-dimensional brace:

$$\left.\frac{V(x = 0.75H, \mathbf{A}^{\text{opt}})}{V(x = 0.5H, \mathbf{A}^{\text{opt}}_{x=0.5H})}\right|_{2D} = \frac{4B^2 + 7H^2}{4B^2 + 8H^2} \tag{29}$$

$$\left.\frac{C(x = 0.75H, \mathbf{A}^{\text{opt}})}{C(x = 0.5H, \mathbf{A}^{\text{opt}}_{x=0.5H})}\right|_{2D} = \left(\frac{4B^2 + 7H^2}{4B^2 + 8H^2}\right)^2 \tag{30}$$

However, this is not true for the (symmetry-constrained) three-dimensional case. The optimization for loads in two different directions results in an optimal structure that is not fully stressed for each of the loads. The ratio of the objectives for the three-dimensional problem is as follows:

$$\left.\frac{V(x = 0.625H, \mathbf{A}^{\text{opt}})}{V(x = 0.5H, \mathbf{A}^{\text{opt}}_{x=0.5H})}\right|_{3D} = \frac{16B^2 + 23H^2}{16B^2 + 24H^2} \tag{31}$$

**Table 1.** Optimal Bracing Point Location in Two and Three Dimensions with Different Objectives

| | Weight-Cost | | Performance | |
|---|---|---|---|---|
| Height $x$ | Volume | Load-Path | Compliance | Displacement |
| 2D | $0.75H$ | $0.75H$ | $0.75H$ | $0.75H$ |
| 3D | $0.625H$ | $0.625H$ | $0.6768H$ | $0.6768H$ |

**Table 2.** Single-Brace Improvement in the Objective Function for the Optimal Bracing Compared to a Midheight Bracing Point

| Improvement over $x = 0.5H$ | Weight-cost | | Performance | |
|---|---|---|---|---|
| | $B = H$ (%) | $1.5B = H$ (%) | $B = H$ (%) | $1.5B = H$ (%) |
| 2D | 8.33 | 10.23 | 15.97 | 19.41 |
| 3D | 2.50 | 3.21 | 9.02 | 11.28 |

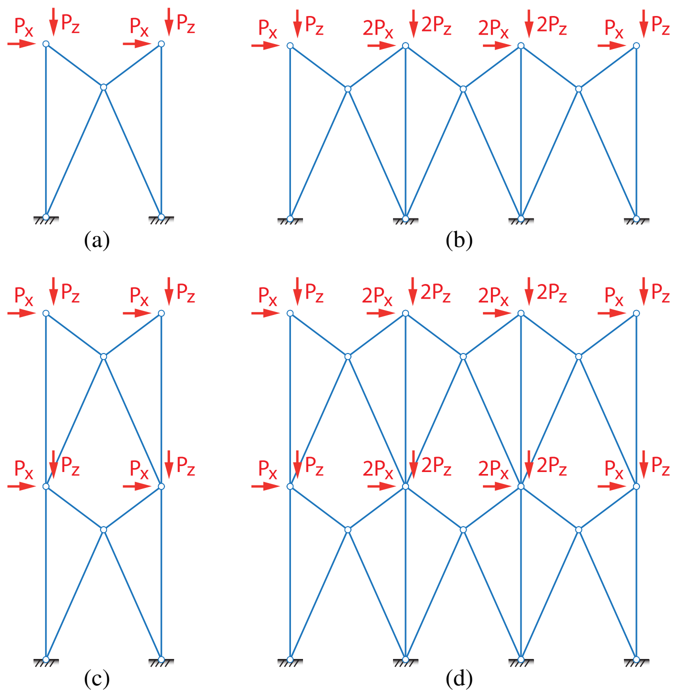


**Fig. 6.** Two-dimensional bracing systems consisting of multiple bays and stories with horizontal and vertical loads: (a) $1 \times 1$ brace; (b) $1 \times 3$ brace; (c) $2 \times 1$ brace; (d) $2 \times 3$ brace

$$\frac{C(x = 0.6768H, \mathbf{A}^{\mathrm{opt}})}{C(x = 0.5H, \mathbf{A}^{\mathrm{opt}}_{x=0.5H})}\bigg|_{3D} = \left[\frac{8B^2 + (7 + 4\sqrt{2})H^2}{8B^2 + (8 + 4\sqrt{2})H^2}\right]^2 \quad (32)$$

The comparison of the improvement in the objective function for a single brace, for the cases of $(B = H)$ and $(1.5B = H)$, is summarized in Table 2.

