

On the Mixed Virtual Element Method (VEM) for Finite Deformations

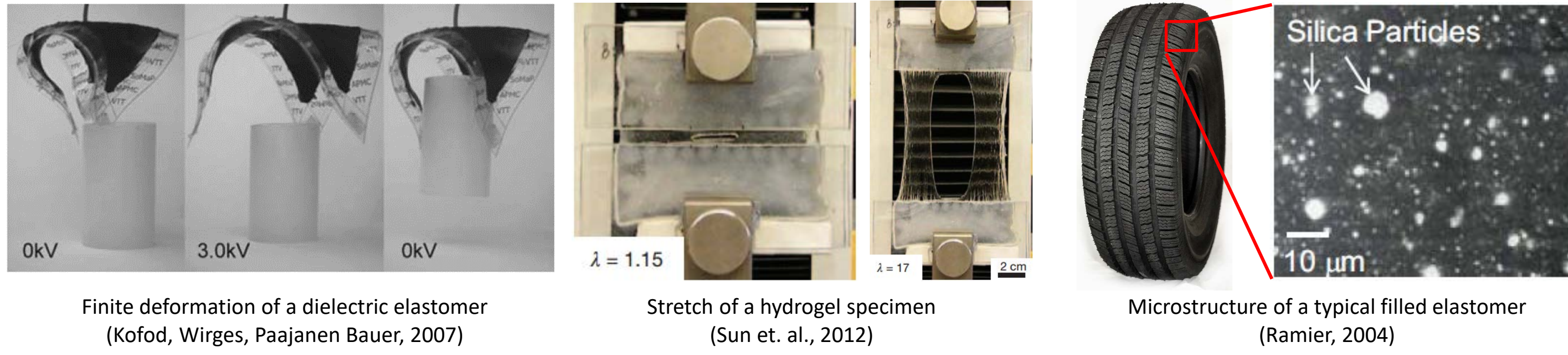
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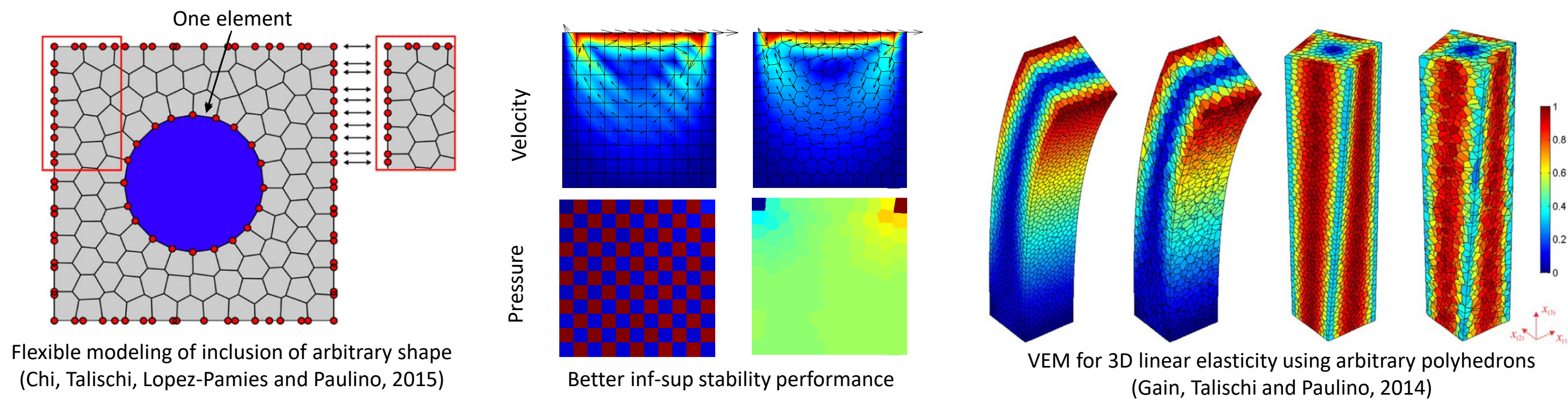


Motivation: Soft Materials

- Large deformations, complex microstructures, and (near) incompressibility.



- Polygonal and polyhedral elements are advantageous to model soft materials.



