A GENERAL TOPOLOGY OPTIMIZATION FRAMEWORK FOR POLYGONAL FINITE ELEMENT MESHES IN ARBITRARY DOMAINS

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Anderson Pereira (PUC-Rio) - Cameron Talischi (UIUC) - Glaucio Paulino (UIUC)

WCSMO-10

MOTIVATION

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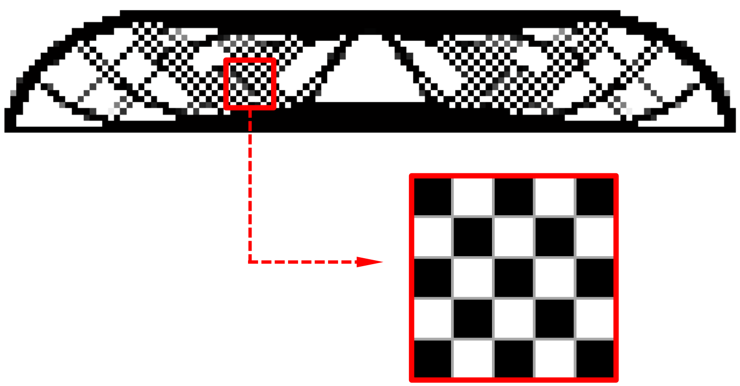

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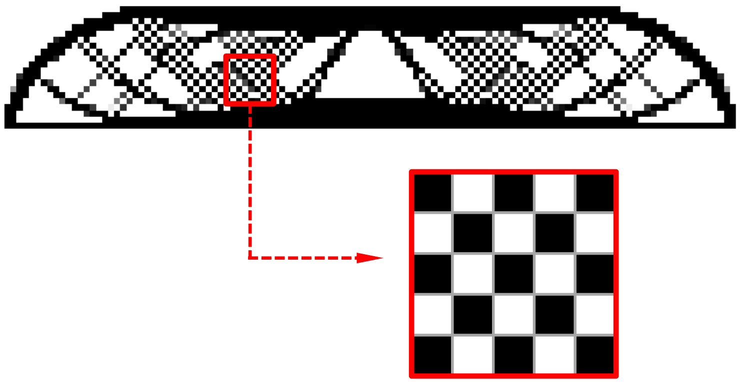

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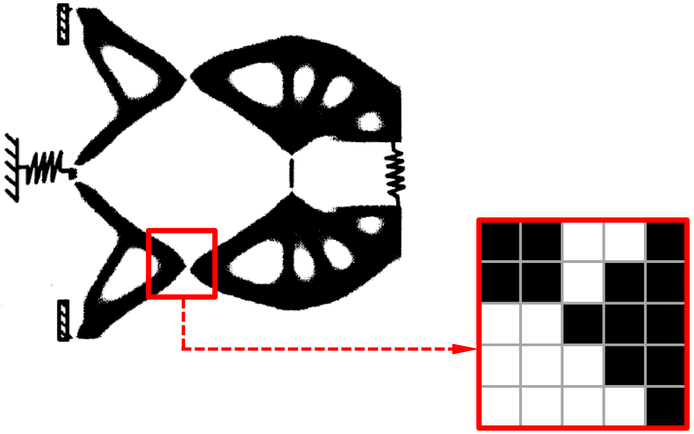

One-node Connections

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- Moreover, the use of simplex elements can cause **bias** in the orientation of members, leading to **mesh-dependent sub-optimal designs**

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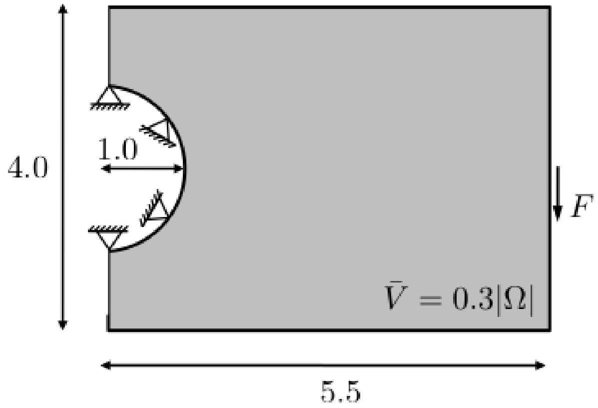
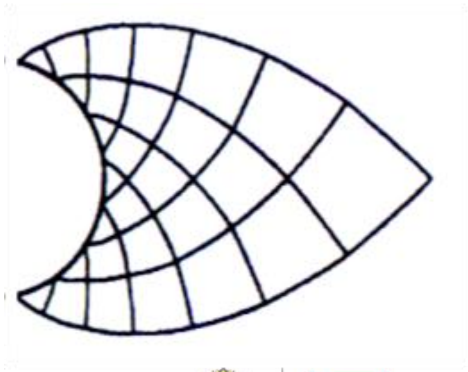
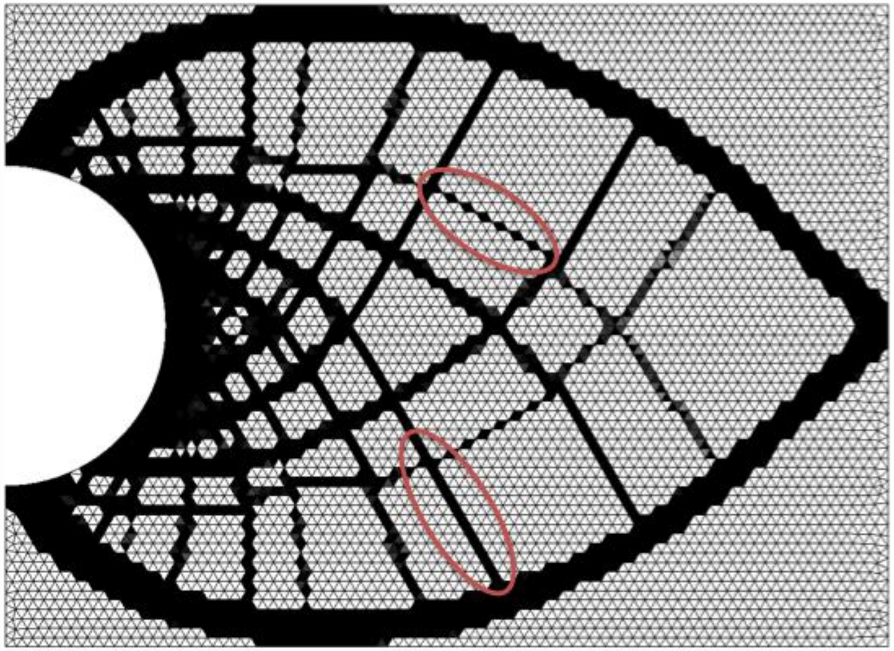
Example: Michell problem



Expected Solution





Solution obtained with **8584** T6 Elements

Main Contribution of this Research

MOTIVATION

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Investigate the behavior of **Polygonal Elements** in the context of Topology Optimization Theory

POLYGONAL FINITE ELEMENTS

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- ✓ **Provide great flexibility in discretizing complex domains**
- ✓ Naturally exclude checkerboard layouts and one-node connections
- ✓ Not biased by the standard FEM simplex geometry (triangles and quads)
- ✓ **Present "better" finite element solution of elasticity problems**
- ✓ **Stable elements w.r.t. Babuska-Brezzi condition**

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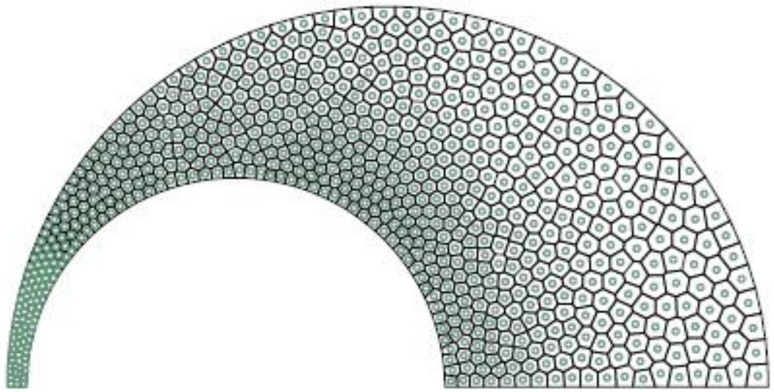

