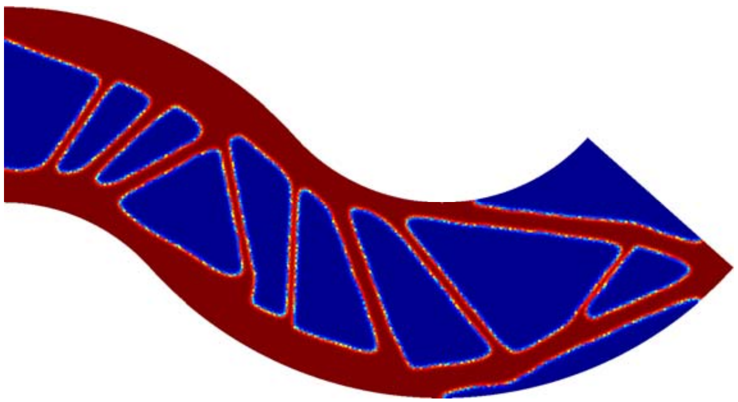
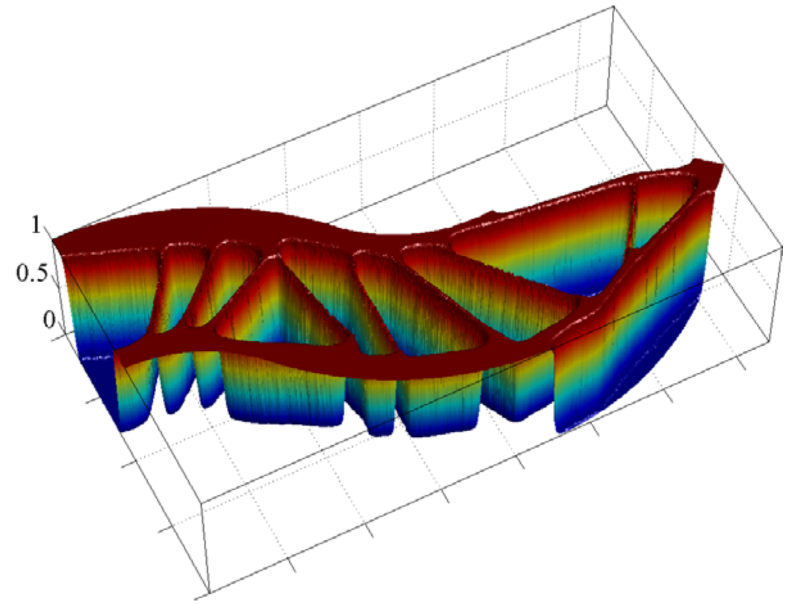
# Phase-field based structural topology optimization using unstructured polygonal meshes



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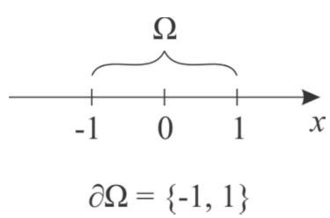
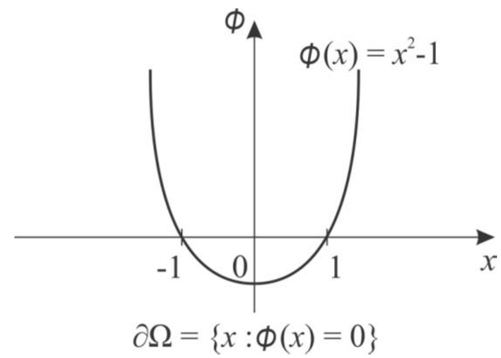
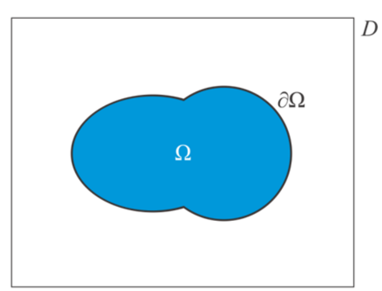
# Introduction & Motivation

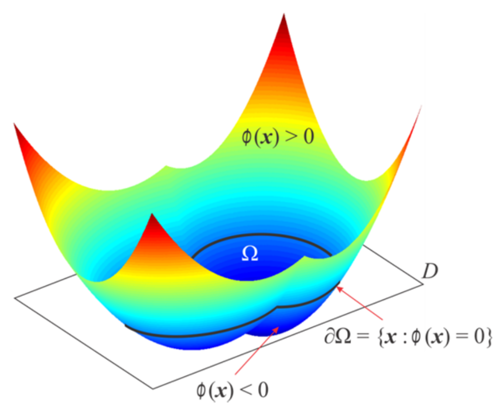
Topology optimization refers to optimum distribution of material in a given design space under certain specified boundary conditions so as to meet certain prescribed performance objective.

**Implicit function methods:** Topology represented in terms of implicit functions and evolved over time using certain PDEs.



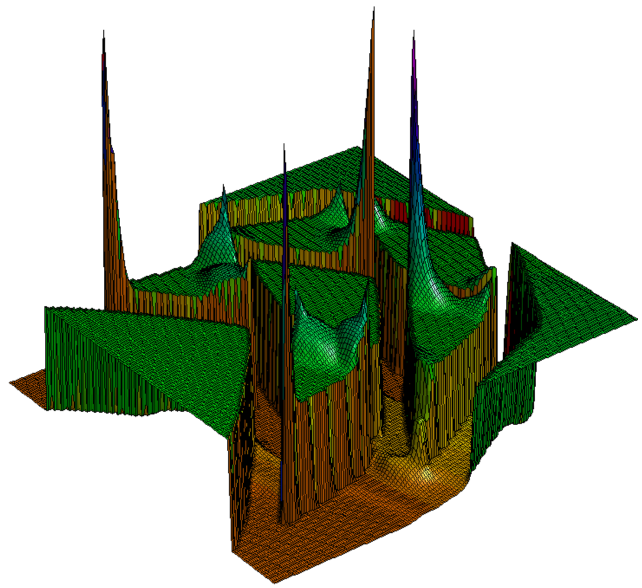



# Introduction & Motivation

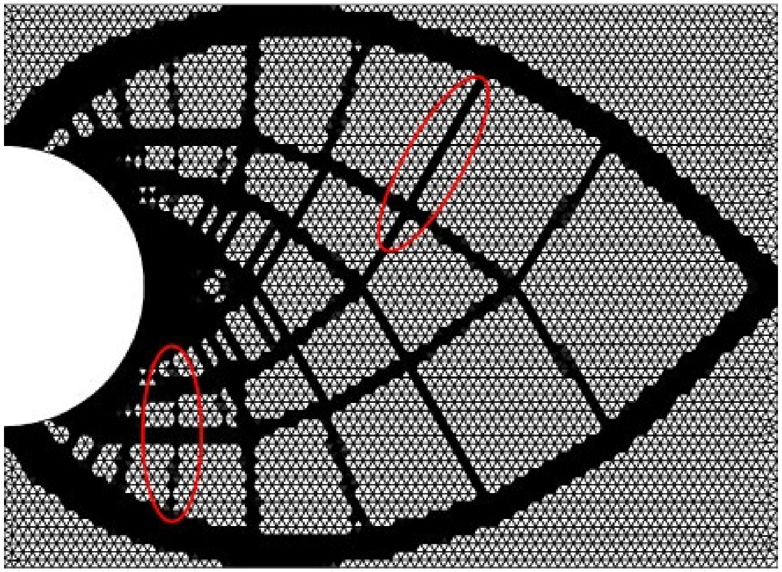
- Implicit function method such as level-set function, although attractive, require periodic reinitializations, for example, to maintain signed distance characteristics for numerical convergence.