- Two VEM highlights:
- No shape functions
 - No approximate numerical integration

Local Displacement Spaces

$$2D: \mathcal{V}(E) \doteq \{ \mathbf{v} \in [H^1(E)]^2 : \Delta \mathbf{v} = 0 \text{ in } E, \mathbf{v}|_{\partial E} \in [C^0(\partial E)]^2 \text{ and } \mathbf{v}|_e \in [\mathcal{P}_1(e)]^2 \forall e \in \partial E \}$$

$$3D: \mathcal{V}(E) \doteq \left\{ \mathbf{v} \in [H^1(E)]^3 : \mathbf{v}|_{\partial E} \in [C^0(\partial E)]^3, \mathbf{v}(\mathbf{X}_i^f) = \sum_{j=1}^{m^f} \beta_j^f \mathbf{v}(\mathbf{X}_j^f) \text{ and } \mathbf{v}|_{T_j^f} \in [\mathcal{P}_1(T_j^f)]^3, j=1, \dots, m^f, \forall f \in \partial E \text{ and } \Delta \mathbf{v} = 0, \text{ in } E \right\}$$

Computable quantities:

- $\Pi_E^0 \nabla \mathbf{v}$: projection of $\nabla \mathbf{v}$ onto its volume average:

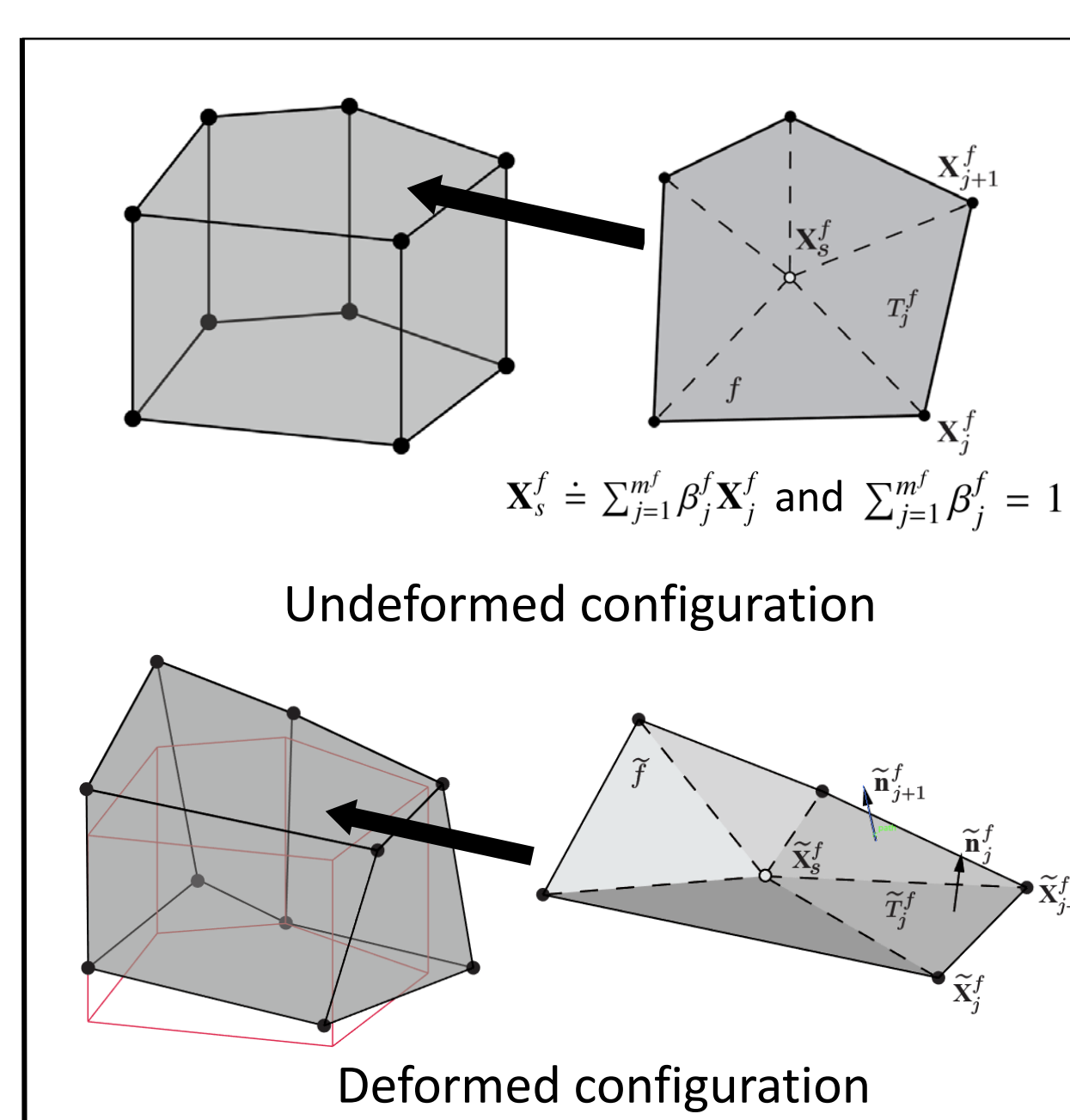
$$\frac{1}{|E|} \int_E \nabla \mathbf{v} d\mathbf{X} = \frac{1}{|E|} \int_{\partial E} \mathbf{v} \otimes \mathbf{n} dS = \frac{1}{|E|} \sum_{f \in \partial E} \sum_{j=1}^{m^f} \{ \mathbf{v}_j^f \mathbf{v}(\mathbf{X}_j^f) \otimes \mathbf{n}_f \} \text{ in } 3D$$

