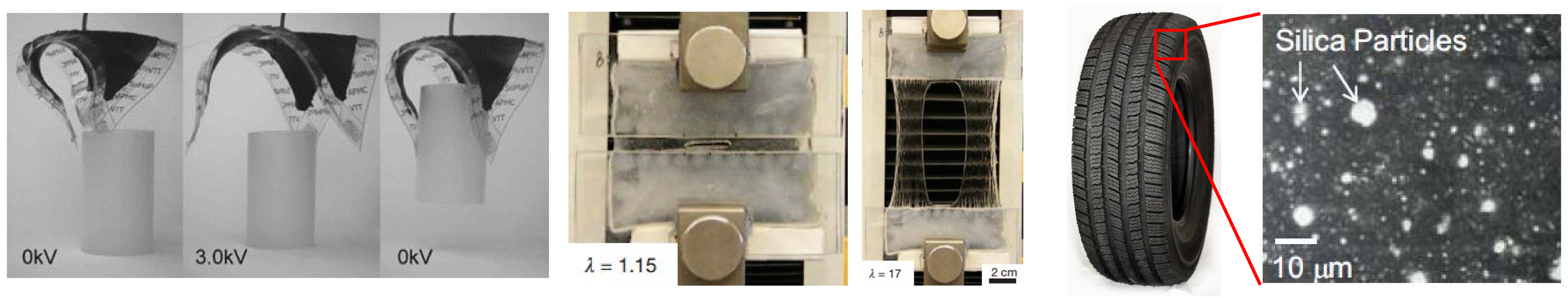
Nonlinear element without explicit shape function using a mimetic-inspired approach

Heng Chi[a], Lourenco Beirão da Veiga[b], Glaucio H. Paulino[a]

a: Georgia Institute of Technology, Atlanta, USA   b: Università di Milano Bicocca, Milano, Italy

Motivation: Soft Materials

- Large deformations, complex microstructures, and (near) incompressibility.

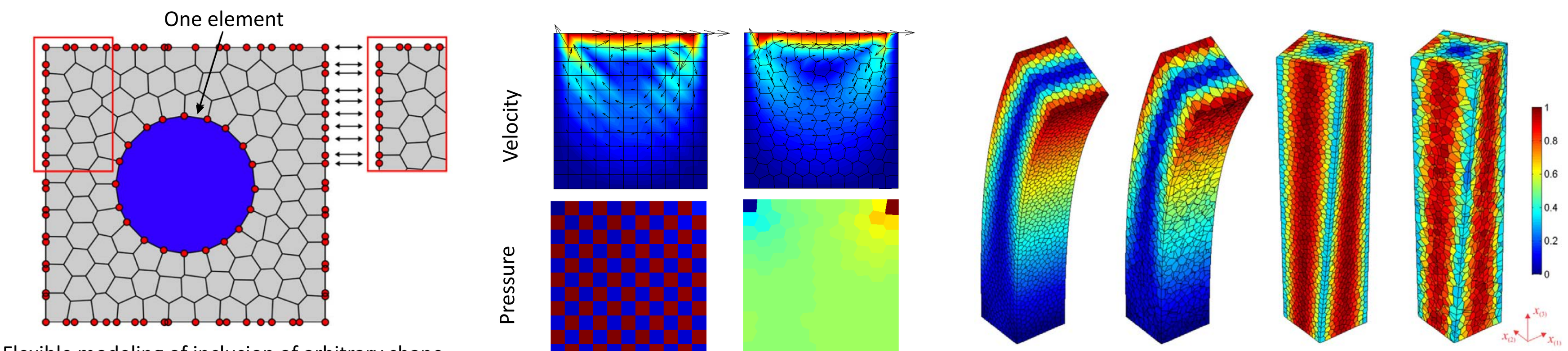


Finite deformation of a dielectric elastomer (Kofod, Wirges, Paajanen Bauer, 2007)

Stretch of a hydrogel specimen (Sun et. al., 2012)

Microstructure of a typical filled elastomer (Ramier, 2004)

- Polygonal and polyhedral elements are advantageous to model soft materials.