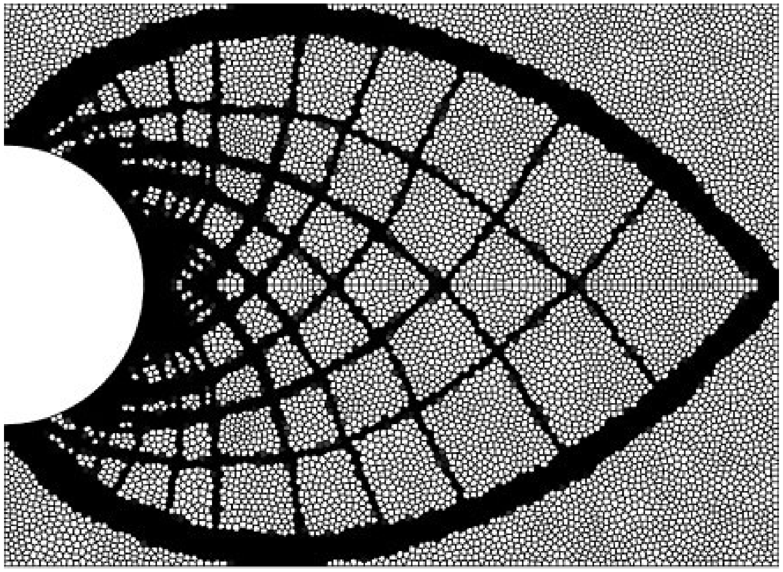
- Reinitializations often performed heuristically.
- Phase field function does not have to be signed distance function so no need of any reinitialization.

# Motivation for using polygonal elements

- For simplicity, uniform grids are often used for topology optimization. Over-constrained geometrical features of structured grids can bias the orientation of the members, leading to mesh dependent, sub-optimal designs.
- Traditional density based topology optimization on Cartesian meshes suffer from numerical anomalies such as checkerboard patterns and one-node connections.



T6 elements

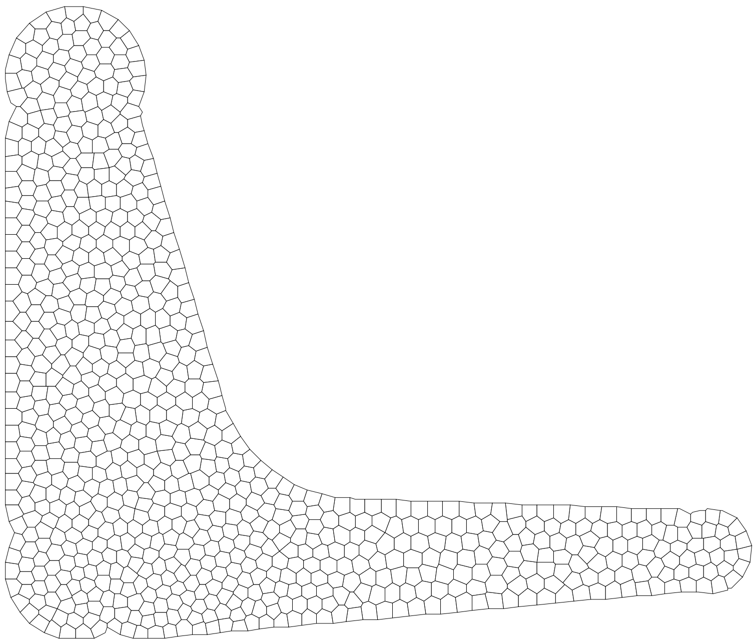


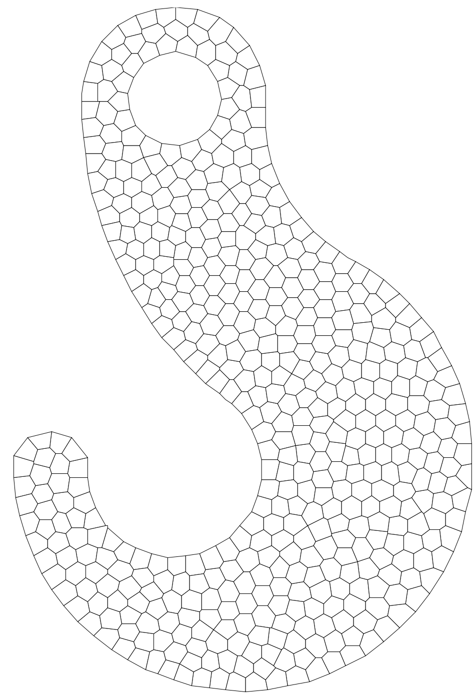
Polygonal elements

**Talischi C., Paulino G. H., Pereira A., and Menezes I. F. M. (2010) Polygonal finite elements for topology optimization: A unifying paradigm. *International Journal for Numerical Methods in Engineering*, 82: 671-698**

# Motivation for using polygonal elements

- Explore general and curved domains rather than the traditional Cartesian domains (box-type) that have been extensively used for topology optimization.