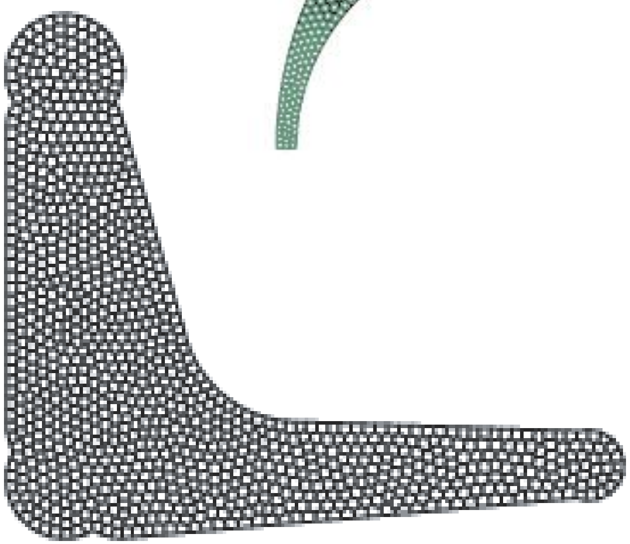

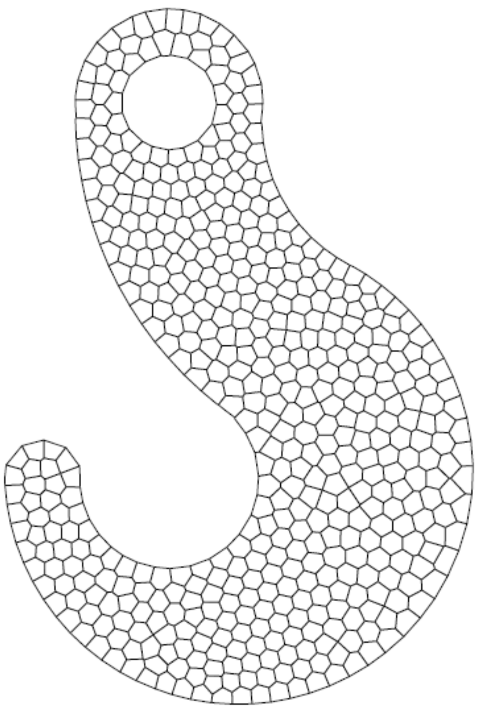

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Q4 Elements

Polygonal Elements

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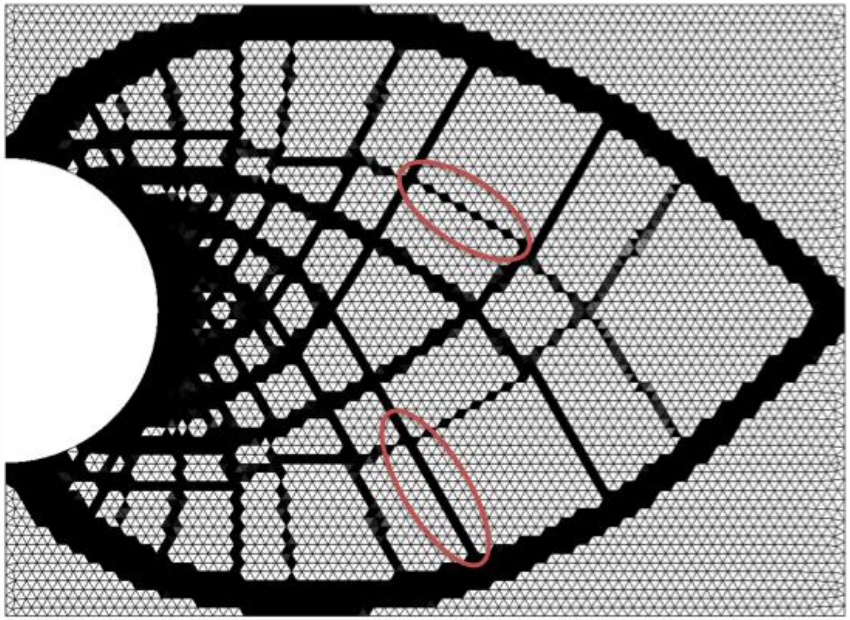
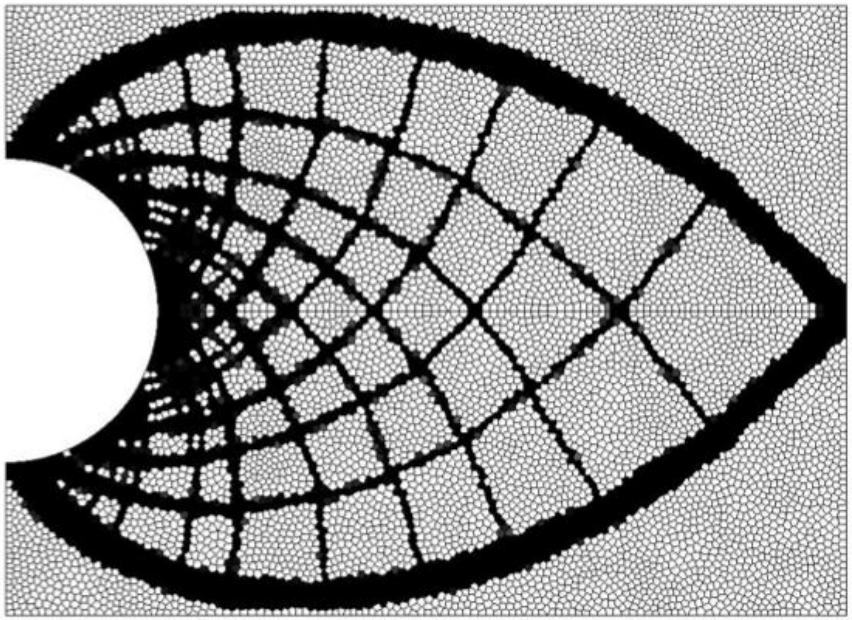
POLYGONAL FINITE ELEMENTS

Not biased by the standard FEM simplex geometry
(based on triangles and quads)



T6 Elements



Polygonal Elements

Talischi, C. , Paulino, G.H., Pereira, A. and Menezes, I.F.M., "*Polygonal Finite Elements for Topology Optimization: A Unifying Paradigm*", **IJNME**, 82(6):671-698, **2010.**

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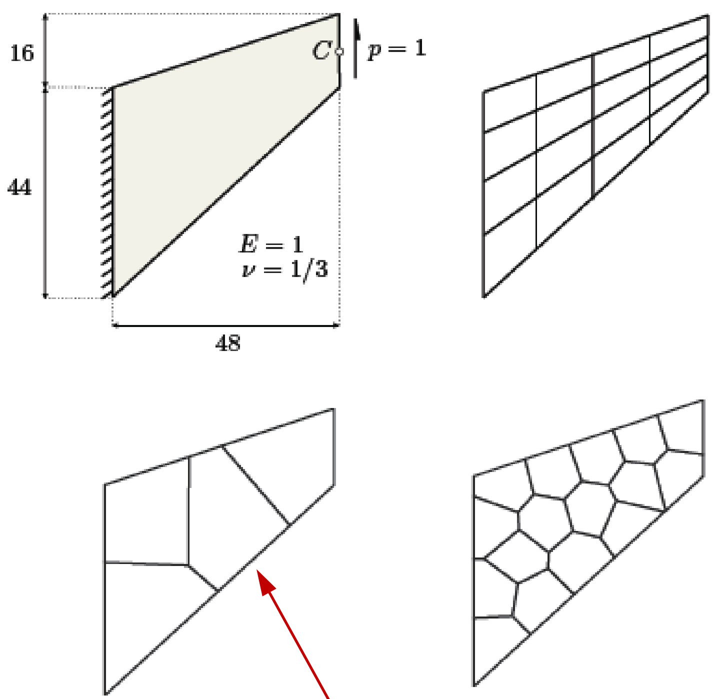
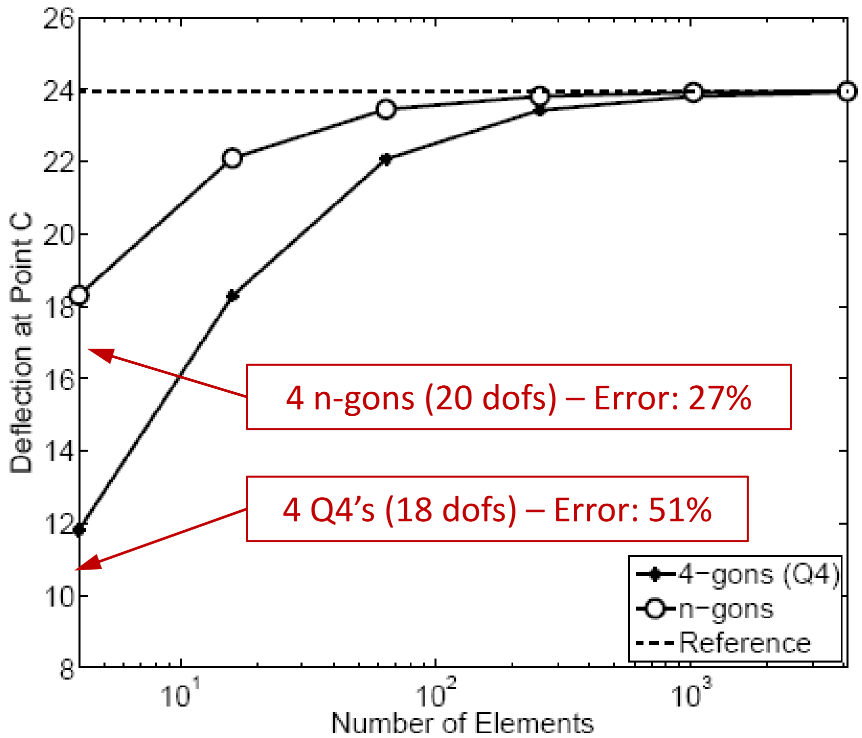


3 quads and 1 pentagon



Cook, Malkus and Plesha, “*Concepts and Applications of FE Analysis”* (4th Edn), Wiley, New York, **2002**.

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Stable elements w.r.t. Babuska-Brezzi condition

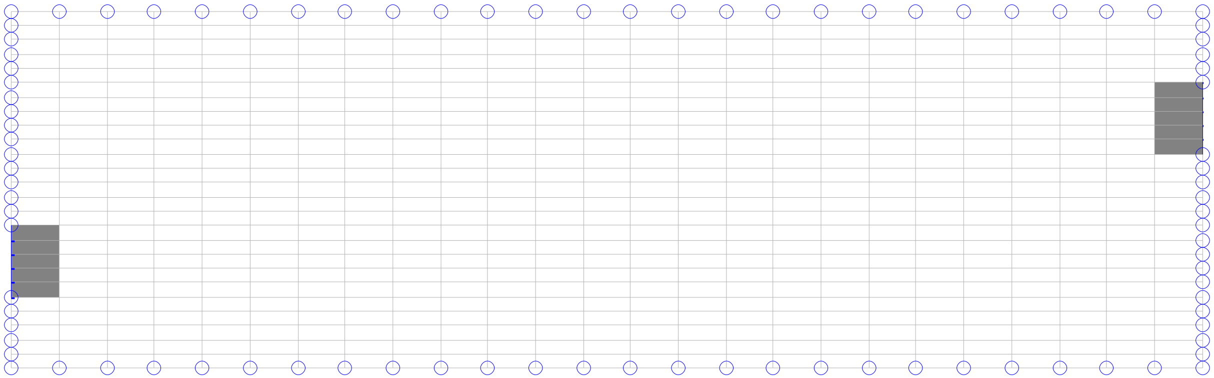
In-Out Flow Problem*: Geometry Description

Interpolation Scheme: **Linear** variation for the Velocity Field and **Constant** Pressure