Flexible modeling of inclusion of arbitrary shape (Chi, Talischi, Lopez-Pamies and Paulino, 2015)

Better inf-sup stability performance

VEM for 3D linear elasticity using arbitrary polyhedrons (Gain, Talischi and Paulino, 2014)

Two VEM highlights:

- No shape functions
- No approximate numerical integration

Local Displacement Spaces

2D: $\mathcal{V}(E) \doteq \left\{ \mathbf{v} \in \left[\mathcal{H}^1(E)\right]^2 : \Delta \mathbf{v} = 0 \text{ in } E,\ \mathbf{v}|_{\partial E} \in \left[C^0(\partial E)\right]^2 \text{ and } \mathbf{v}|_e \in [\mathcal{P}_1(e)]^2\ \forall e \in \partial E \right\}$

3D: $\mathcal{V}(E) \doteq \left\{ \mathbf{v} \in \left[\mathcal{H}^1(E)\right]^3 : \mathbf{v}|_{\partial E} \in \left[C^0(\partial E)\right]^3,\ \mathbf{v}(\mathbf{X}_s^f) = \sum_{j=1}^{m^f} \beta_j^f \mathbf{v}(\mathbf{X}_j^f) \text{ and } \mathbf{v}|_{T_j^f} \in \left[\mathcal{P}_1(T_j^f)\right]^3,\ j = 1, \ldots, m^f,\ \forall f \in \partial E \text{ and } \Delta \mathbf{v} = 0, \text{ in } E \right\}$

Computable quantities:

1. $\Pi_E^0 \nabla \mathbf{v}$: projection of $\nabla \mathbf{v}$ onto its volume average:

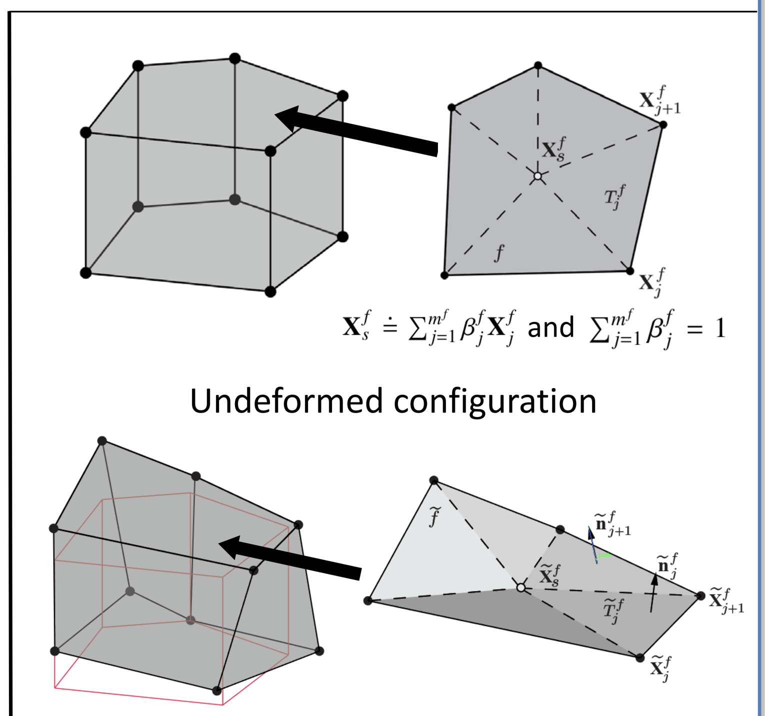
$$\frac{1}{|E|}\int_E \nabla \mathbf{v} d\mathbf{X} = \frac{1}{|E|}\int_{\partial E} \mathbf{v} \otimes \mathbf{n} dS = \frac{1}{|E|} \sum_{f \in \partial E} \sum_{j=1}^{m^f} \left\{ w_j^f \mathbf{v}(\mathbf{X}_j^f) \otimes \mathbf{n}_f \right\} \quad \text{in 3D}$$

2. $\Pi_E^{\nabla} \mathbf{v}$: projection of $\mathbf{v}$ onto linear functions:

$$\nabla\left(\Pi_E^{\nabla}\mathbf{v}\right) = \Pi_E^0(\nabla \mathbf{v}) \quad \text{and} \quad \sum_{i=1}^{m} (\Pi_E^{\nabla}\mathbf{v})(\mathbf{X}_i) = \sum_{i=1}^{m} \mathbf{v}(\mathbf{X}_i)$$