## Multiple Bays/Stories

The previous section deals with a single brace loaded laterally, but in building applications, the bracing system may span several stories or bays (side by side) as in Fig. 6. Additionally, the braces could also be loaded vertically by a load $P_z$, but this load is only taken downwards as opposed to the lateral $P_x$ that may act in any direction.

The best possible solution has a different optimal bracing point for each bay and story. Nonetheless, a unique bracing point for the whole bracing system is desirable for construction and aesthetic reasons. In this section, the optimal bracing point for several different cases is found assuming a single optimal bracing point height for all bays. For statically indeterminate trusses, the cross-sectional areas must be included in the optimization.

### *Single Bay—Multiple Stories*

Single bay braces (several stories high) as in Figs. 6(a and c) are statically determinate, and the optimal bracing point using the load-path formulation requires no member sizing. The addition of vertical loads does not have an effect on the optimal bracing point location. Vertical loads transfer directly to the base through the columns, and therefore affect only the sizes of the columns. The optimal is found to be at $x = 0.75H$ using all of the presented formulations.

### *Limit Case of Infinite Bays—Single Story*

The bracing system is statically indeterminate in this case, and the cross-sectional areas must be included in the optimization. Taking advantage of the symmetry, the analysis can be done as in Fig. 7 for a single braced column. The optimum bracing points are found to be at $x = 0.50H$ using all of the presented formulations. The columns get sized with a zero cross-sectional area, and the addition of vertical loads does not change the location of the optimum bracing points. This can be interpreted as a shear transfer problem across stories, and as expected, the optimal solution is perfectly straight diagonal braces as depicted in Fig. 8.

### *Multiple Bays—Multiple Stories*

The solution to this problem [Fig. 6(d)] is not trivial because of the large number of variables introduced by the cross-sectional areas and the nonlinearity of the problem, and therefore the problem is solved numerically. The stress ratio method introduced in Eq. (7) tends to drive the solution to a local minimum for large enough problems. The cross-sectional areas are

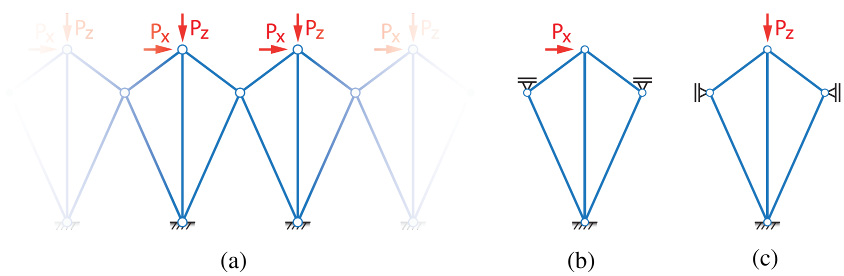


**Fig. 7.** Two-dimensional single-story bracing system with infinite bays: (a) brace with loads; (b) load and boundary conditions for horizontal load; (c) loads and boundary conditions for vertical load