- $\Pi_E^1 \mathbf{v}$: projection of \mathbf{v} onto linear functions:

$$\nabla(\Pi_E^1 \mathbf{v}) = \Pi_E^0(\nabla \mathbf{v}) \text{ and } \sum_{i=1}^m (\Pi_E^1 \mathbf{v})(\mathbf{X}_i) = \sum_{i=1}^m \mathbf{v}(\mathbf{X}_i)$$

- Change of element volume under any \mathbf{v} :

$$J_E(\mathbf{v}) = \frac{1}{d|E|} \int_{\partial E} \bar{\mathbf{X}} \cdot \bar{\mathbf{n}} dS = \frac{1}{6|E|} \sum_{f \in \partial E} \sum_{j=1}^{m^f} \{ \bar{\mathbf{X}}_j^f \cdot \bar{\mathbf{X}}_j^f \wedge \bar{\mathbf{X}}_{j+1}^f \} \text{ in } 3D$$



VEM Approximation

Find (\mathbf{u}, \hat{p}) such that

$$\hat{\Pi}_h(\mathbf{u}_h, \hat{p}_h) = \min_{\mathbf{v}_h \in K_h} \max_{\hat{q}_h \in Q_h} \left\{ \sum_E |E| \Psi(\mathbf{I} + \Pi_E^0 \nabla \mathbf{v}_h) + \frac{1}{2} \sum_E \alpha_E (\mathbf{v}_h) S_{h,E} (\mathbf{v}_h - \Pi_E^1 \mathbf{v}_h, \mathbf{v}_h - \Pi_E^1 \mathbf{v}_h) + \sum_E |E| [\hat{q}_h (J_E(\mathbf{v}_h) - 1) - \hat{U}^* (\hat{q}_h |E|)] - \langle \mathbf{f}, \mathbf{v}_h \rangle_h - \langle \mathbf{t}, \mathbf{v}_h \rangle_h \right\}$$

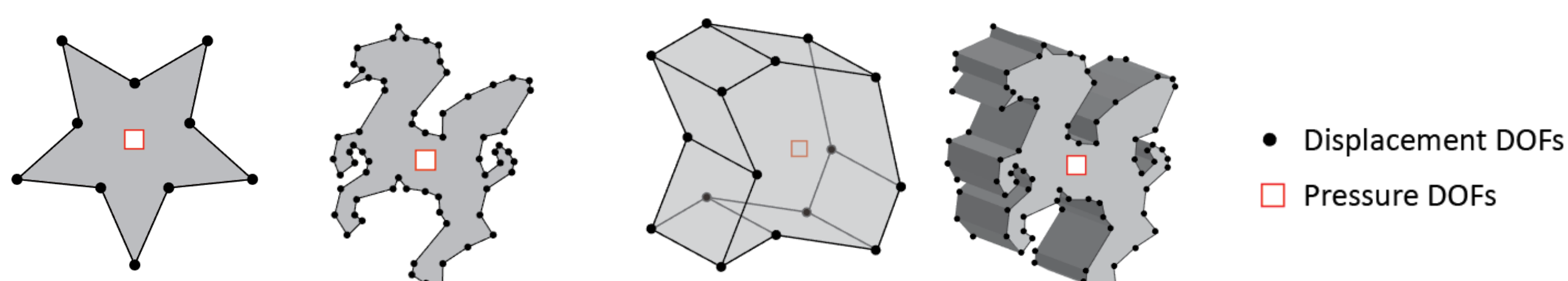
- Use the consistency and stability terms to approximate the exact integral:

$$\int_E \Psi(\mathbf{F}(\mathbf{v}_h)) d\mathbf{X} \approx \underbrace{|E| \Psi(\mathbf{I} + \Pi_E^0 \nabla \mathbf{v}_h)}_{\text{consistency}} + \underbrace{\frac{1}{2} \sum_E \alpha_E (S_h) S_{h,E} (\mathbf{v}_h - \Pi_E^1 \mathbf{v}_h, \mathbf{v}_h - \Pi_E^1 \mathbf{v}_h)}_{\text{stability}}$$