Talischi C., Paulino G. H., Pereira A., and Menezes I. F. M. (2012) PolyMesher: A general-purpose mesh generator for polygonal elements written in Matlab. *Structural and Multidisciplinary Optimization*, 45(3): 309-328

# Presentation Outline

1. **Introduction & Motivation**
2. **Polygonal finite element method**
3. **Phase-field method for topology optimization**
4. **Centroidal Voronoi Tessellation (CVT) based finite volume method to solve the Allen-Cahn equation**
5. **Implementation flow chart**
6. **Numerical examples**
7. **Future research directions**

## Following objective functions will be considered for topology optimization using phase-field method:

$$\inf_{\phi} \overline{J}(\phi) = J_i(\phi) + \lambda P(\phi) \qquad \text{for } i = 1,2$$

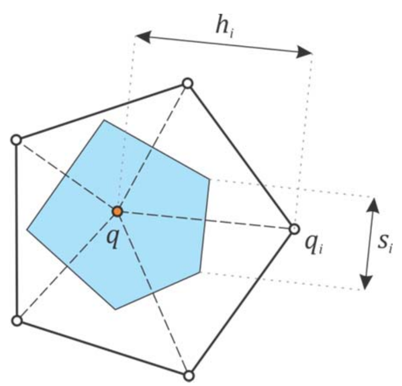
where,

$$J_i(\phi) = \begin{cases} \int_D \boldsymbol{f} \cdot \boldsymbol{u} dD + \int_{\partial D_N} \boldsymbol{g} \cdot \boldsymbol{u} ds & \text{Compliance minimization} \\ -u_{out}(\phi) & \text{Compliant mechanism} \end{cases}$$

$$P(\phi) = \int_D \phi \, dD \qquad \text{Volume constraint}$$

# Brief review of polygonal finite elements used in this work

**Polygonal finite elements:** Finite element space of polygonal elements is constructed using natural neighbor scheme based non-Sibson interpolants (Laplace interpolants)




$$N_i(\boldsymbol{x}) = \frac{\alpha_i(\boldsymbol{x})}{\sum_Q \alpha_j(\boldsymbol{x})}, \qquad \alpha_i(\boldsymbol{x}) = \frac{s_i(\boldsymbol{x})}{h_i(\boldsymbol{x})}, \qquad \boldsymbol{x} \in \Re^2$$

where, $Q = \{q_1, q_2, \ldots, q_n\}$

**Conforming shape functions:**

$$0 \le N_i(\boldsymbol{x}) \le 1, \qquad N_i(\boldsymbol{x}_j) = \delta_{ij}, \qquad \sum_{\mathrm{P}} N_i(\boldsymbol{x}) = 1$$

$$\sum_{\mathrm{P}} \boldsymbol{x}_i N_i(\boldsymbol{x}) = \boldsymbol{x}$$

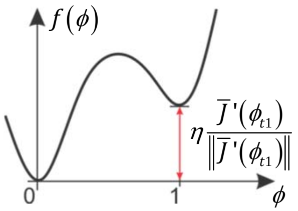

**Belikov VV, Ivanov VD, Kontorovich VK, Korytnik SA, Semenov AY (1997) The non-Sibsonian interpolation: a new method of interpolation of the values of a function on an arbitrary set of points. *Computational Mathematics and Mathematical Physics* 37(1): 9-15**

# Review of the phase-field method employed

**Evolution equation:** Allen-Cahn equation

$$\frac{\partial \phi}{\partial t} = \kappa \nabla^2 \phi - f'(\phi), \qquad \frac{\partial \phi}{\partial \boldsymbol{n}} = 0 \text{ on } \partial D$$

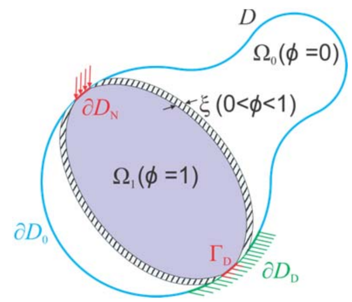
where, $f(0)=0,\quad f(1)=\eta \frac{\bar{J}'(\phi_{t1})}{\left\| \bar{J}'(\phi_{t1}) \right\|},\qquad f'(0)=f'(1)=0$




$$\frac{\partial \phi}{\partial t} = \kappa \nabla^2 \phi + \phi(1-\phi)\left[\phi - \frac{1}{2} - 30\eta \frac{\bar{J}'(\phi_{t1})}{\left\| \bar{J}'(\phi_{t1}) \right\|}\phi(1-\phi)\right]$$

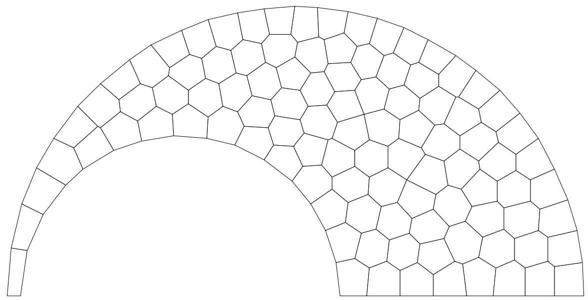
**Effective elasticity tensor:**

$$\boldsymbol{C}^*(\phi) = \begin{cases} \boldsymbol{C} & \boldsymbol{x} \in \Omega_1, \\ \phi^p \boldsymbol{C} & \boldsymbol{x} \in \xi, \\ k_{\min} \boldsymbol{C} & \boldsymbol{x} \in \Omega_0. \end{cases}$$




Takezawa A., Nishiwaki S., and Kitamura M. (2010) Shape and topology optimization based on the phase field method and sensitivity analysis. *Journal of Computational Physics*, 229: 2697-2718

# Centroidal Voronoi Tessellation (CVT) based finite volume method is used to solve the Allen-Cahn equation



**Allen-Cahn equation:**

$$\frac{\partial \phi}{\partial t} = \kappa \nabla^2 \phi - f'(\phi)$$

**Integral form:**

$$\int_{t,D_p} \frac{\partial \phi}{\partial t} dt\, dD = \int_{t,D_p} \kappa \nabla^2 \phi\, dt\, dD - \int_{t,D_p} f'(\phi)\, dt\, dD$$

$$\int_{t,D_p} \frac{\partial \phi}{\partial t} dt\, dD = \int_{t,\Gamma_p} \kappa \nabla \phi \cdot \boldsymbol{n}\, dt\, d\Gamma - \int_{t,D_p} f'(\phi)\, dt\, dD$$

**Vasconcellos J. F. V. and Maliska C. R. (2004) A finite-volume method based on voronoi discretization for fluid flow problems. Numerical Heat Transfer, Part B, 45: 319-342**

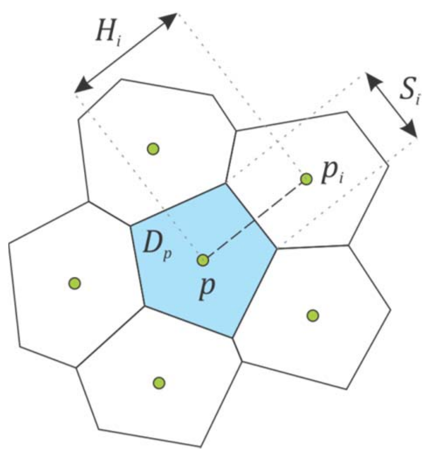
## Centroidal Voronoi Tessellation (CVT) based finite volume method is used to solve the Allen-Cahn equation

**Integral form:**

$$\int_{t,D_p} \frac{\partial \phi}{\partial t}\, dt\, dD = \int_{t,\Gamma_p} \kappa \nabla \phi \cdot \boldsymbol{n}\, dt\, d\Gamma - \int_{t,D_p} f'(\phi)\, dt\, dD$$