3. Change of element volume under any $\mathbf{v}$:

$$J_E(\mathbf{v}) = \frac{1}{d|E|}\int_{\partial \tilde{E}} \tilde{\mathbf{X}} \cdot \tilde{\mathbf{n}} d\tilde{S} = \frac{1}{6|E|} \sum_{f \in \partial E} \sum_{j=1}^{m^f} \left\{ \tilde{\mathbf{X}}_s^f \cdot \tilde{\mathbf{X}}_j^f \wedge \tilde{\mathbf{X}}_{j+1}^f \right\} \quad \text{in 3D}$$



Undeformed configuration

Deformed configuration

VEM Approximation

Find $(\mathbf{u}, \hat{p})$ such that

$$\widehat{\Pi}_h(\mathbf{u}_h, \widehat{p}_h) = \min_{\mathbf{v}_h \in \mathcal{K}_h} \max_{\widehat{q}_h \in Q_h} \left\{ \sum_E |E| \Psi\left(\mathbf{I} + \Pi_E^0 \nabla \mathbf{v}_h\right) + \frac{1}{2}\sum_E \alpha_E(\mathbf{v}_h) S_{h,E}\left(\mathbf{v}_h - \Pi_E^{\nabla}\mathbf{v}_h, \mathbf{v}_h - \Pi_E^{\nabla}\mathbf{v}_h\right) + \sum_E |E| \left[\widehat{q}_h|_E \left(J_E(\mathbf{v}_h) - 1\right) - \widehat{U}^*\left(\widehat{q}_h|_E\right)\right] - <\mathbf{f}, \mathbf{v}_h>_h - <\mathbf{t}, \mathbf{v}_h>_h \right\}$$

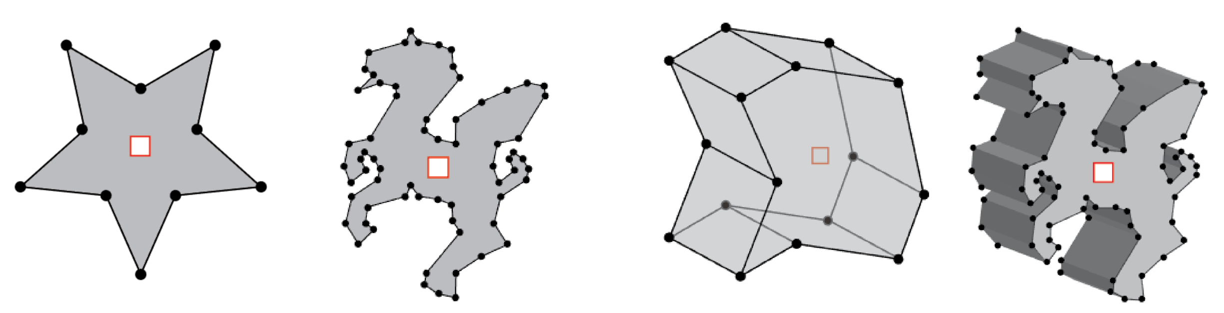
- Use the consistency and stability terms to approximate the exact integral:

$$\int_E \Psi(\mathbf{F}(\mathbf{v}_h)) d\mathbf{X} \approx \underbrace{|E|\Psi\left(\mathbf{I} + \Pi_E^0 \nabla \mathbf{v}_h\right)}_{\text{consistency}} + \underbrace{\frac{1}{2}\sum_E \alpha_E(\mathbf{v}_h) S_{h,E}\left(\mathbf{v}_h - \Pi_E^{\nabla}\mathbf{v}_h, \mathbf{v}_h - \Pi_E^{\nabla}\mathbf{v}_h\right)}_{\text{stability}}$$

