A Paradigm for Higher Order Mixed Polygonal Elements for Finite Elasticity

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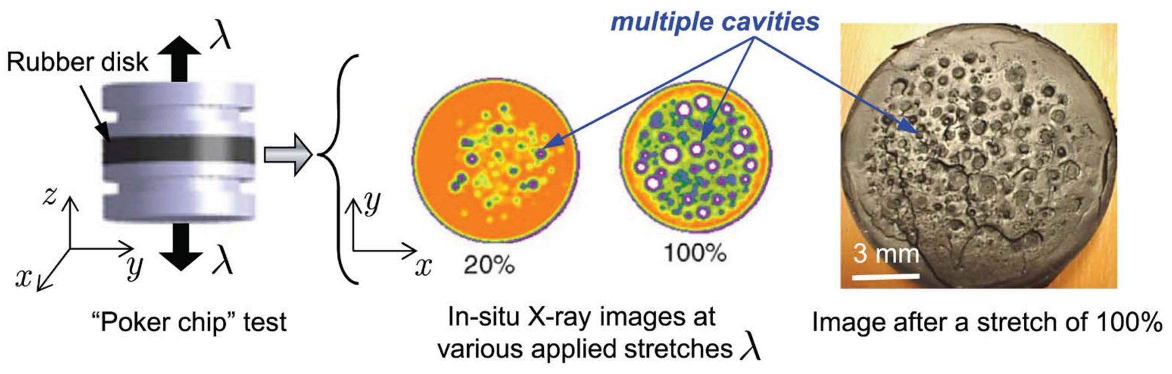
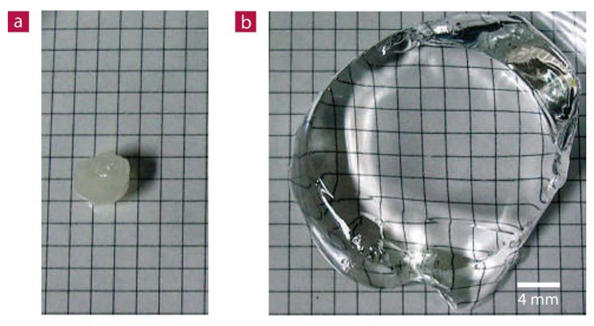
Motivation

- Soft organic materials, such as electro- and magneto-active elastomers and gels, hold tremendous potential for new high-end technologies, e.g. next generation sensors and actuators.
- Soft materials often possess complex microstructures, with underlying local deformations which are typically larger than macroscopic ones.

Bar-Cohen, 2004



Bayraktar, Bessri and Bathias, 2007



Swelling of lipophilic polyelectrolyte gel

Ono et al, 2007

Formulation and Polygonal FEM Approximations

- **Find $(\mathbf{u}^*, p^*)$ such that:**

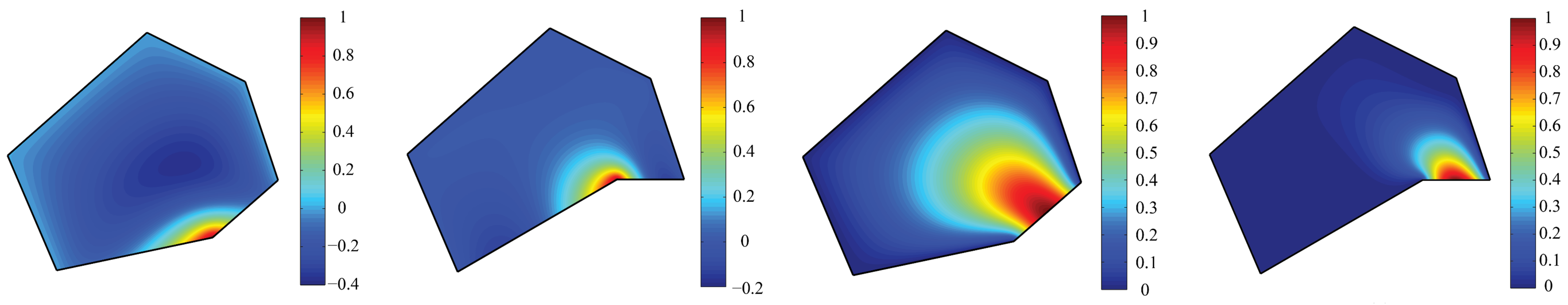
$$\Pi(\mathbf{u}^*, p^*) = \min_{\mathbf{u}} \max_{p} \Pi(\mathbf{u}, p)$$