**Simplifying each term:**

- $$\int_{t,D_p} \frac{\partial \phi}{\partial t}\, dt\, dD = \int_{D_p} \left(\phi^{n+1} - \phi^{n}\right) dD \approx \left(\phi_p^{n+1} - \phi_p^{n}\right) V_p$$




- $$\int_{t,\Gamma_p} \kappa \nabla \phi \cdot \boldsymbol{n}\, dt\, d\Gamma \approx \int_t \sum_P \left[\kappa \nabla \phi^n \cdot \boldsymbol{n}\, S\right]_i dt = \left(\sum_P \left[\left(\kappa \frac{\partial \phi^n}{\partial \boldsymbol{n}}\right)_{p,p_i} S_i\right]\right) \Delta t = P_3$$

$$\left(\frac{\partial \phi^n}{\partial \boldsymbol{n}}\right)_{p,p_i} = \frac{\phi_{p_i}^n - \phi_p^n}{H_i}$$

- $$\int_{t,D_p} f'(\phi)\, dt\, dD \approx V_p \Delta t\, f'\left(\phi_p^n\right) = V_p \Delta t \begin{cases} \phi_p^{n+1}\left(1-\phi_p^n\right) r\left(\phi_p^n\right) & \text{for } r\left(\phi_p^n\right) \le 0 \\ \phi_p^{n}\left(1-\phi_p^{n+1}\right) r\left(\phi_p^n\right) & \text{for } r\left(\phi_p^n\right) > 0 \end{cases}$$

# Centroidal Voronoi Tessellation (CVT) based finite volume method is used to solve the Allen-Cahn equation

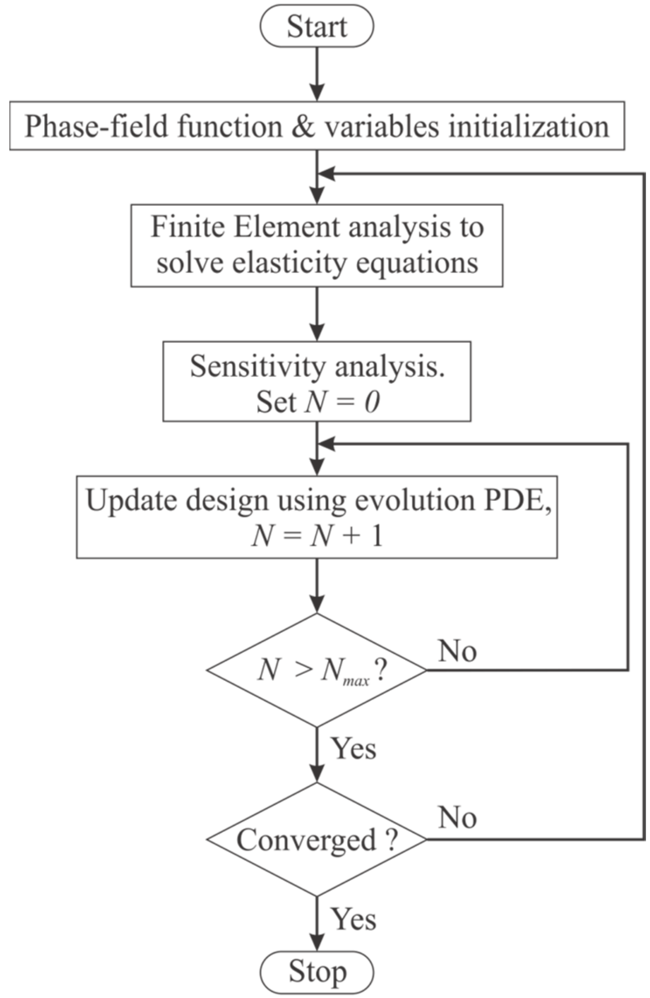
**Semi-implicit updating scheme:**

$$\phi_p^{n+1} = \begin{cases} \dfrac{V_p \phi_p^n + P_3}{V_p\left(1-\left(1-\phi_p^n\right)r\left(\phi_p^n\right)\Delta t\right)}, & \text{for } r\left(\phi_p^n\right) \le 0 \\ \dfrac{V_p \phi_p^n \left(1 + r\left(\phi_p^n\right)\Delta t\right) + P_3}{V_p\left(1+\phi_p^n r\left(\phi_p^n\right)\Delta t\right)}, & \text{for } r\left(\phi_p^n\right) > 0 \end{cases}$$

where,

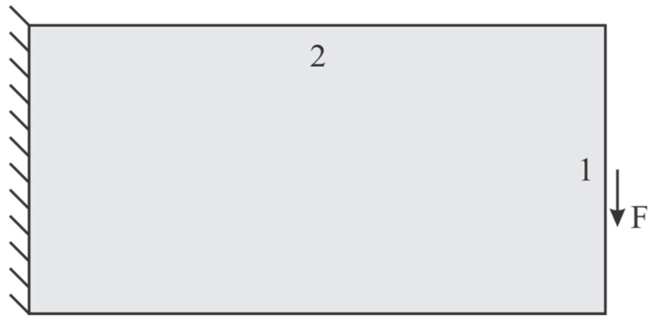
$$r\left(\phi_p^n\right) = \phi_p^n - \frac{1}{2} - 30\eta \frac{\bar{J}'(\phi_{t1})}{\left\|\bar{J}'(\phi_{t1})\right\|} \phi_p^n \left(1-\phi_p^n\right)$$