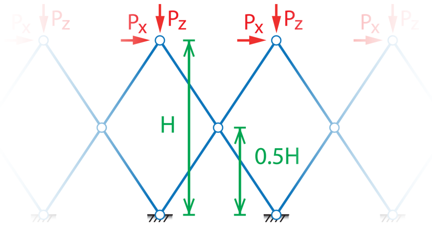


**Fig. 8.** Two-dimensional optimal single story bracing system with infinite number of bays

introduced into the optimization along with the bracing point variable, and the optimization is done using the interior-point method (Karmarkar 1984; Wright 2005). Based on tributary areas, the lateral and vertical loads double at internal nodes. For low levels of vertical loads, the optimal solution lies between the previous solutions $x = 0.50H$ and $x = 0.75H$, and these values act like upper and lower bounds of all possible solutions. This is proven false when a bracing system is subjected to high vertical loads and multiple stories, as the solution may fall below $x = 0.5H$ and slowly converge to it from below. From Fig. 9, it can be inferred that an increase on the number of bays or on the vertical loads will drive the solution closer to $x = 0.5H$. In general, the bracing point location decays asymptotically towards $x = 0.5H$. As an example, the solution for the case with no vertical loads $P_z = 0.0$, and a single story is precisely described by

$$x_{(1-\text{story},\,P_z=0.0)} = \left(\frac{0.25}{\text{bays}} + 0.5\right)H \tag{33}$$

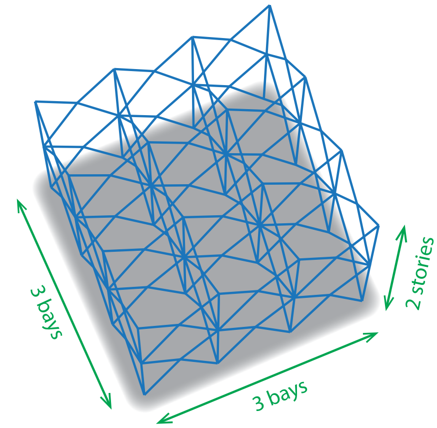


**Fig. 10.** Three-dimensional brace with three bays and two stories (potential uses: stage supports, machine supports, mechanical floors, warehouses, etc.)

and on the limit of infinite bays converges to $x = 0.5H$ as predicted.

### *Three-Dimensional Case*

Three-dimensional braces composed of several bays have their use, for example, in mechanical floors of buildings and machine supports (Fig. 10). Based on tributary areas and compared to corner nodes, loads double at edge nodes and quadruple at interior nodes. The solutions are similar to the two-dimensional case, but with different upper bounds: for cost/weight optimization, the upper bound is at $x = 0.625H = 5/8H$, and for stiffness/performance, the upper bound is at $x = 0.6768H$ for $P_z = 0.0$, but this upper limit gets reduced when increasing the number of stories or vertical load (Fig. 11).