where:

$$\Pi(\mathbf{u}, p) = \int_{\Omega_0} [-W_C(\mathbf{X}, \mathbf{F}(\mathbf{u}), p) + p(\det \mathbf{F}(\mathbf{u}) - 1)] \mathrm{d}\Omega_0 - \int_{\Omega_0} \mathbf{f}_0 \cdot \mathbf{u} \mathrm{d}\Omega_0 - \int_{\partial\Omega_0} \mathbf{t}_0 \cdot \mathbf{u} \, \mathrm{d}\partial\Omega_0$$

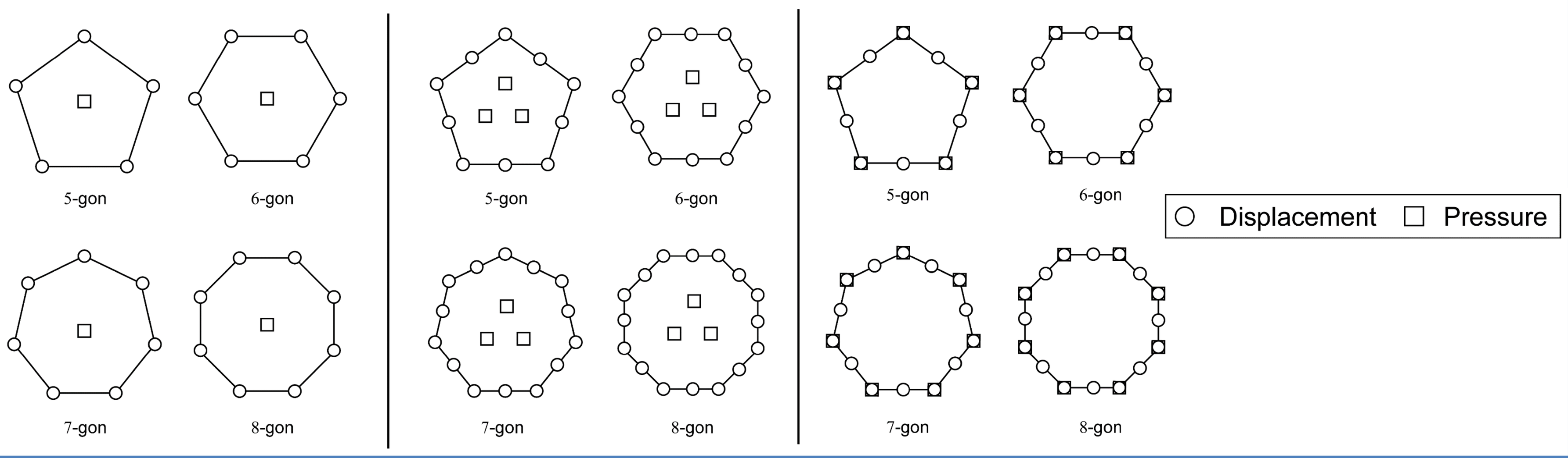
$$W_C(\mathbf{X}, \mathbf{F}, p) = \sup_J \, [p(J-1) - W(\mathbf{X}, \mathbf{F}, J)]$$

- **Polygonal shape functions:**



Linear (Floater, 2003) Quadratic (Rand et al., 2014)

- **Displacement and pressure approximations:**



The Gradient Correction Framework

- **Notations:**

$⨎_E$: quadrature of order $2k-2$ over element E; $\int_E$: exact integral over element E

$\int_{\partial E}$: quadrature of order $2k-1$ over the boundary of E; $\mathcal{M}_k(E)$: element local space