# Implementation flow chart



# Numerical examples: Rectangular domain

## Cantilever beam problem

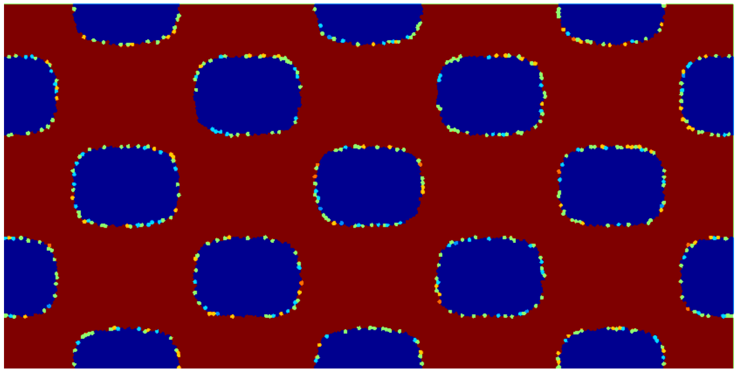


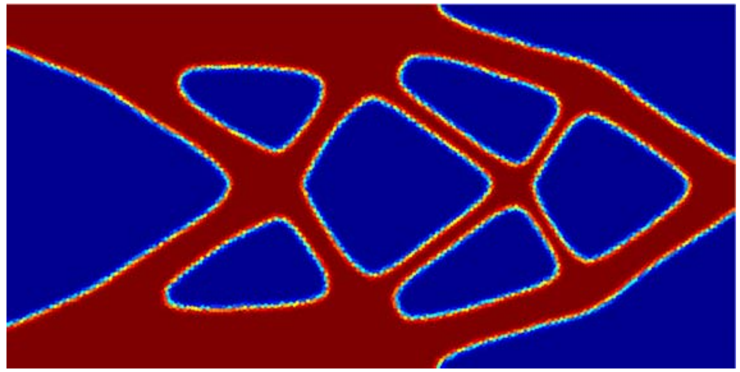

- **Objective:** Compliance minimization
- **Domain size:** 2x1 with 20,000 polygonal elements
- For each FE iteration, 20 Allen-Cahn equation updates using CVT based FV method
- $\kappa = 2\times 10^{-5}, \quad \eta = 10, \quad k_{\min} = 10^{-4}, \quad \lambda = 95$