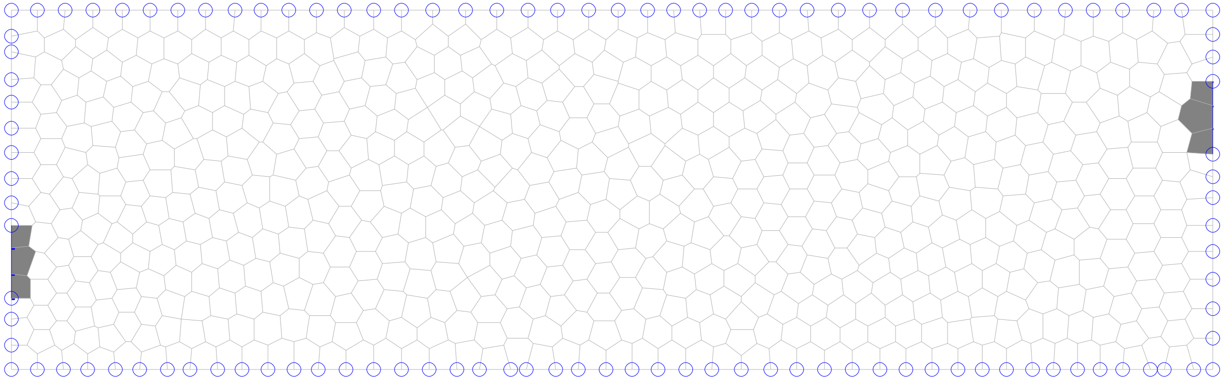
** Mixed Variational Formulation (pressure-velocity) of the Stokes Flow Problem*

Stable elements w.r.t. Babuska-Brezzi condition

Finite Element Meshes



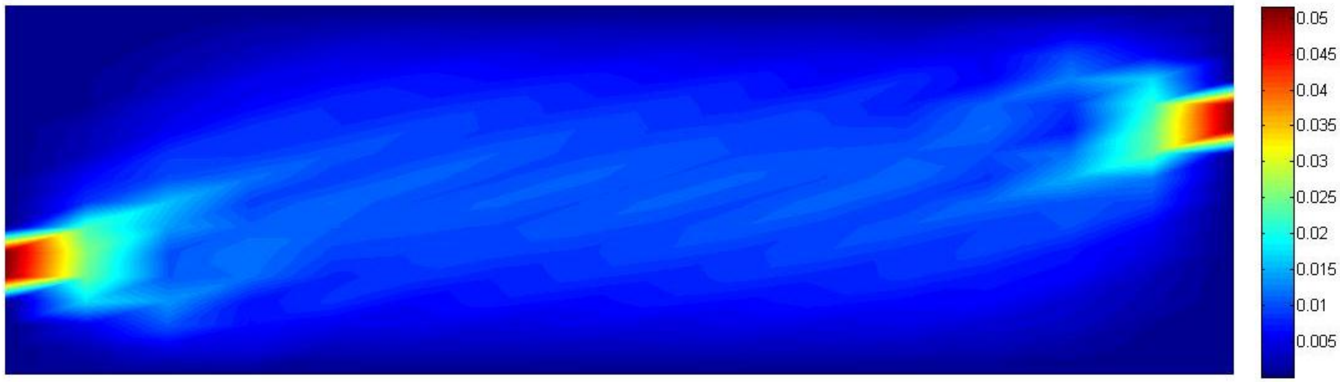
Q4 Elements



Polygonal Elements

Stable elements w.r.t. Babuska-Brezzi condition

Velocity Fields



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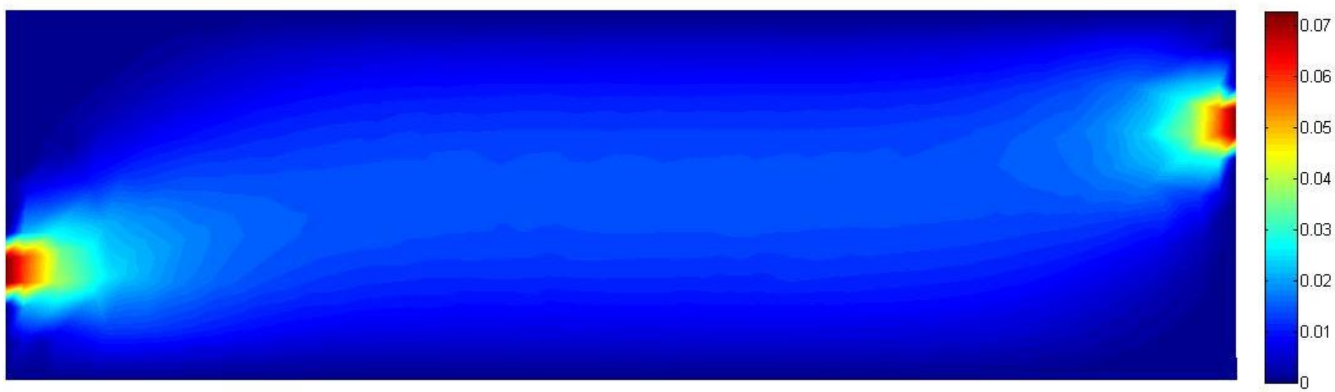
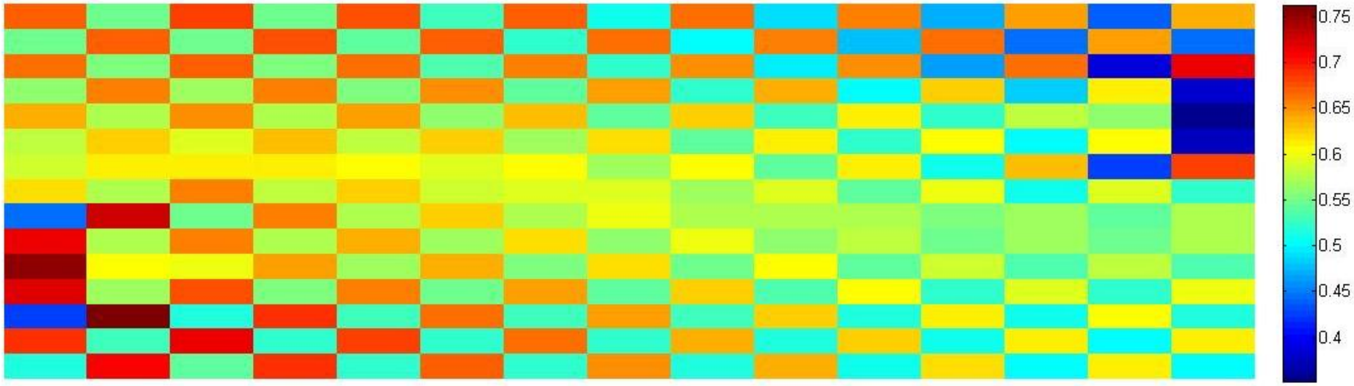


Polygonal Elements

Stable elements w.r.t. Babuska-Brezzi condition

Pressure Fields



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ELEMENT FORMULATION

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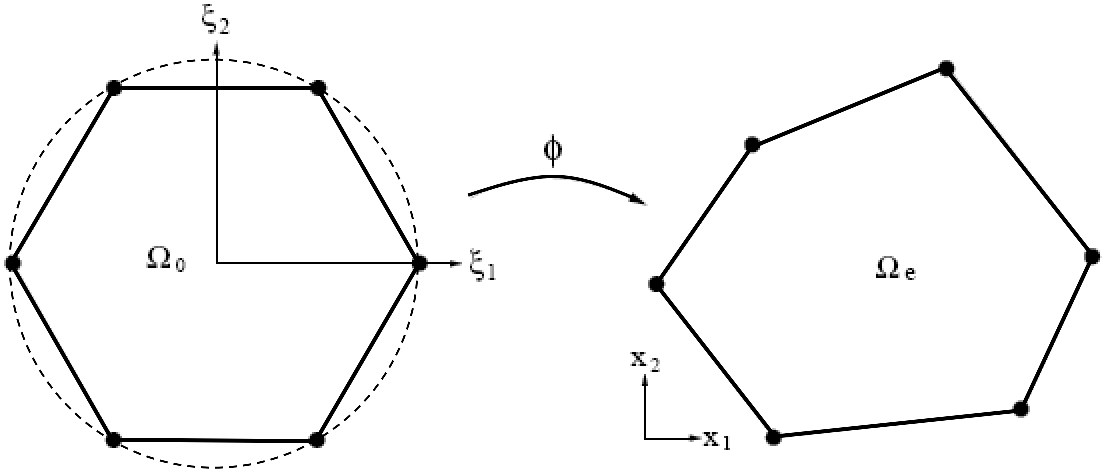
Constructed using **Laplace Shape Functions**



Isoparametric Mapping

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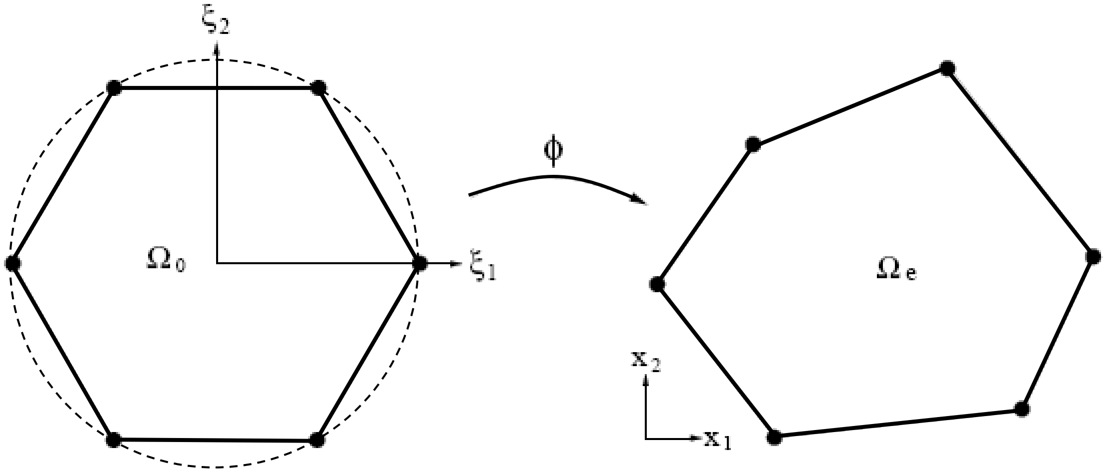

Isoparametric Mapping

“can be viewed as an extension of the common triangles and linear quads to all convex n-gons”