$\mathcal{P}_k(E)$: polynomial space of order k

- **The correction to the gradient is defined as:**

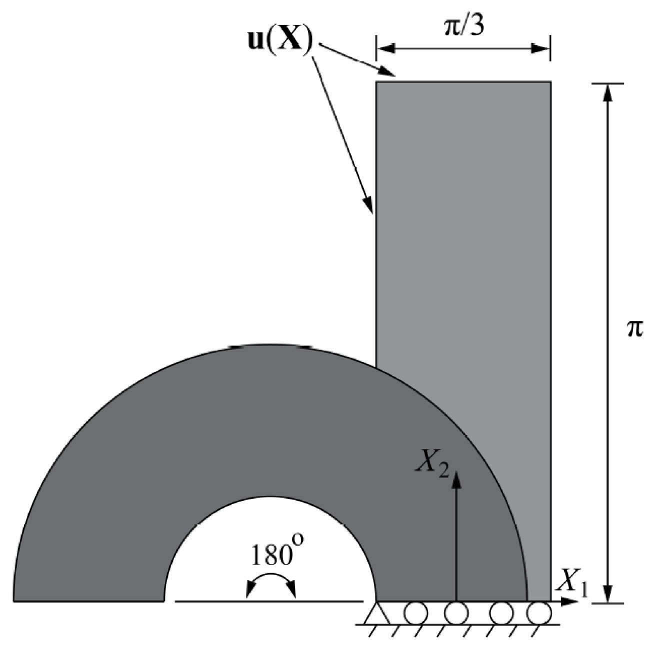
For any given $v \in \mathcal{M}_k(E)$, the corrected gradient $\nabla_{E,k} v$ is the one satisfies:

- $\nabla_{E,k} v - \nabla v \in [\mathcal{P}_{k-1}(E)]^2$
- $⨎_E \mathbf{p} \cdot \nabla_{E,k} v \mathrm{d}\mathbf{X} = \int_{\partial E} (\mathbf{p} \cdot \mathbf{N}) \, v \mathrm{d}S - ⨎_E v \, \mathrm{Div} \mathbf{p} \mathrm{d}\mathbf{X}, \quad \forall \mathbf{p} \in [\mathcal{P}_{k-1}(E)]^2$

Conclusion

- The gradient correction leads to optimal convergence for linear and quadratic polygonal elements
- Polygonal elements have advantages in modeling inclusions with arbitrary geometries, as well as incorporating periodic boundaries.
- Polygonal elements (both linear and quadratic) appear to be more tolerant to large local deformations than classical finite elements.