# Numerical examples: Rectangular domain

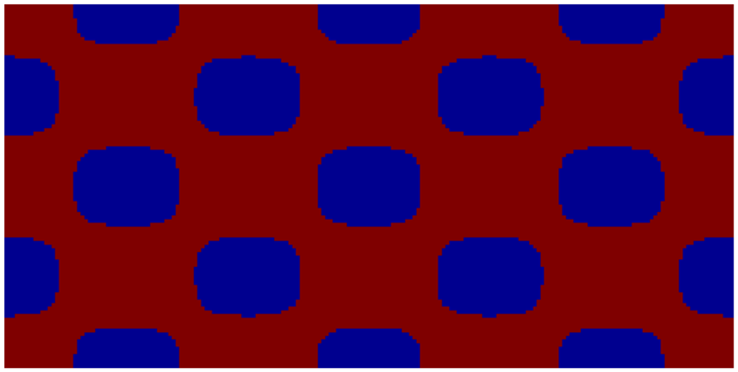
**Cantilever beam problem**

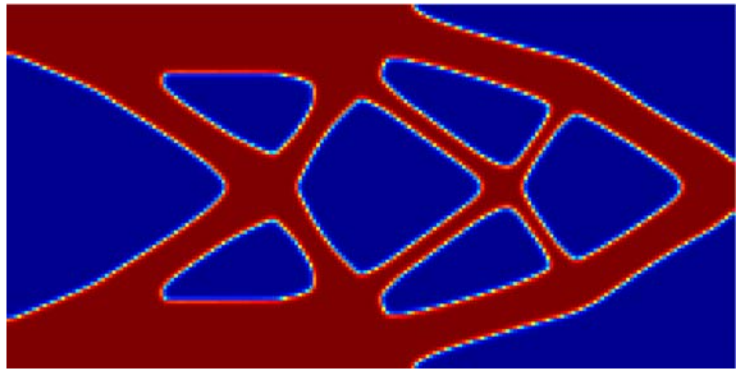
Polygonal Elements





Q4 Elements



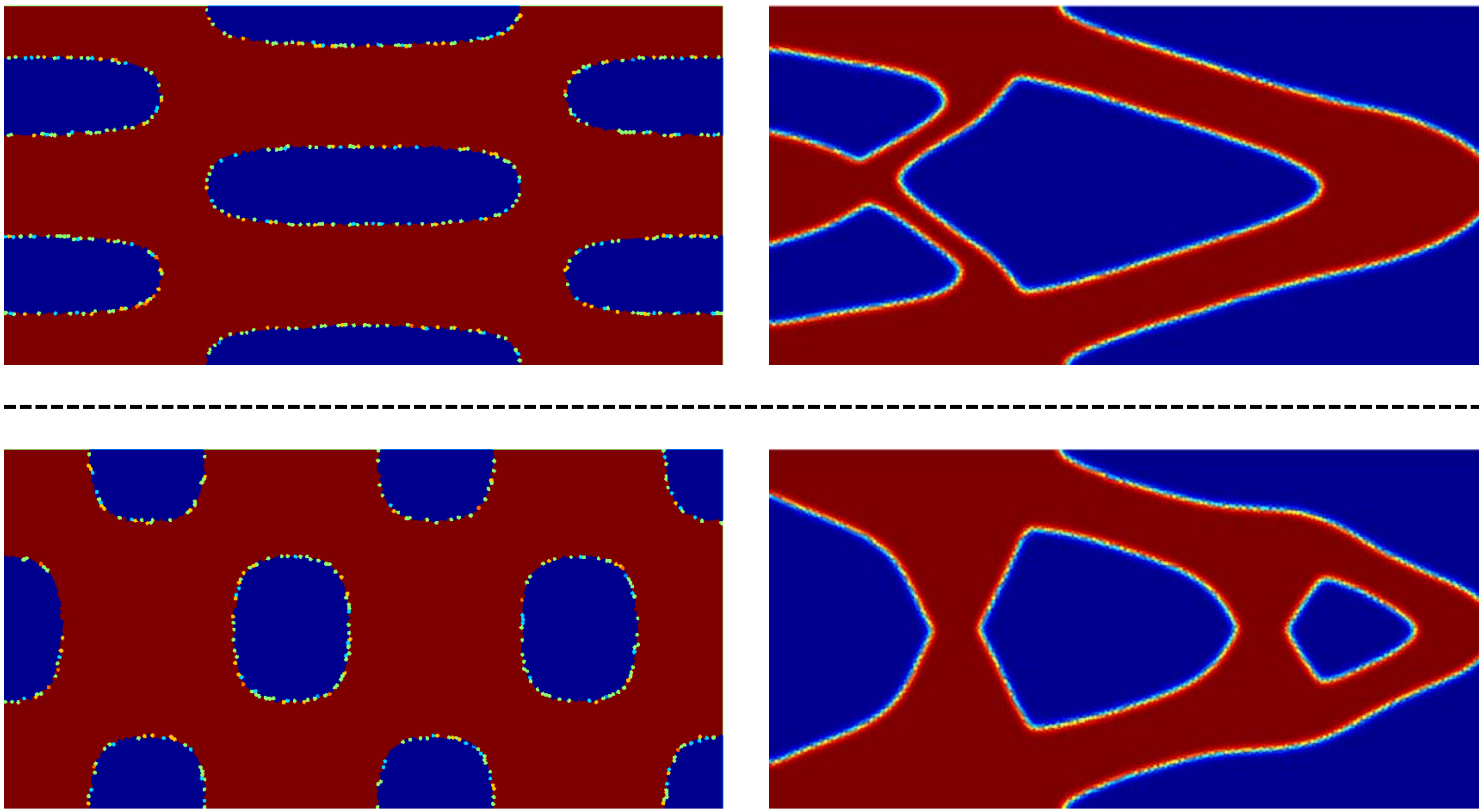


Initial configuration

Converged topology

# Numerical examples: Rectangular domain

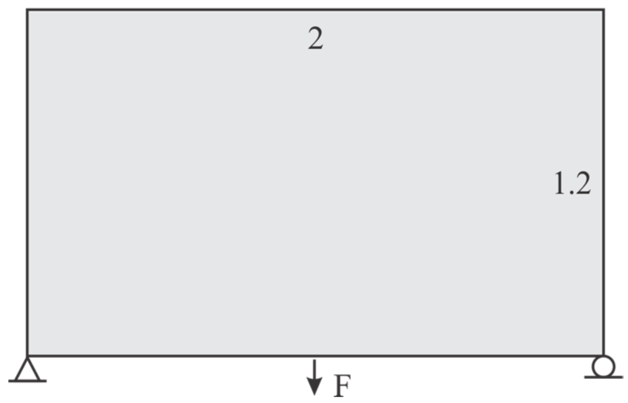
**Cantilever beam problem with <u>different initial guesses</u>**



Initial configuration                Converged topology

# Numerical examples: Rectangular domain

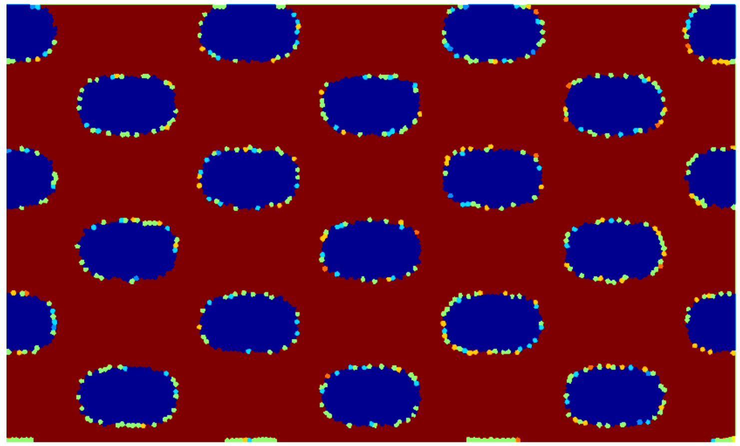
**Bridge problem – Study the influence of diffusion coefficient, $\kappa$**

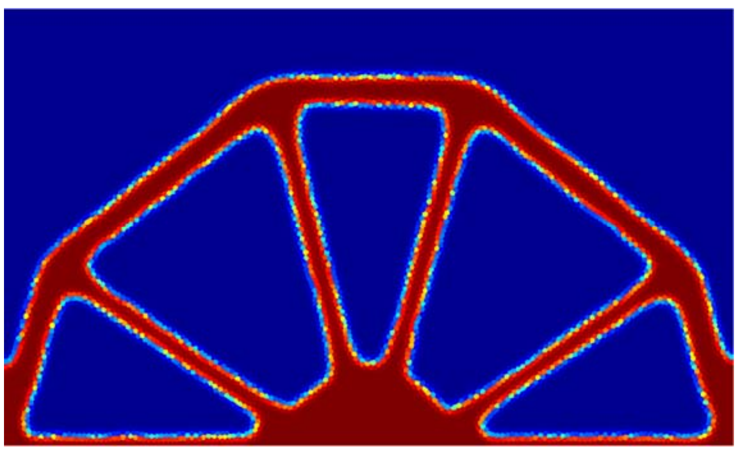



- **<u>Objective</u>:** Compliance minimization
- **<u>Domain size</u>:** 2x1.2 with 15,360 polygonal elements
- For each FE iteration, 20 Allen-Cahn equation updates using CVT based FV method
- $\kappa = 2\times10^{-5}, \quad 10\times10^{-5}$

# Numerical examples: Rectangular domain

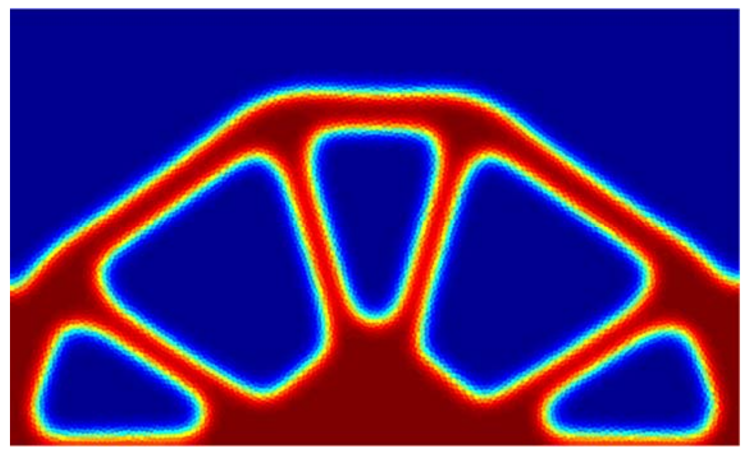
**Bridge problem – Study the influence of diffusion coefficient, $\kappa$**





$\kappa = 2\times10^{-5}$

$\phi \in [0.01, 0.99] = 28.2\%$

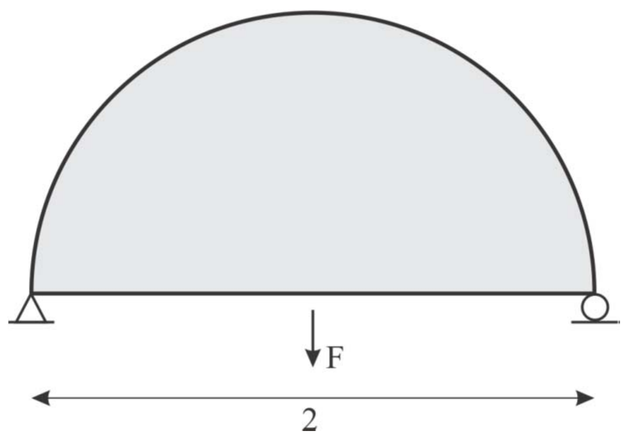


$\kappa = 10\times10^{-5}$

$\phi \in [0.01, 0.99] = 46.3\%$

# Numerical examples: non-Cartesian domain

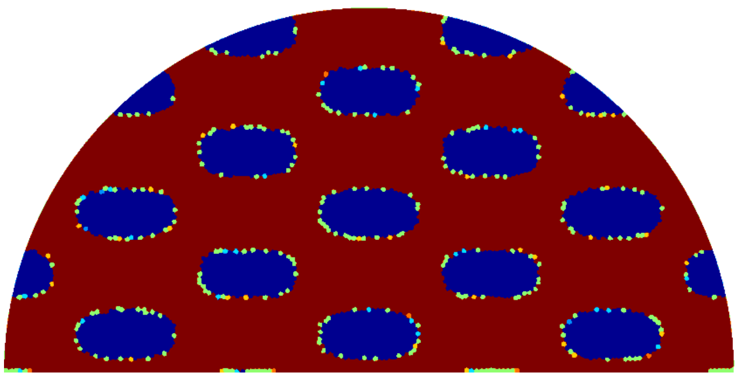
## Bridge problem on semi-circular design domain