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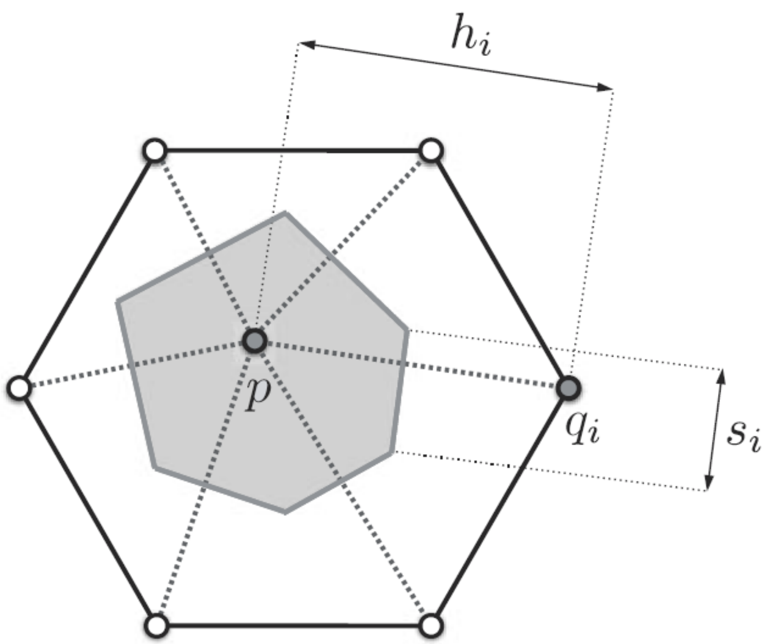
ELEMENT FORMULATION

Laplace Shape Functions



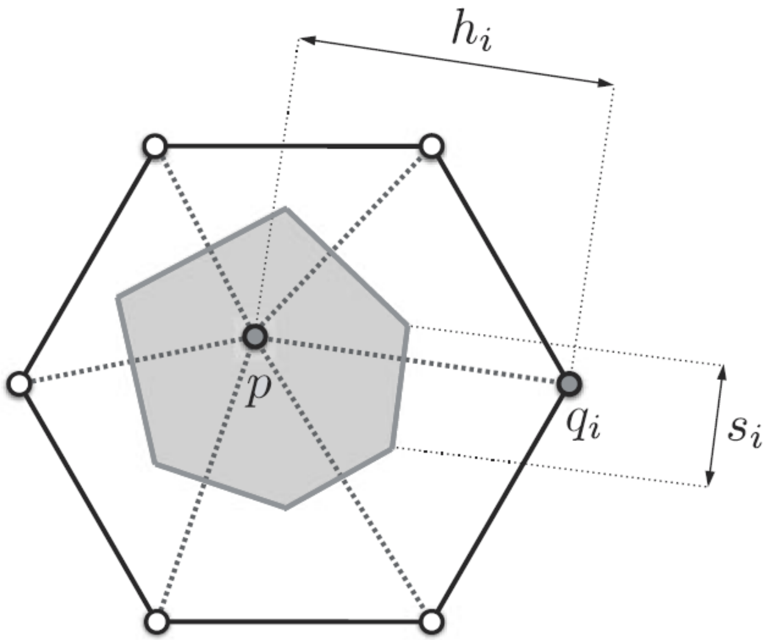
$$\phi_i(\boldsymbol{x}) = \frac{w_i(\boldsymbol{x})}{\sum_j w_j(\boldsymbol{x})}$$

$$w_i(\boldsymbol{x}) = \frac{s_i(\boldsymbol{x})}{h_i(\boldsymbol{x})}$$

Properties:

$$\begin{cases} 0 \leqslant \phi_i(\boldsymbol{x}) \leqslant 1 \\ \sum_i \phi_i(\boldsymbol{x}) = 1 \end{cases}$$

Laplace Shape Functions



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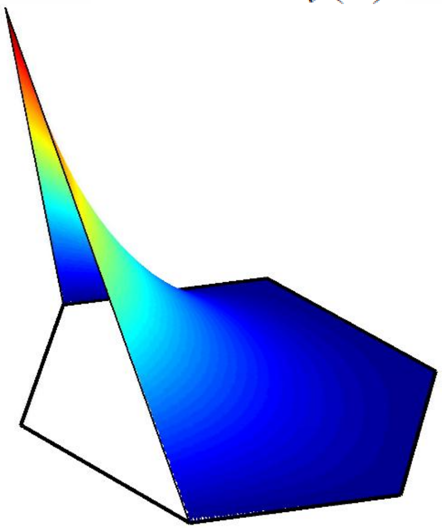

Typical Laplace Shape Function

PROPOSED FRAMEWORK

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PolyTop† – General Topology Optimization Framework using Unstructured Polygonal Finite Element Meshes

† Talischi, C., Paulino, G.H., Pereira, A., and Menezes, I.F.M., "*PolyTop: a Matlab implementation of a general topology optimization framework using unstructured polygonal finite element meshes*", **JSMO**, 45:329–357, **2012.** doi:10.1007/s00158-011-0696-x

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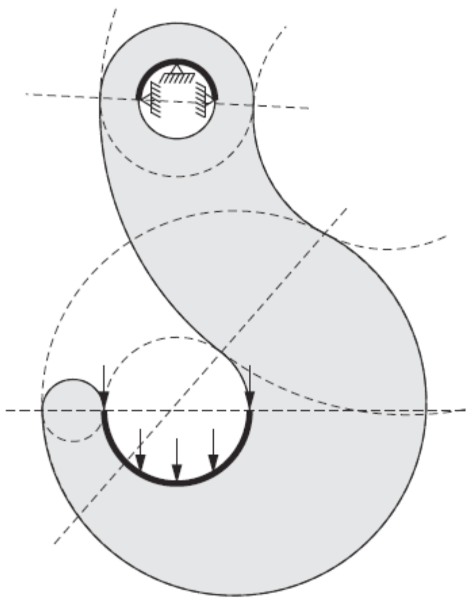

Geometry & BC's

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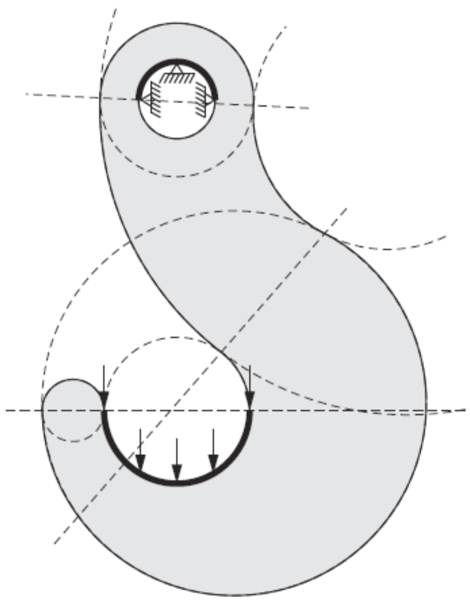

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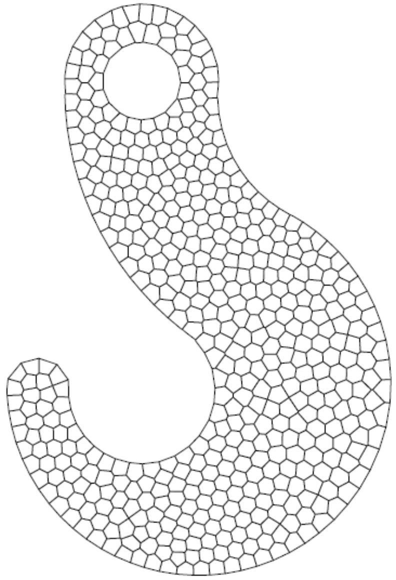

Polygonal Mesh

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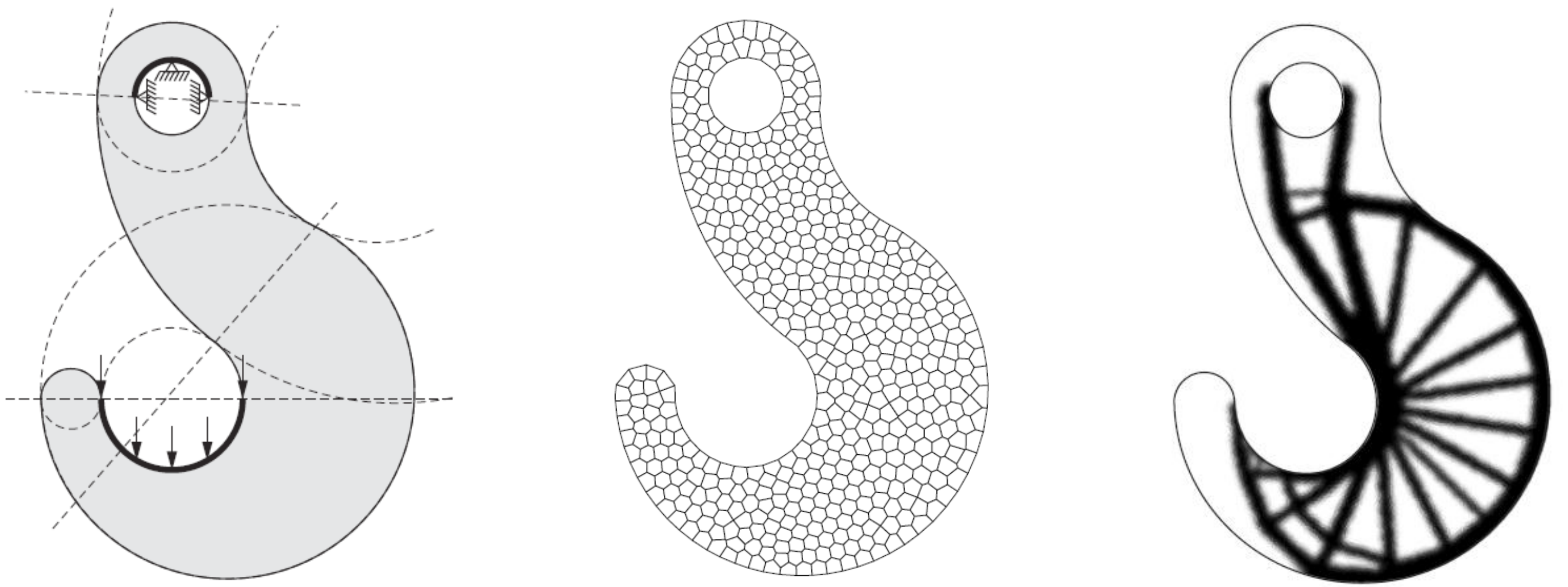

Geometry & BC's   Polygonal Mesh   PolyTop

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PolyTop FEATURES

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- Developed in MATLAB
- Possesses **189** lines, of which 116 lines pertain to the Finite Element Analysis (including 81 lines for the element stiffness calculations for polygonal elements)

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- Modular Framework

  - analysis routine and the optimization algorithm are separated from the specific choice of topology optimization formulation

  - the finite element and sensitivity analysis routines contain no information related to the formulation and thus can be extended, developed and modified independently