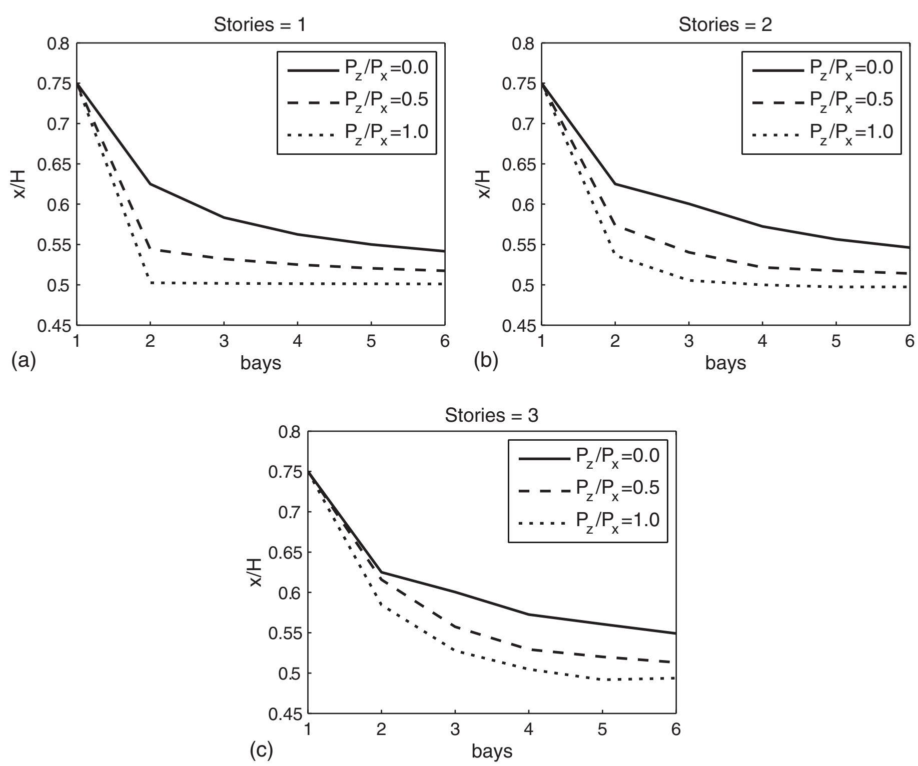


**Fig. 9.** Multiple bays—multiple stories optimal bracing locations in two dimensions: (a) one story high; (b) two stories high; (c) three stories high

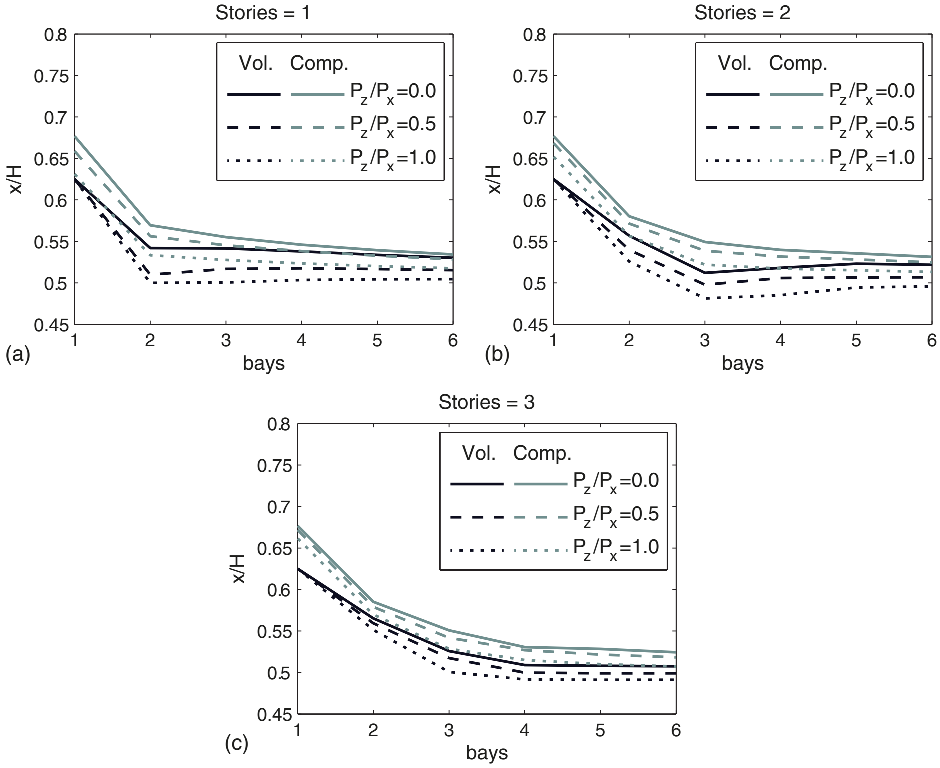


**Fig. 11.** Multiple bays—multiple stories optimal bracing locations in three dimensions: (a) one story high; (b) two stories high; (c) three stories high

## Conclusions

Optimal structures, and consequently bracing systems, are said to be optimal in accordance to the objective function used. Four typical formulations (using different objective functions) for structural optimization were presented. Similarities, differences, and connections between them were highlighted and explored. Compared to geometrical optimization, the member sizing problem is better known and understood. Therefore, the focus of this work was placed on the bracing point location, leaving the member sizing to be done a posteriori (although, in obtaining the bracing points for statically indeterminate braces, the members were sized).