Convergence Test



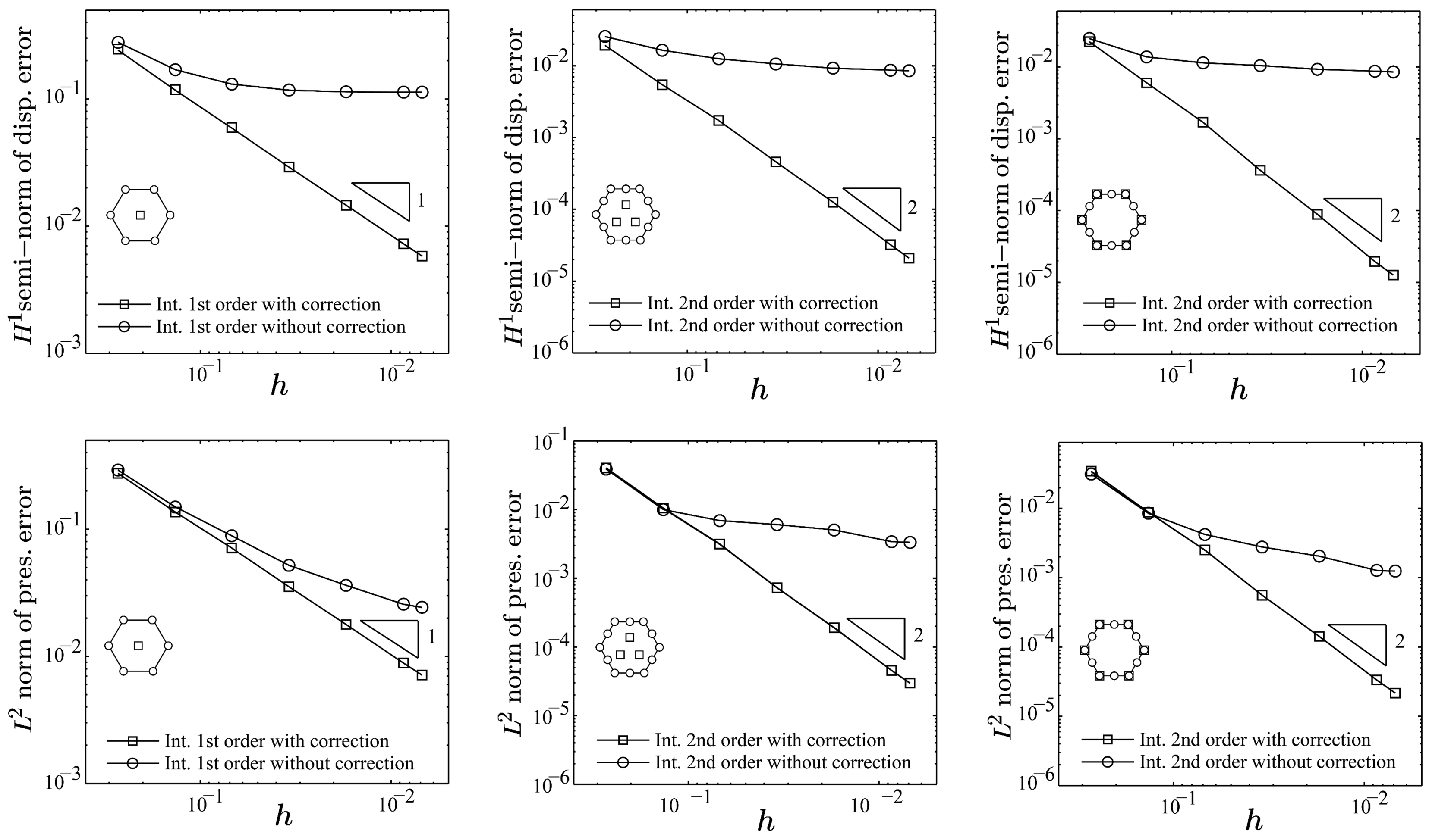
- **Analytical Solution**

$$u_1 = r(X_1)\cos(X_2) - r\left(-\frac{\pi}{6}\right) - \frac{\pi}{6} - X_1$$

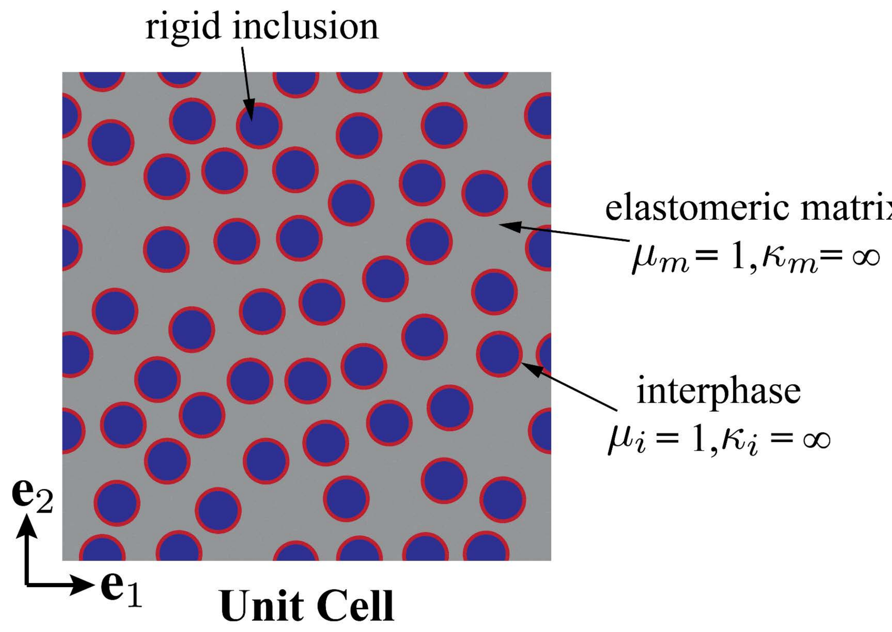
$$u_2 = r(X_1)\sin(X_2) - X_2$$