CODE STRUCTURE

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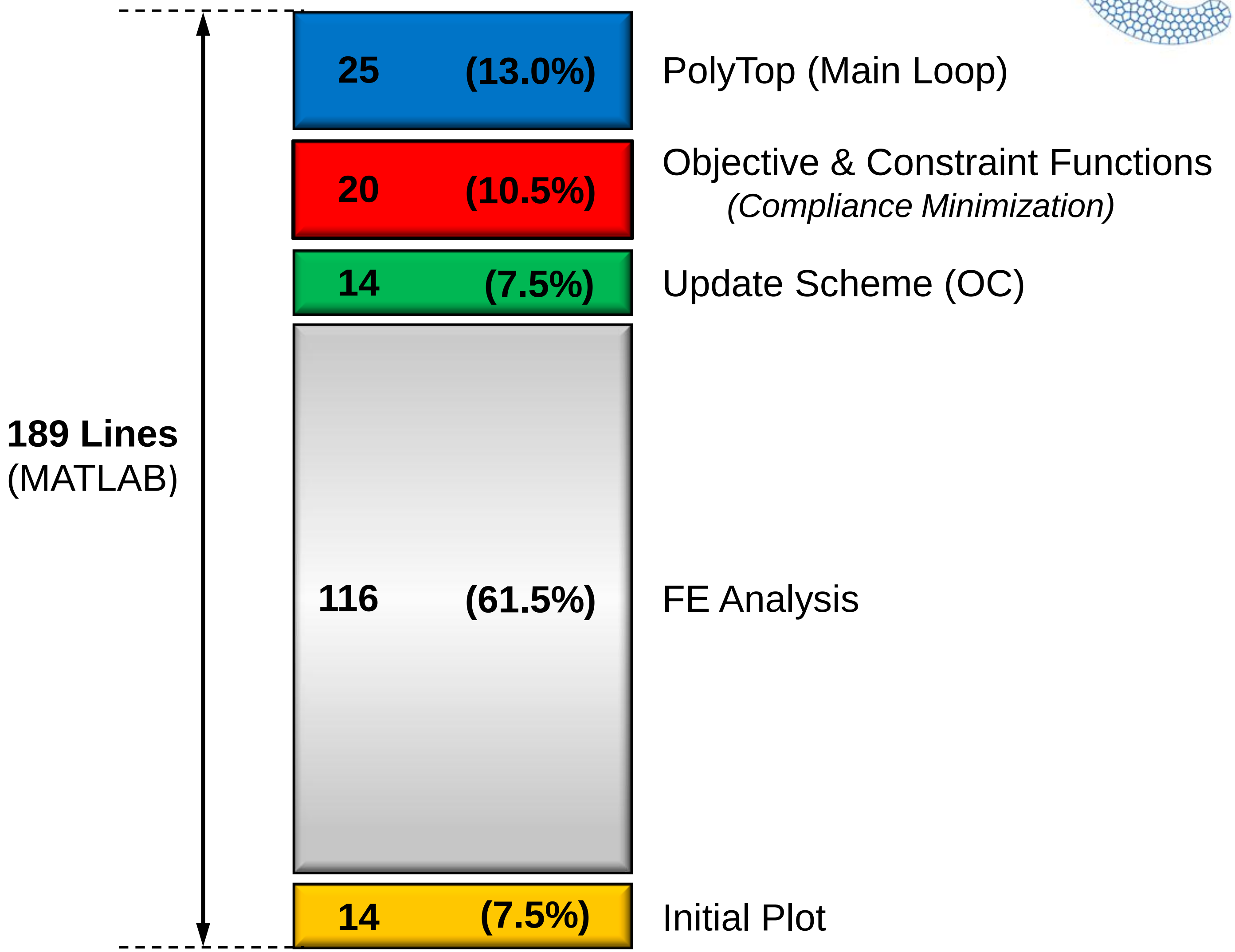

CODE STRUCTURE

Input Data Structures

The input to the main kernel PolyTop consists of two MATLAB structure arrays containing the analysis and optimization fields:

fem	opt
fem.NNode	opt.zMin
fem.NElem	opt.zMax
fem.Node	opt.zIni
fem.Element	opt.MatIntFnc
fem.Supp	opt.P
fem.Load	opt.MaxIter
fem.ShapeFnc	opt.Tol
...	...

CODE MODULARITY AND FLEXIBILITY

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- **Material interpolation functions (e.g. SIMP, RAMP)**
- Different optimizers (e.g. OC, MMA, SLP)
- Analysis routine (e.g. FEM or other method)
- Objective Function (e.g. Compliance, Compliant Mechanism)

CODE MODULARITY AND FLEXIBILITY

Material Interpolation Functions

```
function [E,dEdy,V,dVdy] = MatIntFnc(y,type,param)
eps = 1e-4;  %Ersatz stiffness
switch(type)
  case('SIMP')
    penal = param;
    E = eps+(1-eps)*y.^penal;
    V = y;
    dEdy = (1-eps)*penal*y.^(penal-1);
    dVdy = ones(size(y,1),1);
  case('RAMP')
    q = param;
    E = eps+(1-eps)*y./(1+q*(1-y));
    V = y;
    dEdy = ((1-eps)*(q+1))./(q-q*y+1).^2;
    dVdv = ones(size(y));
end
```

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Select a Different Optimizer

```
%-------------------------------------------- OPTIMALITY CRITERIA UPDATE
function [zNew,Change] = UpdateScheme(dfdz,g,dgdz,z0,opt)
zMin=opt.zMin; zMax=opt.zMax;
move=opt.OCMove*(zMax-zMin); eta=opt.OCEta;
l1=0; l2=1e6;
while l2-l1 > 1e-4
  lmid = 0.5*(l1+l2);
  B = -(dfdz./dgdz)/lmid;
  zCnd = zMin+(z0-zMin).*B.^eta;
  zNew = max(max(min(min(zCnd,z0+move),zMax),z0-move),zMin);
  if (g+dgdz'*(zNew-z0)>0),  l1=lmid;
  else                        l2=lmid;  end
end
Change = max(abs(zNew-z0))/(zMax-zMin);
```

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Change the Objective Function

Compliance:
(self-adjoint problem)

$$f = \mathbf{F}^T\mathbf{U} \quad ; \quad \frac{\partial f}{\partial \mathbf{E}} = -\mathbf{U}^T \frac{\partial \mathbf{K}}{\partial \mathbf{E}} \mathbf{U} = -\sum \mathbf{U}_i \left(\mathbf{k}_l\right)_{ij} \mathbf{U}_j$$

Compliant:
(non self-adjoint problem)

$$f = \mathbf{L}^T\mathbf{U} \quad ; \quad \left.\begin{array}{r} \dfrac{\partial f}{\partial \mathbf{E}} = -\boldsymbol{\lambda}^T \dfrac{\partial \mathbf{K}}{\partial \mathbf{E}} \mathbf{U} \\ \\ \mathbf{K}\boldsymbol{\lambda} = \mathbf{L} \end{array}\right\} -\sum \boldsymbol{\lambda}_i \left(\mathbf{k}_l\right)_{ij} \mathbf{U}_j$$

CODE MODULARITY AND FLEXIBILITY

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PolyTop can be easily extended to handle compliant mechanism design by changing **10** existing lines and adding **7** new ones!

CODE MODULARITY AND FLEXIBILITY

Change the Objective Function

```
< f = dot(fem.F,U);
< temp = cumsum(-U(fem.i).*fem.k.*U(fem.j));
< dfdE = [temp(1);temp(2:end)-temp(1:end-1)];
< while l2-l1 > 1e-4
<   B = -(dfdz./dgdz)/lmid;
<   fem.F = zeros(2*fem.NNode,1);  %external load vector
<   fem.F(2*fem.Load(1:NLoad,1)-1) = fem.Load(1:NLoad,2);
<   fem.F(2*fem.Load(1:NLoad,1))   = fem.Load(1:NLoad,3);







< K = sparse(fem.i,fem.j,E(fem.e).*fem.k);
< U = zeros(2*fem.NNode,1);
```

Compliance

```
> f = fem.DOut(3)*U(fem.DofDOut,1);
> temp = cumsum(fem.DOut(3)*U(fem.i,1).*fem.k.*U(fem.j,2));
> dfdE = -[temp(1);temp(2:end)-temp(1:end-1)];
> while (l2-l1)/(l2+l1) > 1e-4 && l2>1e-40
>   B = max(1e-10,-(dfdz./dgdz)/lmid);
>   fem.F = zeros(2*fem.NNode,2);
>   fem.F(2*fem.Load(1:NLoad,1)-1,1) = fem.Load(1:NLoad,2);
>   fem.F(2*fem.Load(1:NLoad,1),1)   = fem.Load(1:NLoad,3);
>   fem.DofDOut = 2*fem.DOut(1)-2+fem.DOut(2);
>   fem.F(fem.DofDOut,2) = -1;
>   NSpring = size(fem.Spring,1);
>   s = zeros(2*fem.NNode,2);  %spring vector
>   s(2*fem.Spring(1:NSpring,1)-1) = fem.Spring(1:NSpring,2);
>   s(2*fem.Spring(1:NSpring,1))   = fem.Spring(1:NSpring,3);
>   fem.s = spdiags(s(:),0,2*fem.NNode,2*fem.NNode);
> K = sparse(fem.i,fem.j,E(fem.e).*fem.k) + fem.s;
> U = zeros(2*fem.NNode,2);
```