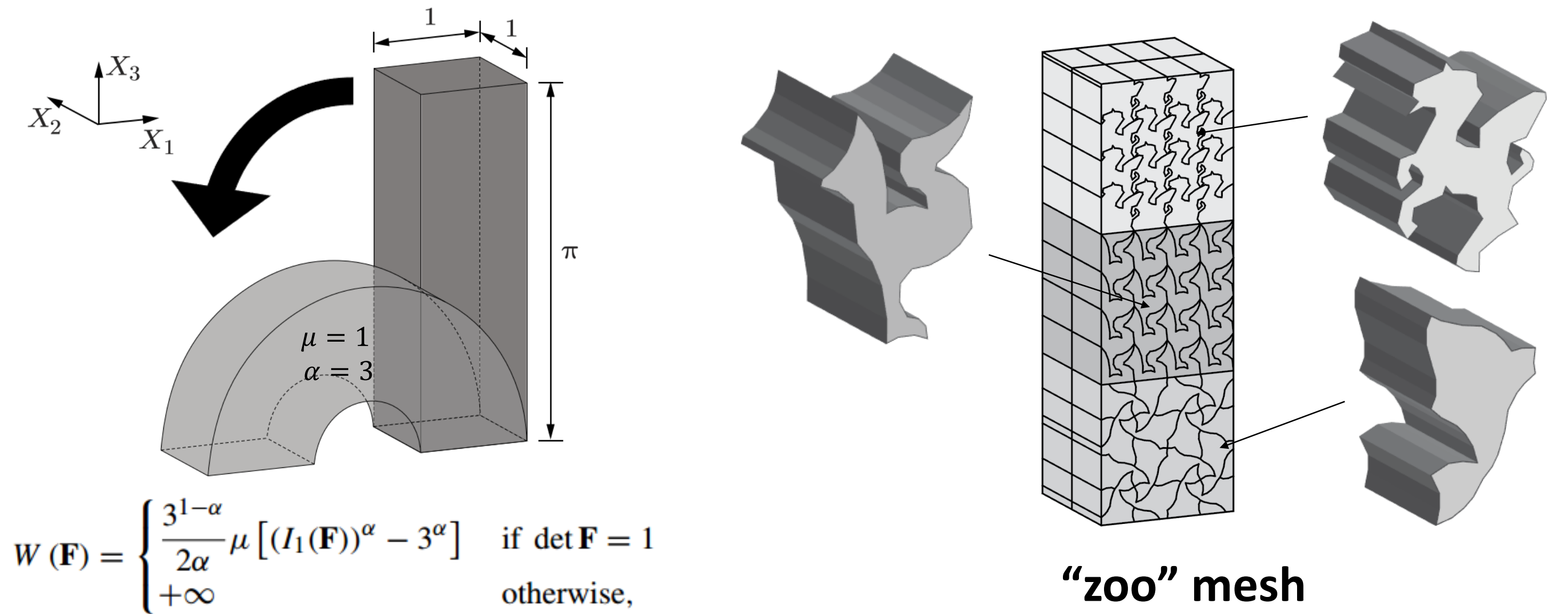
- Closed-form expressions of the deformation-evolving stability parameter for isotropic solids:

$$\alpha_E(I_1, I_2, J) = \frac{4I_1}{9} \frac{\partial^2 \Phi}{\partial I_1 \partial I_1} + \frac{(4I_1 I_2 + 12J^2)}{9} \frac{\partial^2 \Phi}{\partial I_2 \partial I_2} + \frac{I_2}{9} \frac{\partial^2 \Phi}{\partial J \partial J} + \frac{16I_2}{9} \frac{\partial^2 \Phi}{\partial I_1 \partial I_2} + \frac{3J}{4} \frac{\partial^2 \Phi}{\partial I_1 \partial J} + \frac{4J I_1}{9} \frac{\partial^2 \Phi}{\partial I_2 \partial J} + 2 \frac{\partial \Phi}{\partial I_1} + \frac{8I_1}{9} \frac{\partial \Phi}{\partial I_2} \text{ in } 3D$$

- Examples of the mixed virtual elements (Pegasus credit: M. C. Escher)