The optimal bracing point location in two dimensions is the same regardless of the objective function (formulation) used. In most cases, the optimal bracing point is found to be in a feasible region delimited by $x = 0.50H$ and an upper limit or bound. The optimal bracing point location may fall below $x = 0.50H$ if the structure is subjected to high vertical loads. If the bracing system is subjected to vertical loads, and/or if the bracing system is composed of multiple bays, then the optimal bracing point location approaches the lower limit. The upper limit is $x = 0.75H = 3/4H$ for the 2D case, regardless of the objective function. For the three-dimensional case, the upper limit is $x = 0.625H = 5/8H$ for cost/weight optimized structures, and for the stiffness/performance case, the upper limit is $x = 0.6768H$ or a lower value if subjected to vertical loads.

The symmetry constraint in three-dimensional trusses breaks the fully stressed condition for structures optimized for stiffness/performance. In other words, material is not being used efficiently (or at full capacity), but the resulting structure will still be the stiffest.

The purpose of this paper is to provide insight for the initial guess of cost-effective or high-performing lateral braces. These findings can aid the engineer in the initial stages of design and also provide guidance to improve common engineering practices that often put the bracing point at the middle $x = 0.5H$, or worse, at the top $x = H$. A natural extension of his work is the consideration of the dynamic behavior of the structure.

## Appendix. Extension to Nonsquare Three-Dimensional Braces

The analysis of three-dimensional braces is limited to square bases to narrow the scope of the paper. However, the authors extended this work to nonsquare braces. As an example, the optimal bracing point for a single nonsquare three-dimensional brace, optimized for volume/cost considering different loads in each direction $P_1$ and $P_2$ is:

$$x = \frac{H}{4}\left[2 + \frac{\max(\lambda_1, \lambda_2)}{\lambda_1 + \lambda_2}\right] \qquad y = \frac{H}{4}\left[2 - \frac{\max(\lambda_1, \lambda_2)}{\lambda_1 + \lambda_2}\right] \quad (34)$$

with $\lambda_1 = P_1/B_1$, $\lambda_2 = P_2/B_2$, and the width of the bay in the $x_1$ and $x_2$ directions being $B_1$ and $B_2$, respectively (Fig. 5).

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## Notation

*The following symbols are used in this paper:*

- $A$, $\mathbf{A}$ = cross-sectional area and vector of all cross-sectional areas, respectively;
- $B$ = brace width;
- $C$ = compliance of the truss (strain energy);
- $E$ = elastic modulus;
- $\mathbf{f}$ = nodal force vector;
- $H$ = brace height;
- $I$ = member's area moment of inertia;
- $\mathbf{K}$ = stiffness matrix associated with the structure;
- $L$, $\mathbf{L}$ = truss member length and vector of all member lengths, respectively;
- $\mathcal{L}$ = Lagrangian function;
- $N$ = member's axial load;
- $n_d$, $n_n$, $n_e$ = structure's dimensions, number of nodes, and elements, respectively;
- $P$ = lateral or vertical load on the bracing system;
- $r$ = member's radius of gyration = $\sqrt{I/A}$;
- $u$, $\mathbf{u}$ = nodal displacement and vector of displacements, respectively;
- $V$ = volume of the structure;
- $x$, $y$ = bracing point height and clear, respectively; $x + y = H$;
- $Z$ = Michell's number (performance index);
- $\alpha$ = diagonal member's cost variable; $\alpha = 1$ in 2D and $\alpha = 2$ for 3D braces;
- $\gamma$ = limit stress in tension and compression ratio; $\gamma = -\sigma_t/\sigma_c$;
- $\Delta$ = displacement (or linear combination of displacements) to be minimized;
- $\kappa$ = column effective length factor (used in buckling calculations);
- $\lambda$ = Lagrange multiplier; and
- $\sigma$ = internal stress.

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