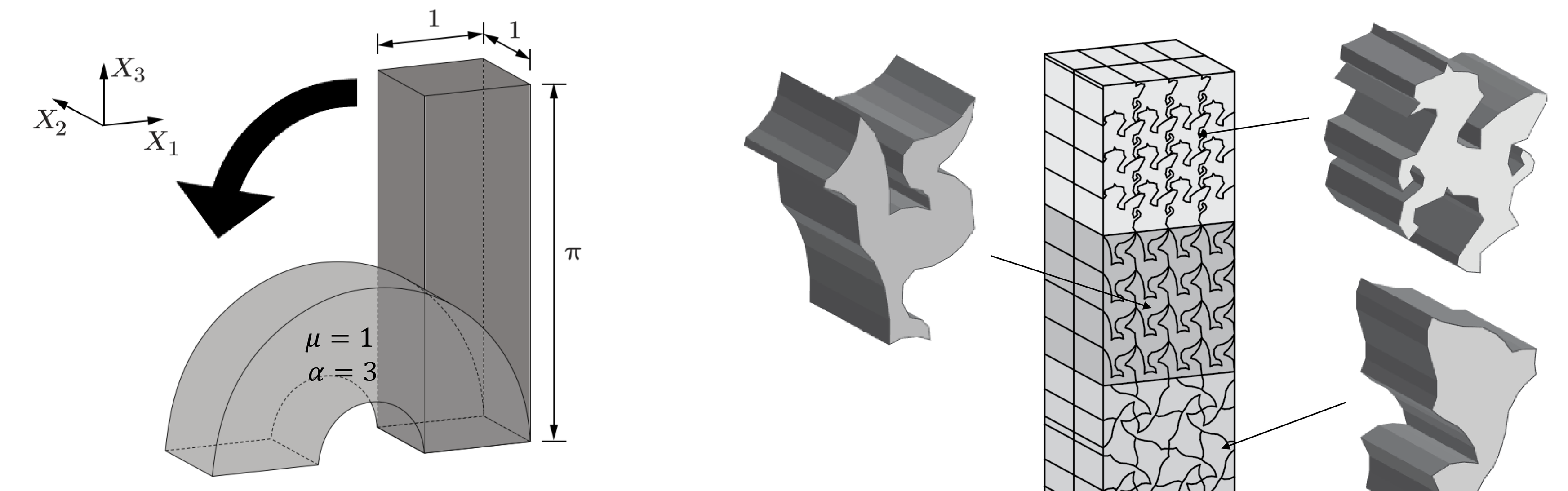
- Closed-form expressions of the deformation-evolving stability parameter for isotropic solids:

$$\alpha_E(I_1, I_2, J) = \frac{4I_1}{9}\frac{\partial^2 \Phi}{\partial I_1 \partial I_1} + \frac{(4I_1 I_2 + 12J^2)}{9}\frac{\partial^2 \Phi}{\partial I_2 \partial I_2} + \frac{I_2}{9}\frac{\partial^2 \Phi}{\partial J \partial J} + \frac{16 I_2}{9}\frac{\partial^2 \Phi}{\partial I_1 \partial I_2} + \frac{3J}{4}\frac{\partial^2 \Phi}{\partial I_1 \partial J} + \frac{4J I_1}{9}\frac{\partial^2 \Phi}{\partial I_2 \partial J} + 2\frac{\partial \Phi}{\partial I_1} + \frac{8 I_1}{9}\frac{\partial \Phi}{\partial I_2} \quad \text{in 3D}$$

- Examples of the mixed virtual elements (Pegasus credit: M. C. Escher)