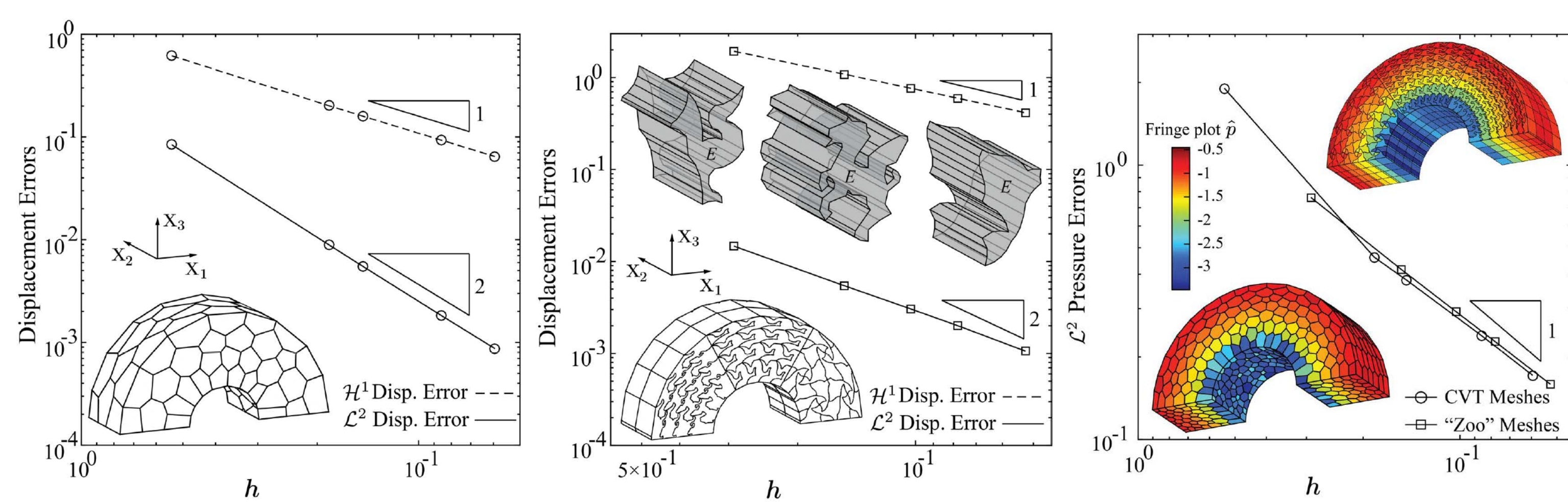


VEM Convergence Study

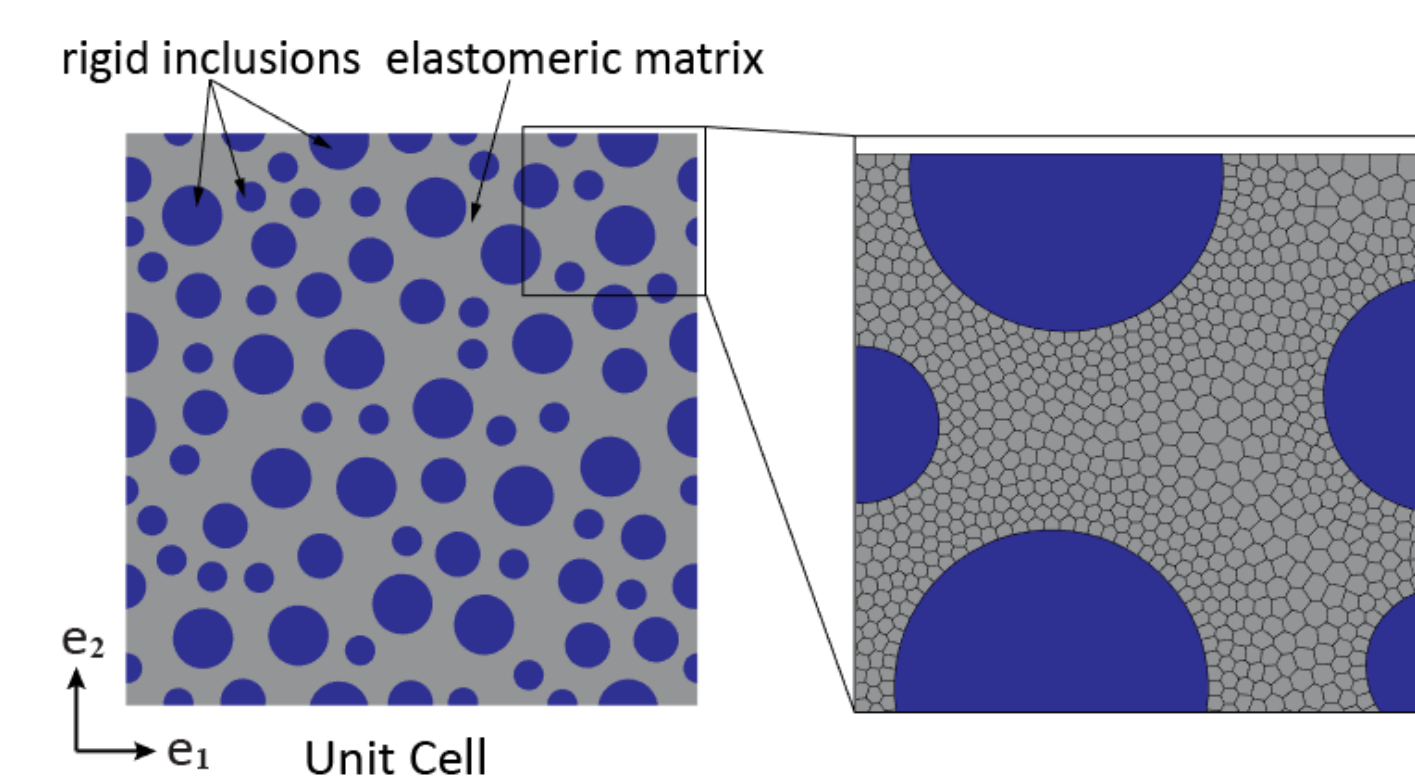


$$W(\mathbf{F}) = \begin{cases} \frac{3^{1-\alpha}}{2\alpha} \mu [(I_1(\mathbf{F}))^\alpha - 3^\alpha] & \text{if } \det \mathbf{F} = 1 \\ +\infty & \text{otherwise,} \end{cases}$$

Convergence of the displacement and pressure fields: convex and concave meshes



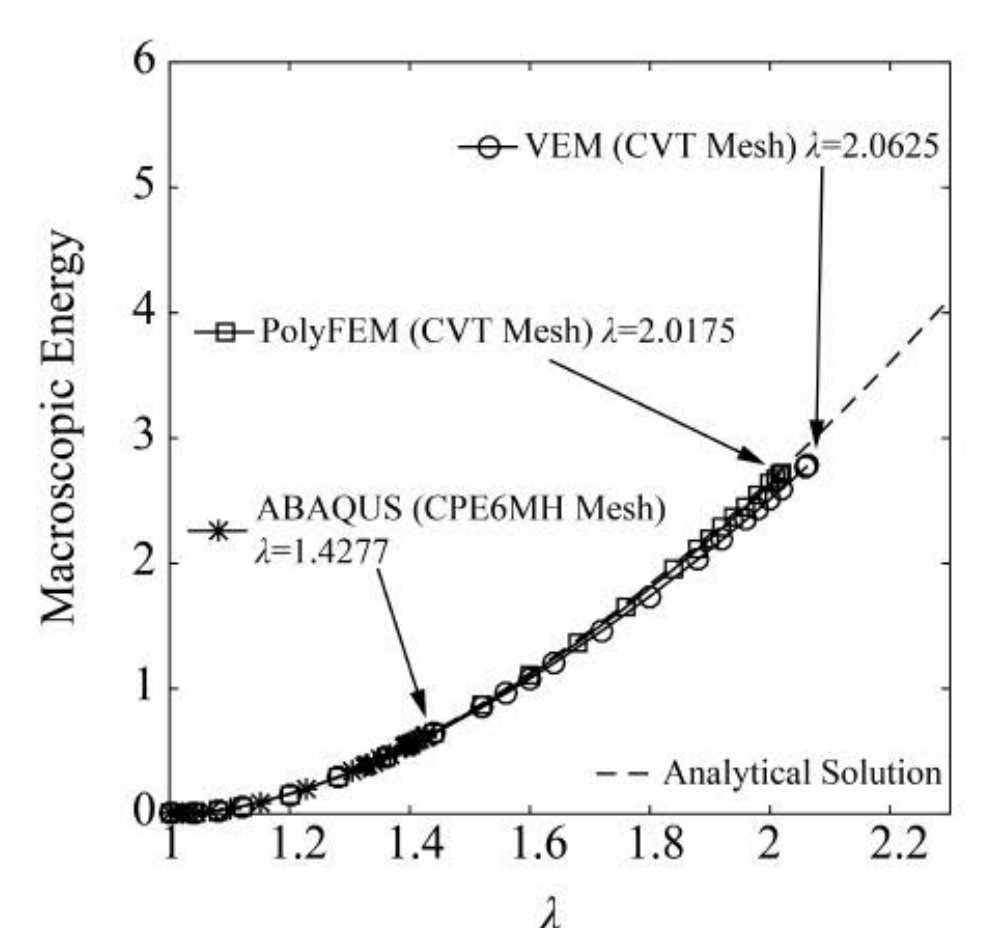
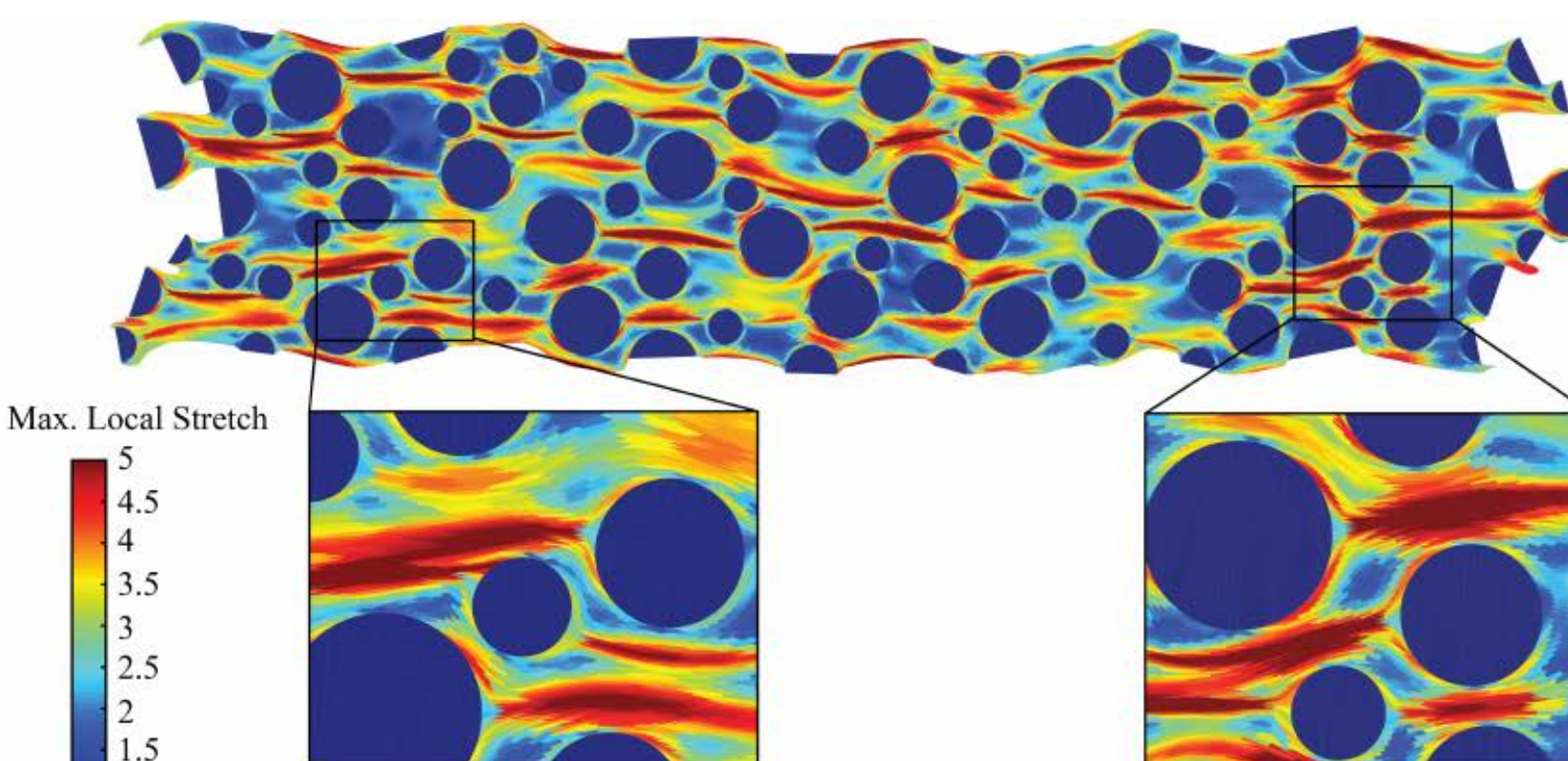
Particle Reinforced Elastomers