$$r(X_1) = \sqrt{2X_1 + \beta} \quad \text{with} \quad \beta = \sqrt{\frac{4\pi^2}{9} + 4}$$

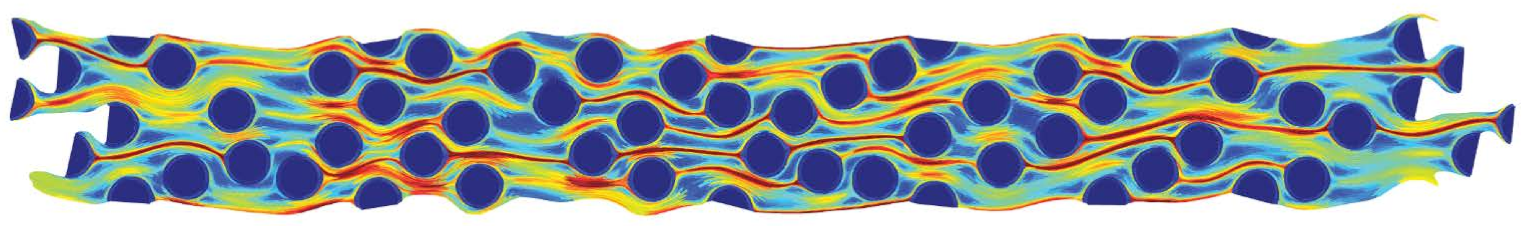
$$p = -\frac{\mu}{2}\left[\frac{1}{r(X_1)^2} - r(X_1)^2 + \mu\beta\right]$$