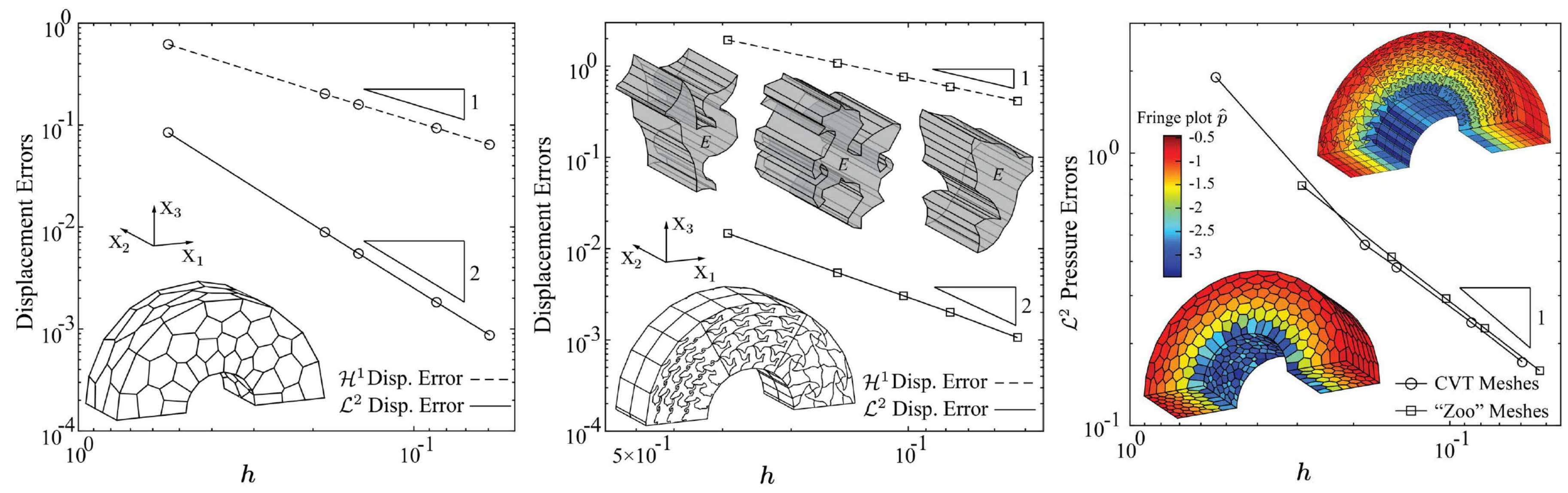
VEM Convergence Study



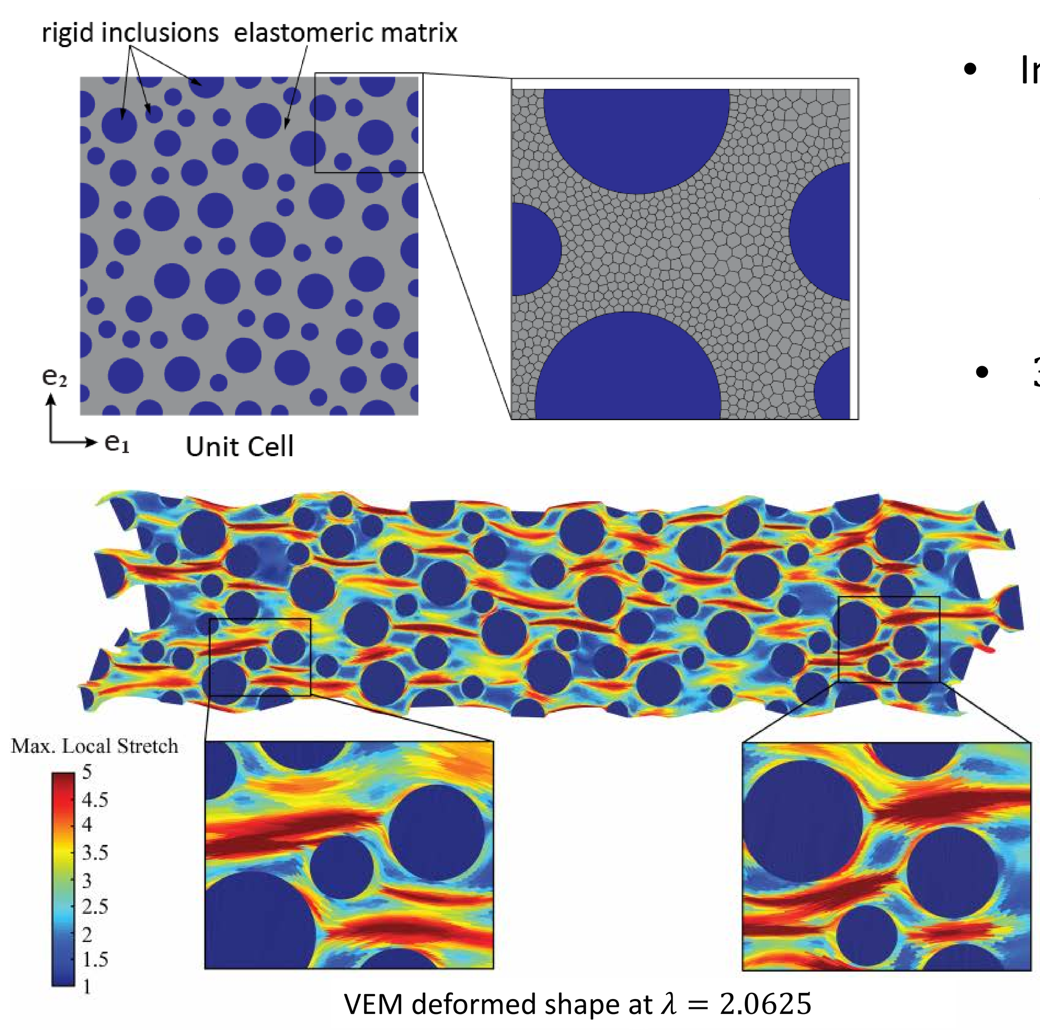
$$W(\mathbf{F}) = \begin{cases} \frac{3^{1-\alpha}}{2\alpha}\mu\left[(I_1(\mathbf{F}))^{\alpha} - 3^{\alpha}\right] & \text{if } \det \mathbf{F} = 1 \\ +\infty & \text{otherwise,} \end{cases}$$