- Incompressible neo-Hookean matrix

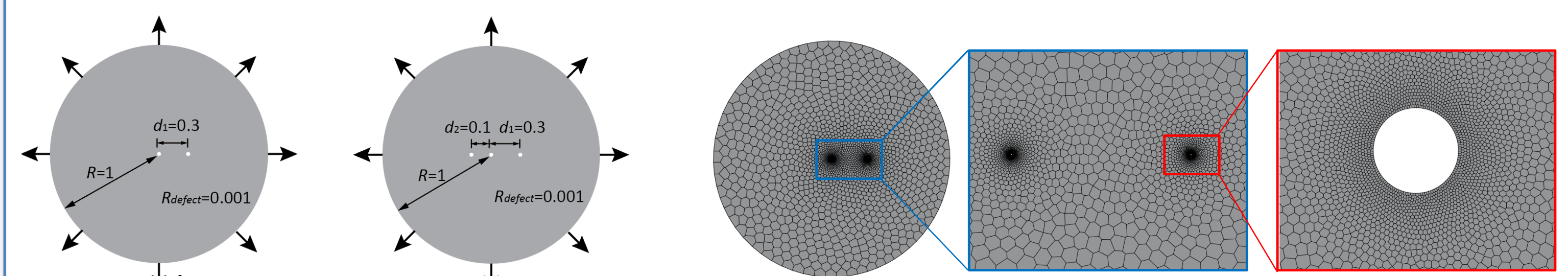
$$W(\mathbf{F}) = \begin{cases} \frac{\mu}{2} (\mathbf{F} : \mathbf{F} - 3) & \text{if } \det \mathbf{F} = 1 \\ +\infty & \text{otherwise} \end{cases}$$