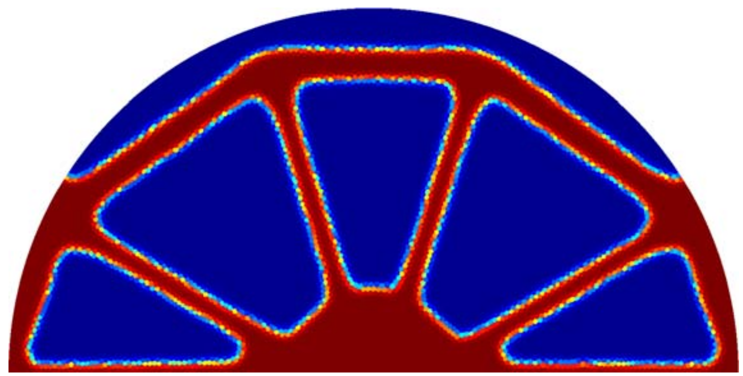

- **Objective:** Compliance minimization
- **Domain size:** 11,000 polygonal elements
- For each FE iteration, 20 Allen-Cahn equation updates using CVT based FV method
- $\kappa = 2\times10^{-5}, \quad \eta = 10, \quad k_{\min} = 10^{-4}, \quad \lambda = 60$

# Numerical examples: non-Cartesian domain

**Bridge problem on semi-circular design domain**



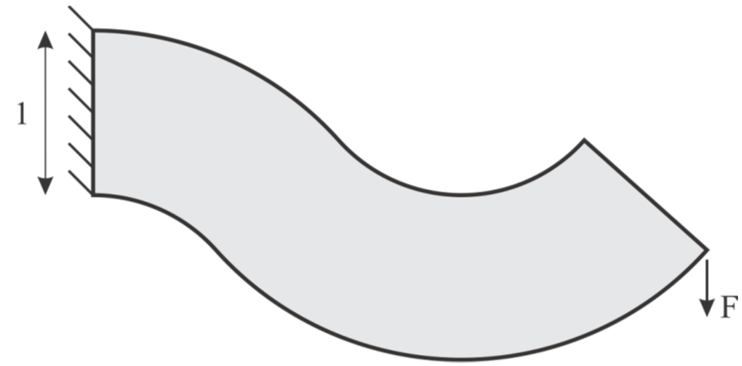
Initial configuration



Converged topology

# Numerical examples: non-Cartesian domain

## Curved cantilever beam problem



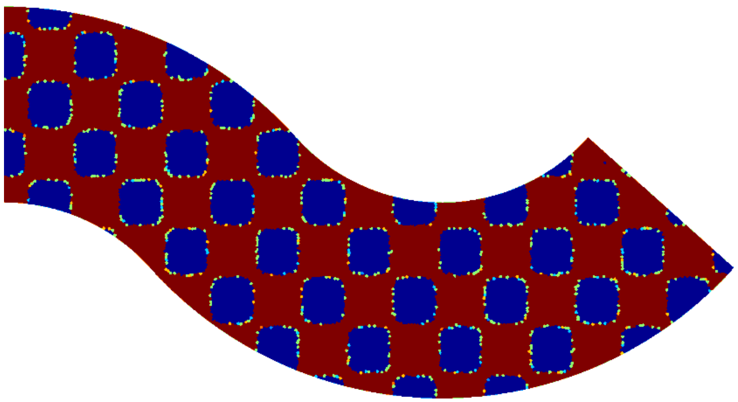

- **Objective:** Compliance minimization
- **Domain size:** 20,000 polygonal elements
- For each FE iteration, 20 Allen-Cahn equation updates using CVT based FV method
- $\kappa = 2\times10^{-5}, \quad \eta = 10, \quad k_{\min} = 10^{-4}, \quad \lambda = 250$

# Numerical examples: non-Cartesian domain

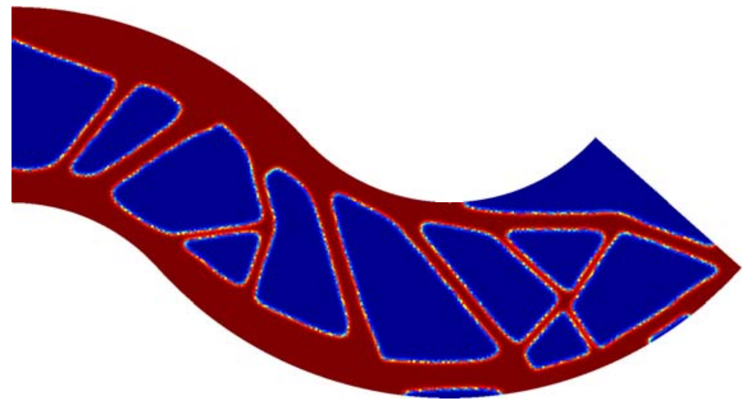
**Curved cantilever beam problem**

# Numerical examples: non-Cartesian domain

**Curved cantilever beam problem**



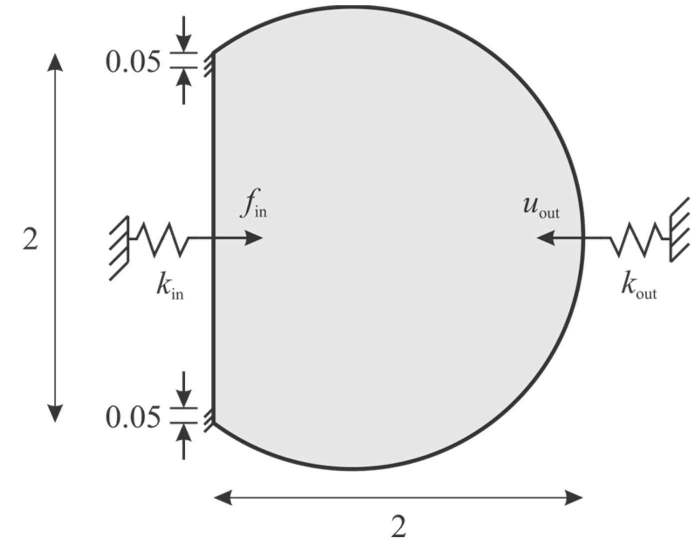
Initial configuration



Converged topology

# Numerical examples: non-Cartesian domain

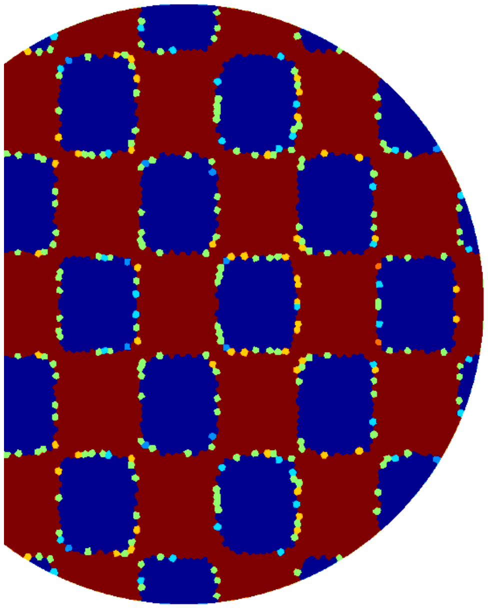
## Inverter problem on circular segment design domain