"zoo" mesh

Convergence of the displacement and pressure fields: convex and concave meshes



Particle Reinforced Elastomers

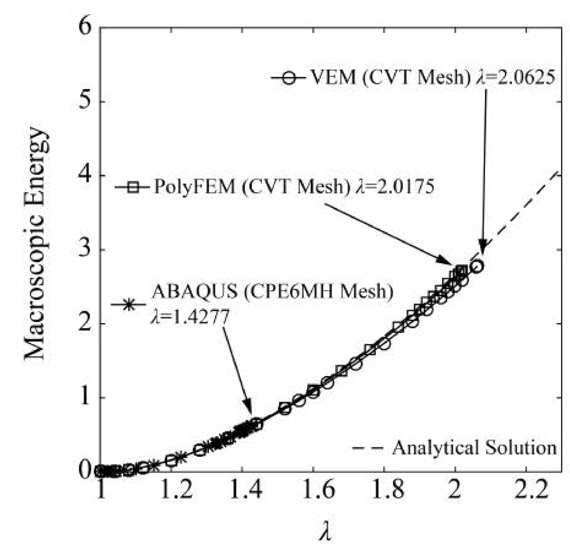


VEM deformed shape at $\lambda = 2.0625$

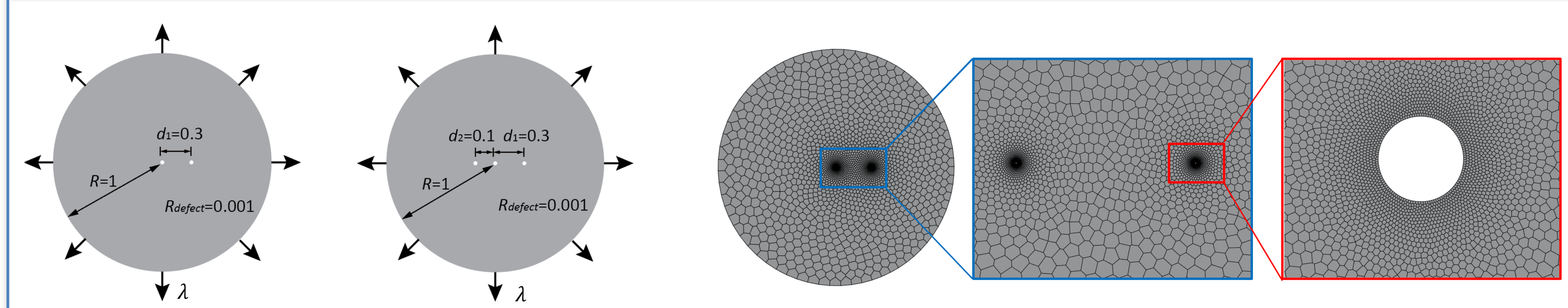
- Incompressible neo-Hookean matrix

$$W(\mathbf{F}) = \begin{cases} \frac{\mu}{2}(\mathbf{F} : \mathbf{F} - 3), & \text{if } \det \mathbf{F} = 1 \\ +\infty & \text{otherwise} \end{cases}$$