- 35% of rigid circular reinforcement

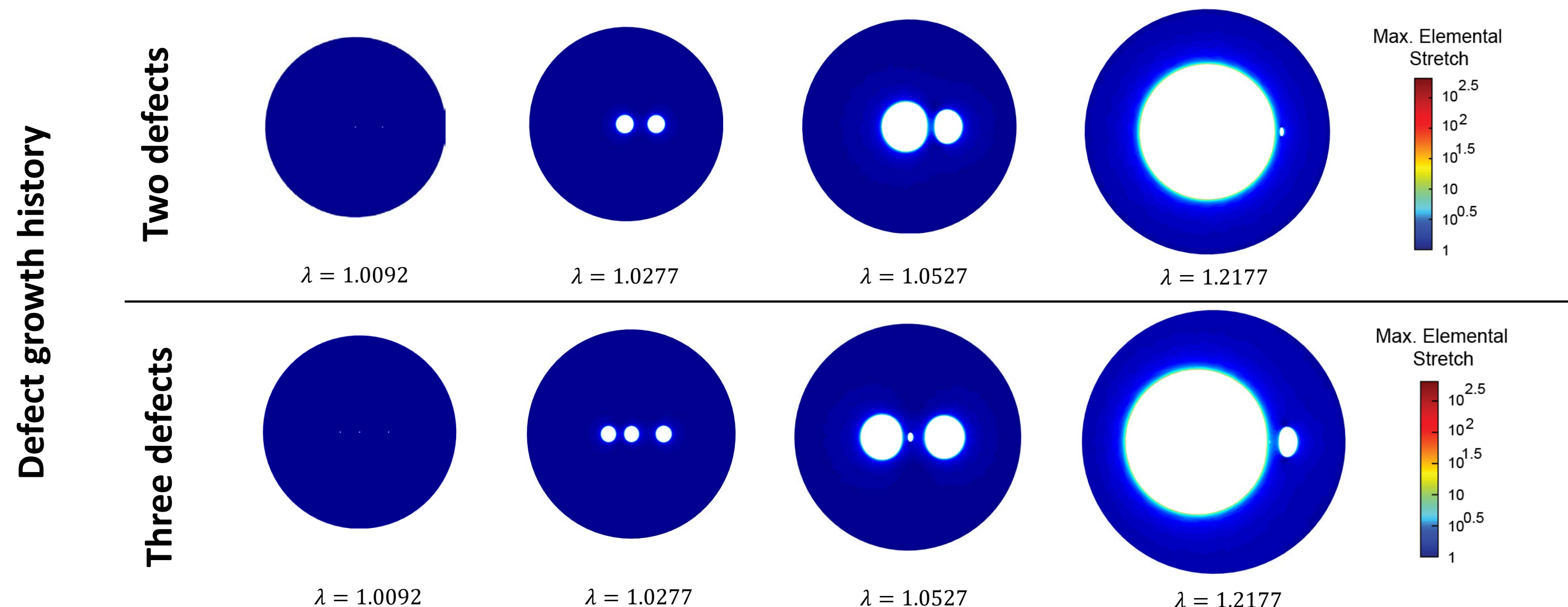


VEM Approximation

Cavitation Instability



Circular disks with two and three defects Graded Voronoi mesh (scale bridging)



References

- Chi, H., Beirão da Veiga, L., and Paulino, G.H. 2017. Some basic formulations of the Virtual Element Method (VEM) for finite deformations. *CMAME*, 318, 148-192.
- Beirão da Veiga, L., Brezzi, F., Cangiani, A., Manzini, G., Marini, L. D., and Russo, A. 2013. Basic principles of virtual element methods. *M3AS*, 23(01), 199-214.
- Chi, H., Talisch, C., Lopez-Pamies, O., Paulino, G.H. 2015. Polygonal finite elements for finite elasticity. *IJNME*, 101, 305-328

Conclusions

- The first work in the literature to address finite deformation using the VEM (2D & 3D).
- VEM is able to address practical engineering problems involving finite deformations.
- The VEM approximation can handle 2D and 3D arbitrary (non-convex) element geometries.
- The deformation-evolving stability term is key to capture extremely large and heterogeneous deformations.