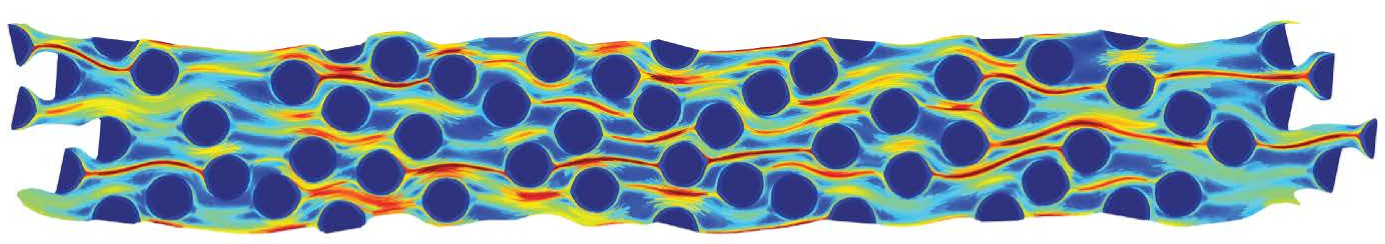
Particle Reinforced Elastomers



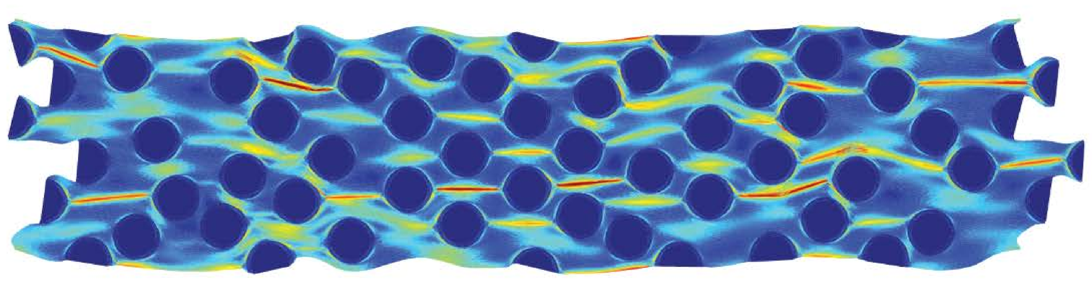
- Neo-Hookean matrix and interphase
- $N = 50$ particles with a total volume fraction of $c_p = 25\%$
- The thickness of the interphase: $t = 0.2R_p$, where R_p is the radius of the particle.
- The effective volume fraction is $c = c_p + c_i = 36\%$



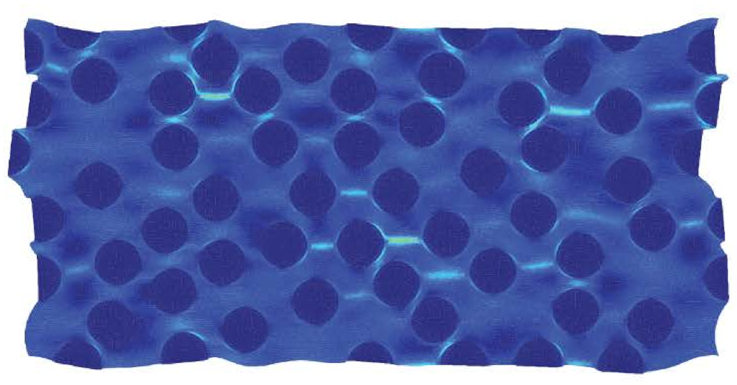
(a) $\mathcal{M}_2 - \mathcal{M}_1$ element: λmax=2.9132



(b) $\mathcal{M}_2 - \mathcal{P}_1^D$ element: λmax=2.6456

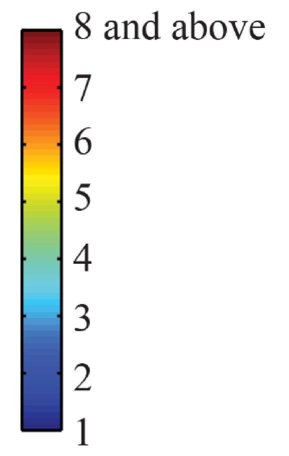


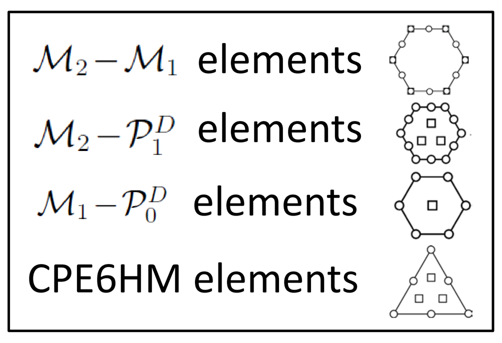
(c) $\mathcal{M}_1 - \mathcal{P}_0^D$ element: λmax=2.0836



(d) CPE6HM element: λmax=1.4308

Max. Principle Stretch





Reference

1. H. Chi, C. Talischi, O. Lopez-Pamies, and G. H. Paulino, "A paradigm for higher-order polygonal elements in finite elasticity using a gradient correction scheme." *CMAME,* submitted
2. C. Talischi, A. Pereira, I.F. Menezes, and G. H. Paulino, "Gradient correction for polygonal and polyhedral finite elements." *IJNME.* Vol 102. pp. 728-747. 2015.
3. H. Chi, C. Talischi, O. Lopez-Pamies, and G. H. Paulino, "Polygonal finite elements for finite elasticity." *IJNME.* Vol. 101, pp. 305-328. 2015.
4. C. Talischi, G.H. Paulino, A. Pereira, I.F.M. Menezes, "PolyMesher: A general-purpose mesh generator for polygonal elements written in Matlab." *JSMO.* Vol. 45, No. 3, pp. 309-328, 2012.