Compliant Mechanism

CODE MODULARITY AND FLEXIBILITY

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< K = sparse(fem.i,fem.j,E(fem.e).*fem.k);
< U = zeros(2*fem.NNode,1);
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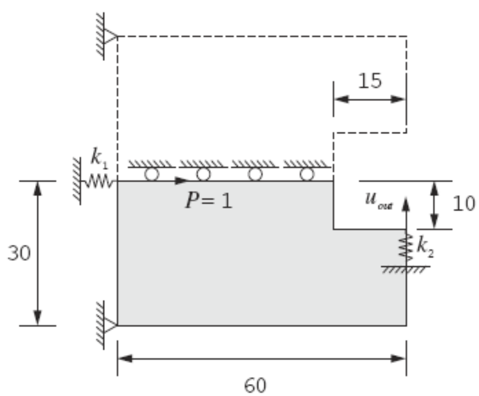
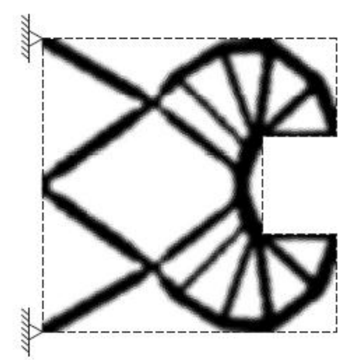
Compliance

```
> f = fem.DOut(3)*U(fem.DofDOut,1);
> temp = cumsum(fem.DOut(3)*U(fem.i,1).*fem.k.*U(fem.j,2));
> dfdE = -[temp(1);temp(2:end)-temp(1:end-1)];
> while (l2-l1)/(l2+l1) > 1e-4 && l2>1e-40
>   B = max(1e-10,-(dfdz./dgdz)/lmid);
>   fem.F = zeros(2*fem.NNode,2);
>   fem.F(2*fem.Load(1:NLoad,1)-1,1) = fem.Load(1:NLoad,2);
>   fem.F(2*fem.Load(1:NLoad,1),1)   = fem.Load(1:NLoad,3);
>   fem.DofDOut = 2*fem.DOut(1)-2+fem.DOut(2);
>   fem.F(fem.DofDOut,2) = -1;
>   NSpring = size(fem.Spring,1);
>   s = zeros(2*fem.NNode,2);  %spring vector
>   s(2*fem.Spring(1:NSpring,1)-1) = fem.Spring(1:NSpring,2);
>   s(2*fem.Spring(1:NSpring,1))   = fem.Spring(1:NSpring,3);
>   fem.s = spdiags(s(:),0,2*fem.NNode,2*fem.NNode);
> K = sparse(fem.i,fem.j,E(fem.e).*fem.k) + fem.s;
> U = zeros(2*fem.NNode,2);
```

Compliant Mechanism

Example
(Compliant Mechanism):

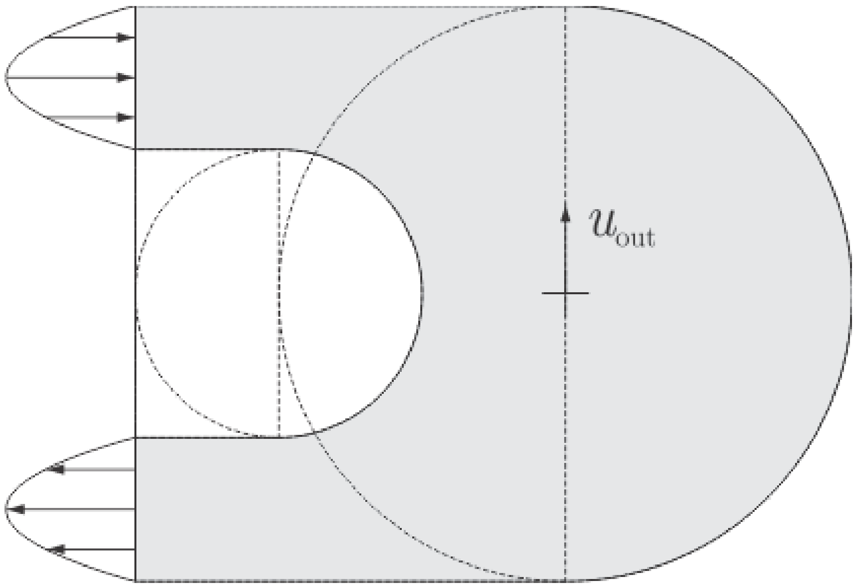




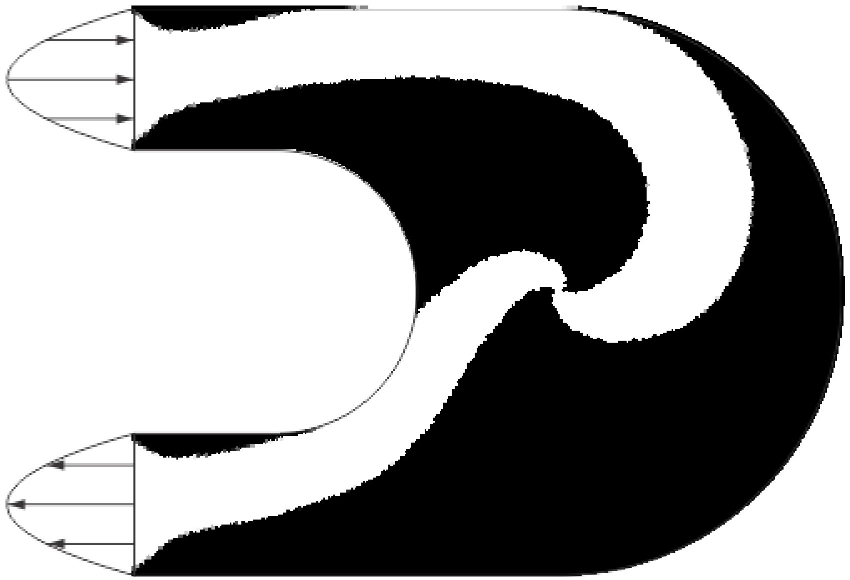
CODE MODULARITY AND FLEXIBILITY

Example: Fluid Mechanics (Stokes Flow)

Maximize the y-velocity at a given location:



Problem Description

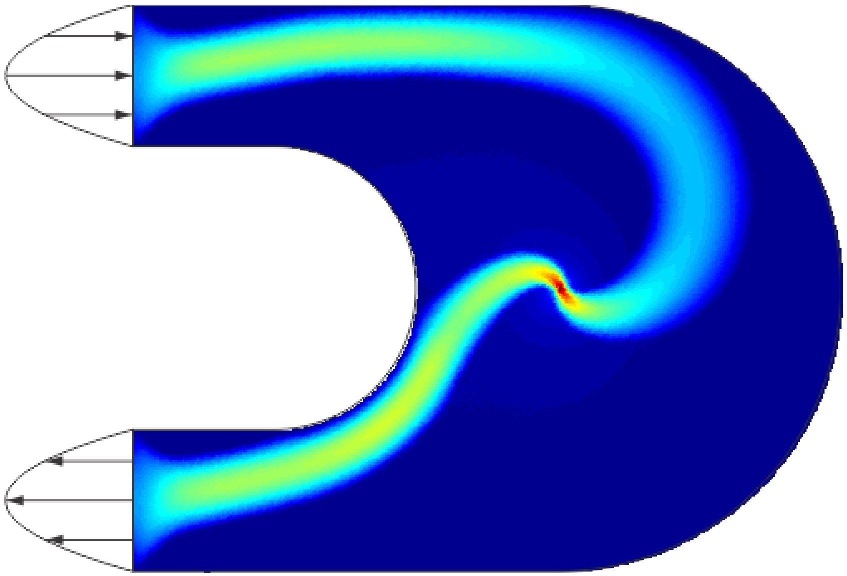


Optimal Solution

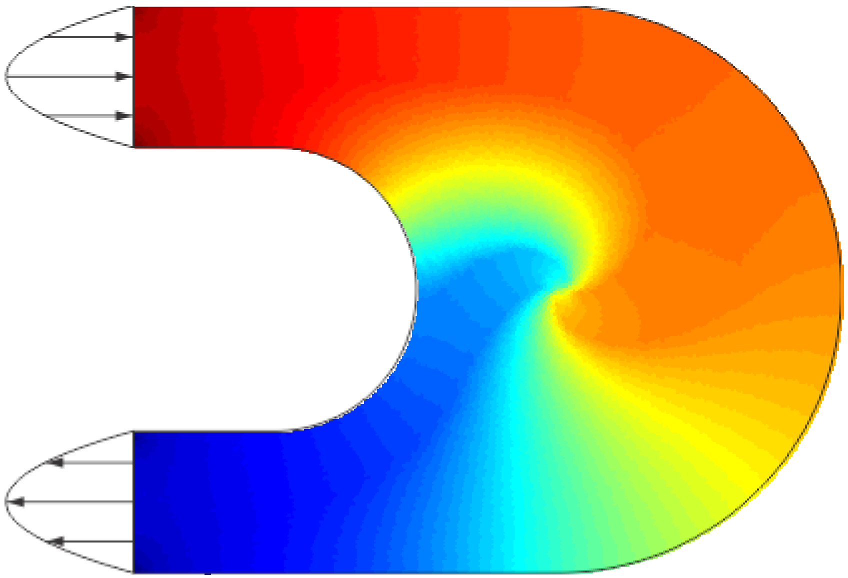
CODE MODULARITY AND FLEXIBILITY

Example: Fluid Mechanics (Stokes Flow)

Maximize the y-velocity at a given location:



Velocity Field



Pressure Field

CODE EFFICIENCY

CODE EFFICIENCY

Comparison with the 88-line code*

Mesh Size	90x30	150x50	300x100	600x200
PolyTop	11.9	**31.5**	**135.5**	**764.1**
88-line	**10.9**	33.0	252.2	3092.9

(time in sec for 200 optimization iterations)

* Andreassen E., Clausen A., Schevenels M., Lazarov B., Sigmund O., "*Efficient topology optimization in MATLAB using 88 lines of code*", **JSMO**, 43(1):1–16, **2011.** doi:10.1007/s00158-010-0594-7

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Mesh Size	90x30	150x50	300x100	600x200
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(time in sec for 200 optimization iterations)

† Design Volume
(OC Update Function)

$$V(\mathbf{z}) = \sum_{\ell=1}^{N} (\mathbf{Pz})_\ell = \mathbf{1}^T(\mathbf{Pz}) = \left(\mathbf{1}^T\mathbf{P}\right)\mathbf{z} = \left(\mathbf{P}^T\mathbf{1}\right)^T\mathbf{z}$$

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MESH GENERATION – BASIC IDEAS