- 35% of rigid circular reinforcement

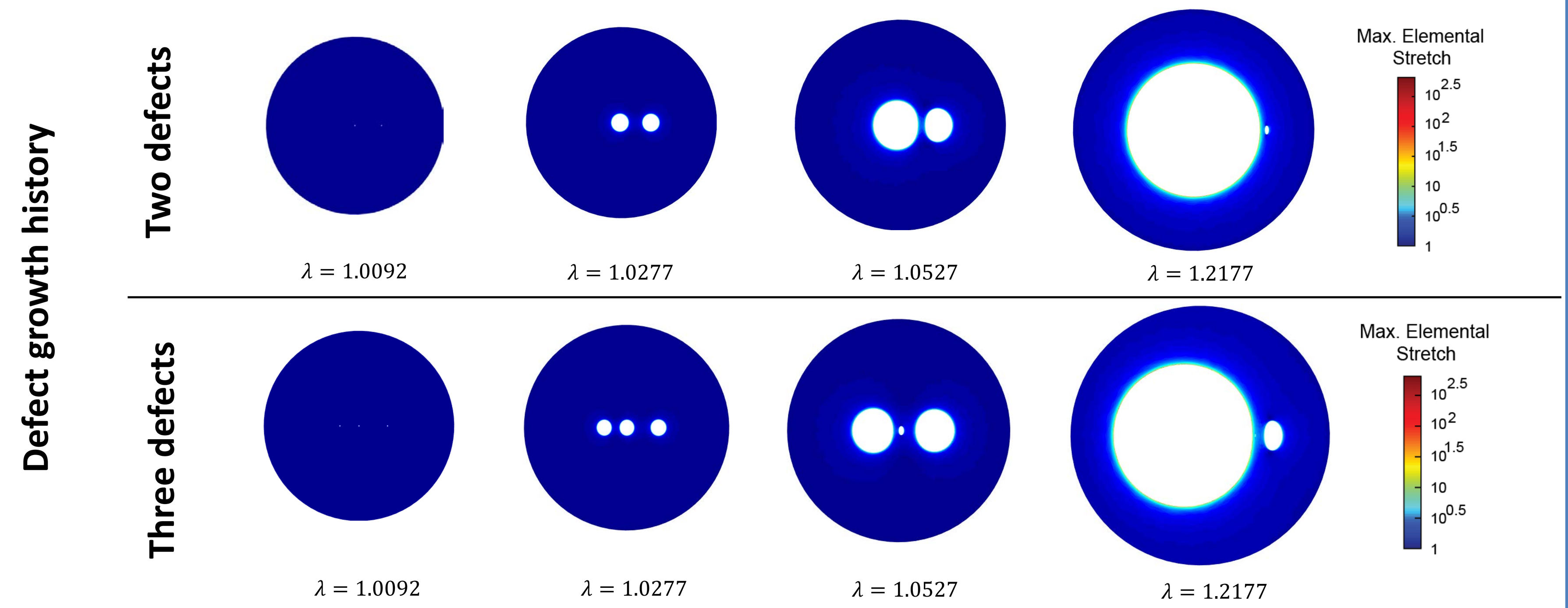


Cavitation Instability



Circular disks with two and three defects

Graded Voronoi mesh (scale bridging)



References

1. Chi, H., Beirao da Veiga, L., and Paulino, G.H. 2017. Some basic formulations of the Virtual Element Method (VEM) for finite deformations. *CMAME*, 318, 148-192.
2. Beirão da Veiga, L., Brezzi, F., Cangiani, A., Manzini, G., Marini, L. D., and Russo, A. 2013. Basic principles of virtual element methods. *M3AS*, 23(01), 199-214.
3. Chi, H., Talischi, C., Lopez-Pamies, O., Paulino, G.H. 2015. Polygonal finite elements for finite elasticity. *IJNME*, 101, 305–328

Conclusions

- The first work in the literature to address finite deformation using the VEM (2D & 3D).
- VEM is able to address practical engineering problems involving finite deformations.
- The VEM approximation can handle 2D and 3D arbitrary (non-convex) element geometries.
- The deformation-evolving stability term is key to capture extremely large and heterogeneous deformations.