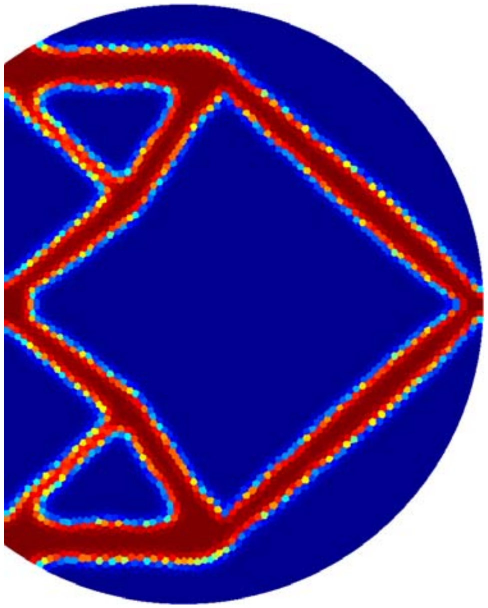

- **<u>Objective</u>:** Compliant mechanism
- **<u>Domain size</u>:** 6,000 polygonal elements
- For each FE iteration, 20 Allen-Cahn equation updates using CVT based FV method
- $\kappa = 10 \times 10^{-5}, \quad \eta = 10, \quad k_{\min} = 10^{-4}$

# Numerical examples: non-Cartesian domain

**Curved cantilever beam problem**



Initial configuration



Converged topology

## Summary and Conclusions

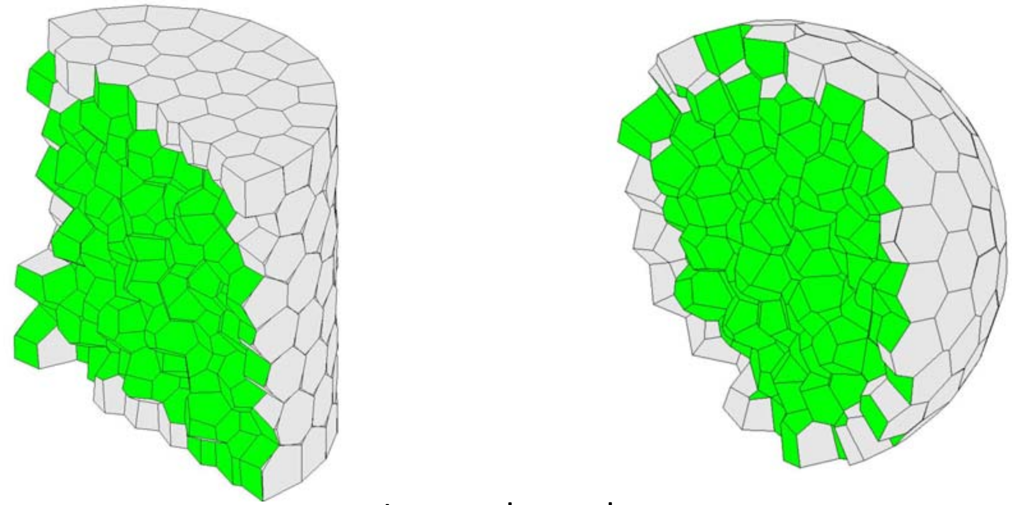
- **Phase-field based topology optimization with polygonal elements offer a general framework for topology optimization on arbitrary domains.**

- **Meshes based on simplex geometry such as quads/bricks or triangles/tetrahedrons introduce intrinsic bias in standard FEM, but polygonal/polyhedral meshes do not.**

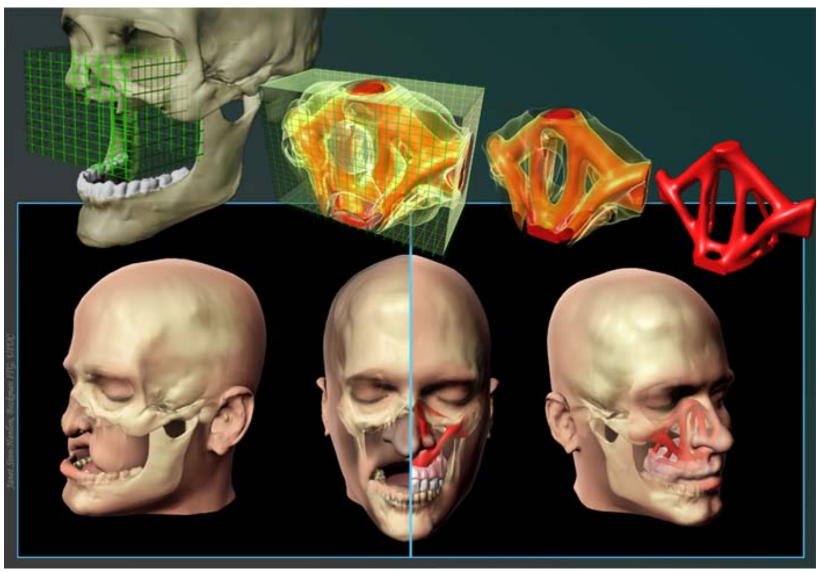
- **Polygonal/polyhedral meshes based on Voronoi tessellation not only provide greater flexibility in discretizing non-Cartesian design domains but also remove numerical artifacts such as one-node connections and checkerboard pattern in density based methods.**

# We are looking at the following future research directions:

**Implementation of phase-field method in three-dimensions using polyhedral meshes**



Leonardo et al.



**Phase field method using polygonal meshes paves the way for medical engineering applications including craniofacial segmental bone replacement**

Sutradhar A, Paulino GH, Miller MJ, Nguyen TH (2010) Topology optimization for designing patient-specific large craniofacial segmental bone replacements. Proceedings of the National Academy of Sciences 107(30): 13222-13227

# Questions and Comments?