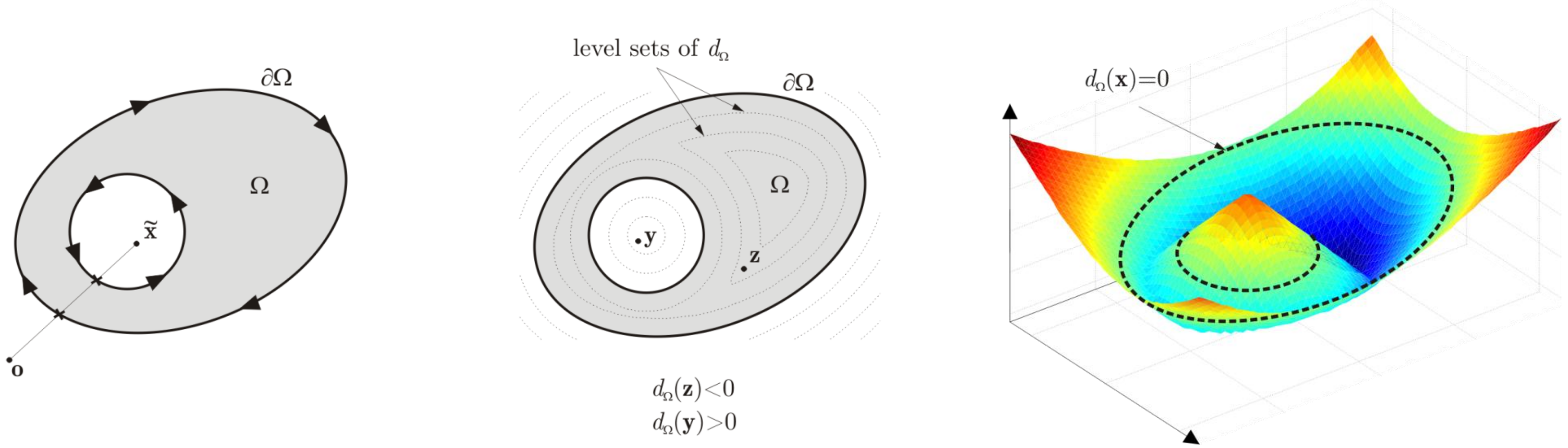
MESH GENERATION – BASIC IDEAS

- **Based on the Concept of Voronoi Tesselation**
- **Implicit Representation (signed distance functions)**

MESH GENERATION – BASIC IDEAS

Implicit Representation

- The implicit representation of the domain is one of the main ingredients of our meshing algorithm:



- The signed distance function contains all the essential information about the meshing domain needed in our mesh algorithm:

$$d_\Omega(\mathbf{x}) = s_\Omega(\mathbf{x}) \min_{\mathbf{y}\in\partial\Omega} \|\mathbf{x}-\mathbf{y}\| \qquad s_\Omega(\mathbf{x}) := \begin{cases} -1, & \mathbf{x}\in\Omega \\ +1, & \mathbf{x}\in\mathbb{R}^2\backslash\Omega \end{cases}$$

PROPOSED FRAMEWORK

PROPOSED FRAMEWORK

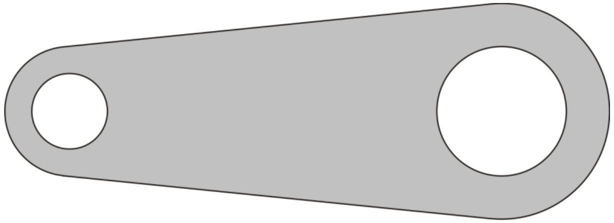
PolyMesher† – General-purpose Mesh Generator for Polygonal Elements

- Developed in MATLAB
- Possesses **133** lines, of which 69 lines pertain to the mesh post processing

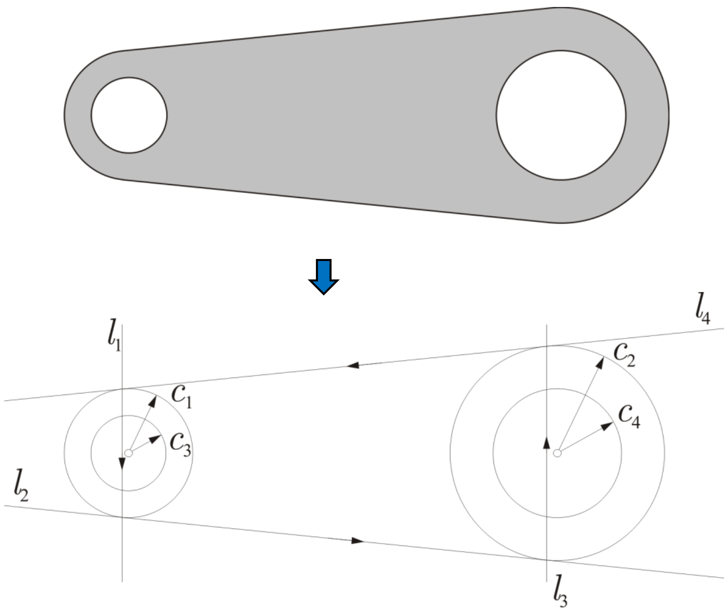
† Talischi, C., Paulino, G.H., Pereira, A., and Menezes, I.F.M., "*PolyMesher: a general-purpose mesh generator for polygonal elements written in Matlab*", **JSMO**, 45:309–328, **2012.** doi:10.1007/s00158-011-0706-z

PolyMesher EXAMPLE

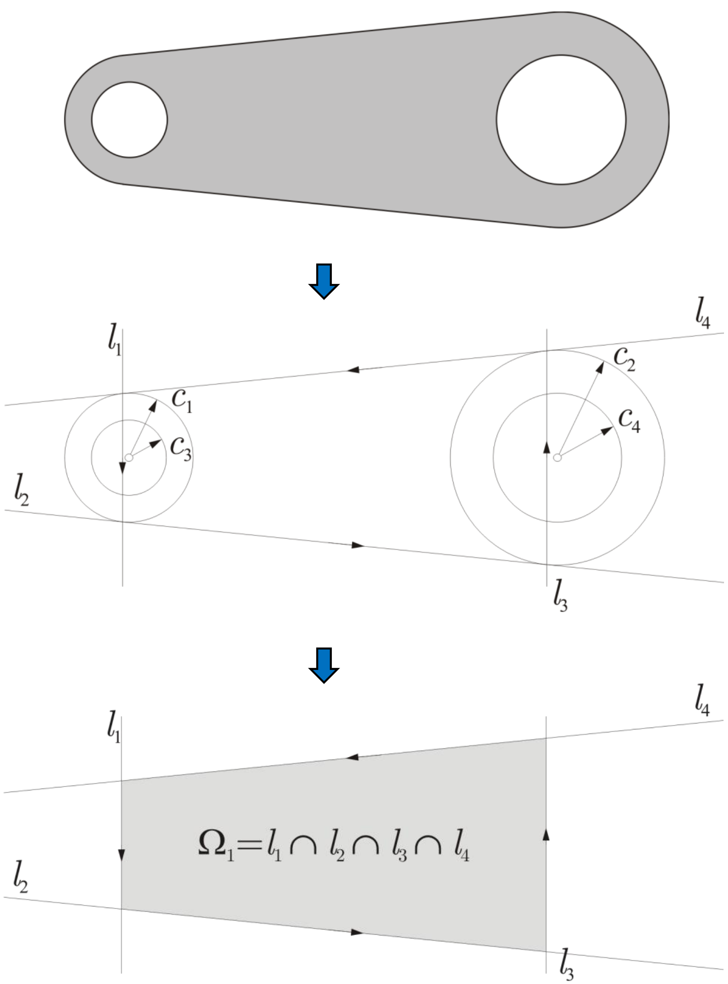
PolyMesher EXAMPLE: WRENCH



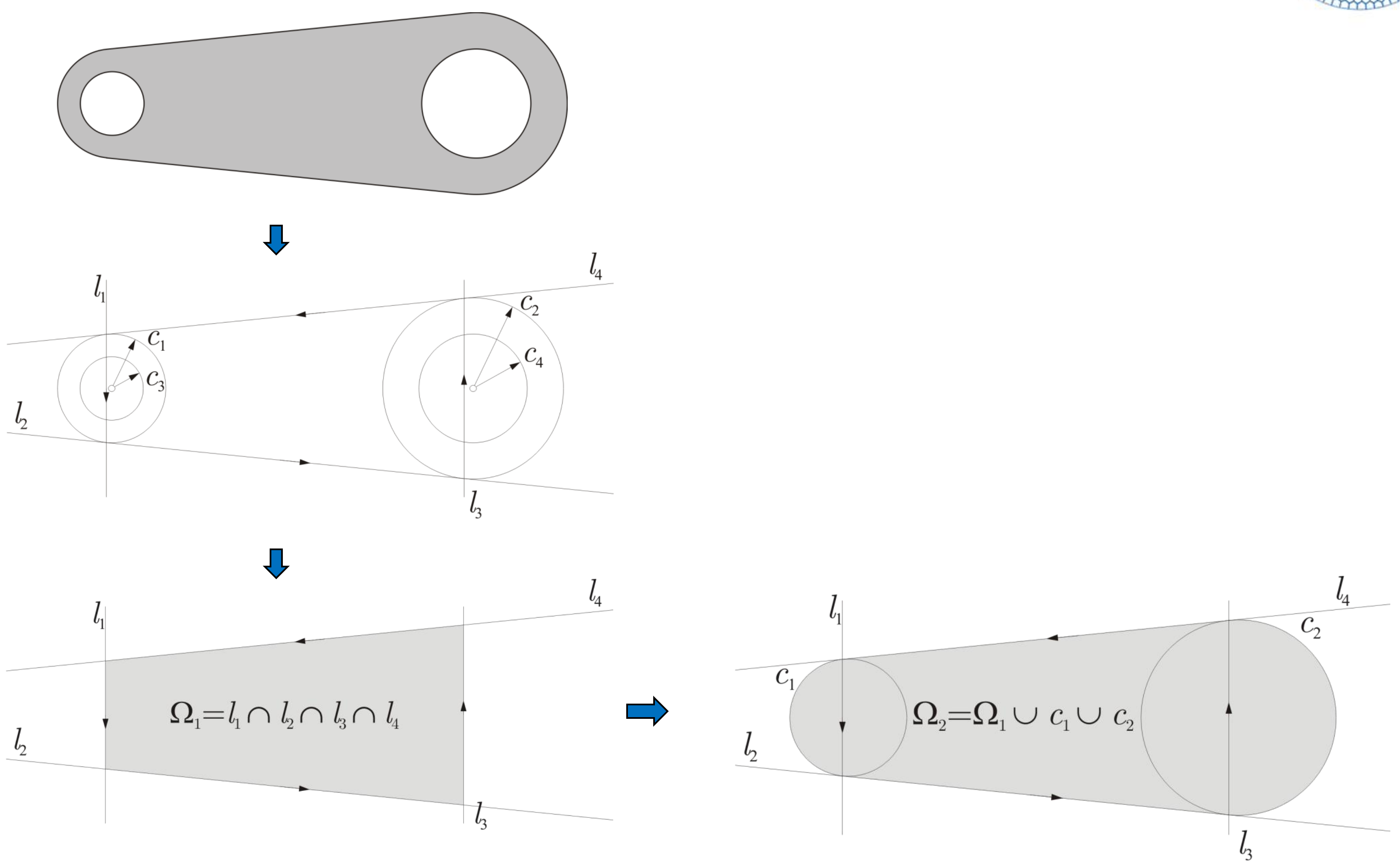
PolyMesher EXAMPLE: WRENCH



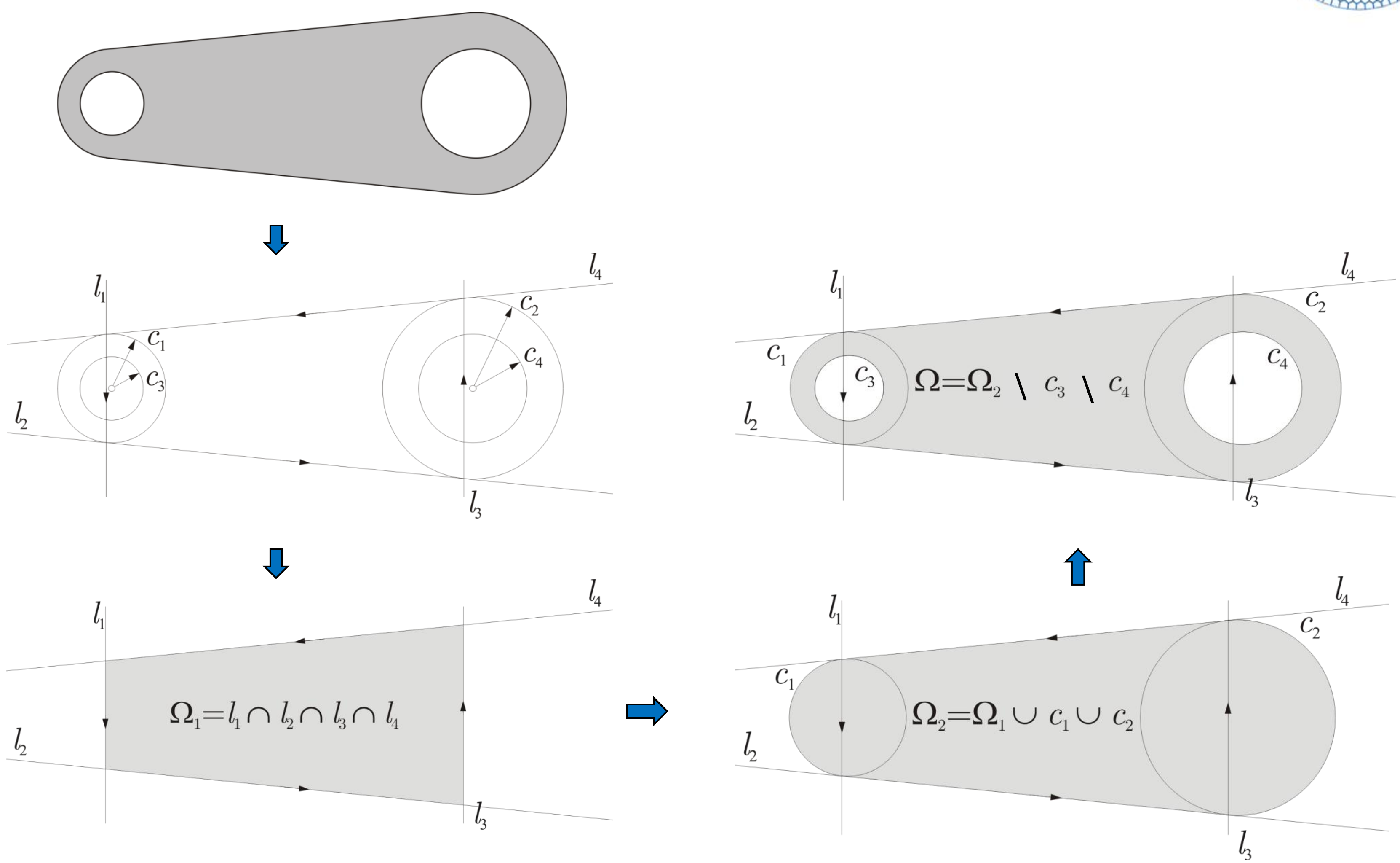
PolyMesher EXAMPLE: WRENCH



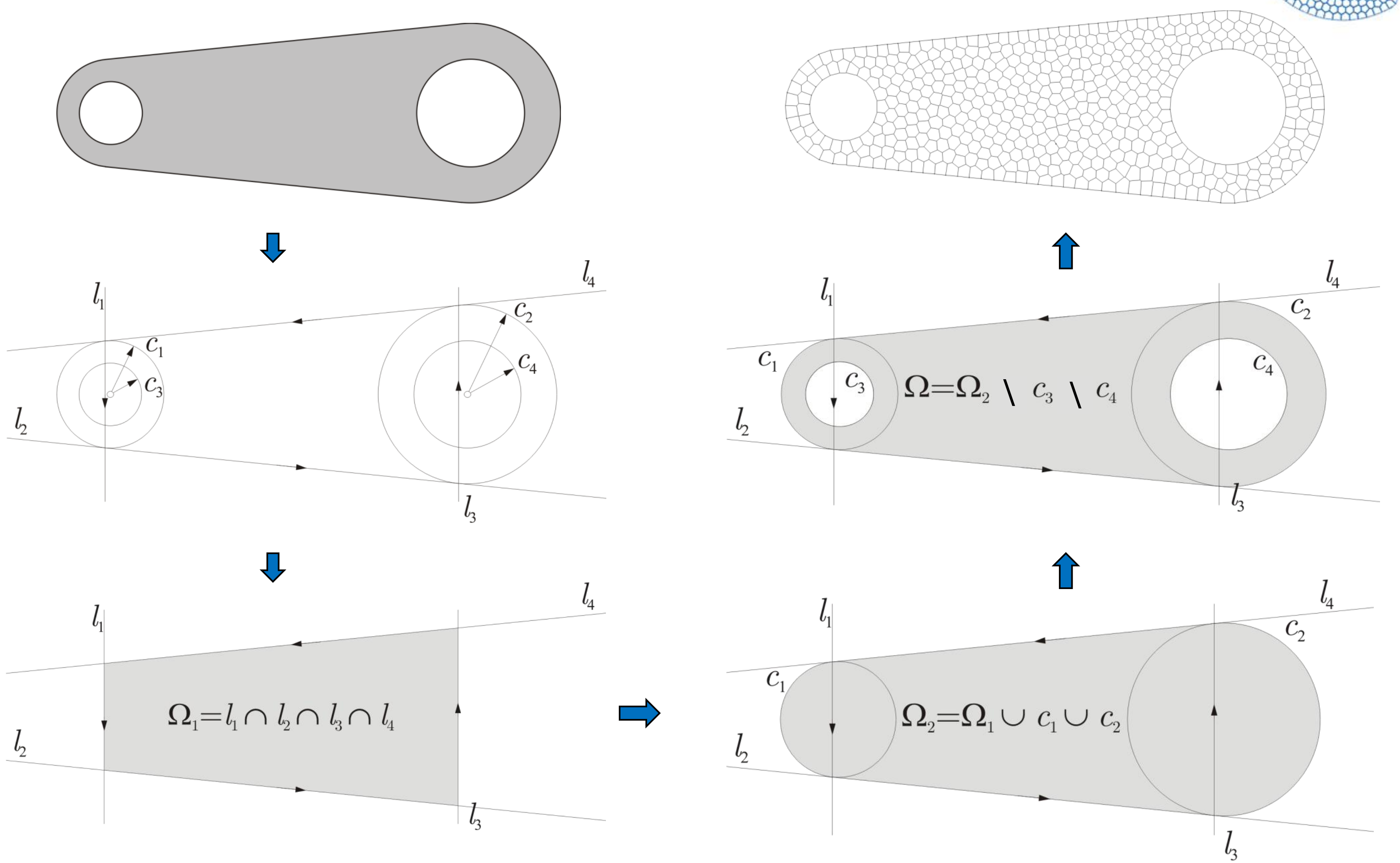
PolyMesher EXAMPLE: WRENCH



PolyMesher EXAMPLE: WRENCH



PolyMesher EXAMPLE: WRENCH



MATLAB DEMO

CONCLUDING REMARKS

CONCLUDING REMARKS

- The development of PolyMesher was motivated by the desire to present a complete, self-contained, efficient and useful code in MATLAB, including domain description and discretization algorithms;
- Due to the modularity and flexibility of the PolyTop code, analysis routine and optimization algorithm can be separated from the specific choice of topology optimization formulation;
- We would like to point out that PolyTop offers room for further exploration of finite elements and topology optimization formulations both for research and for practical engineering applications;
- We hope that the community can make use of PolyMesher & PolyTop in ways that we cannot anticipate.

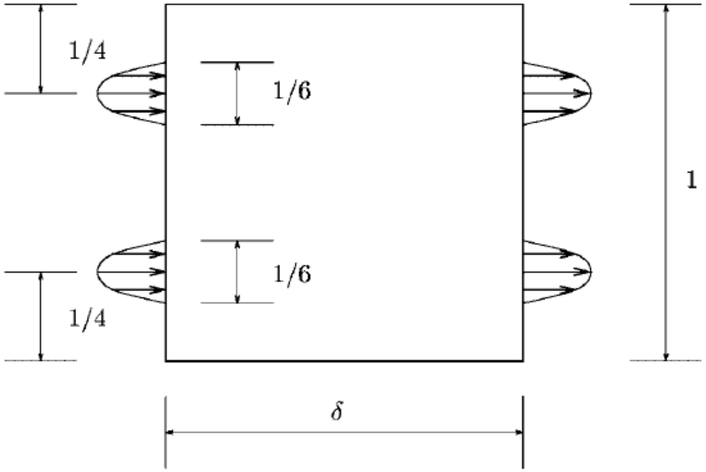
QUESTIONS ?

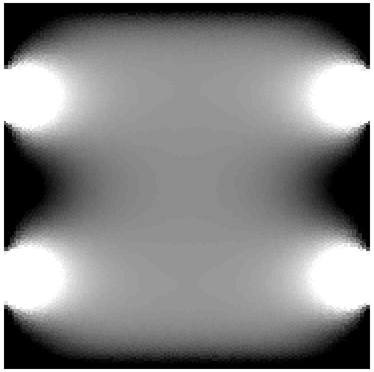
SUPPLEMENTARY MATERIAL

ONGOING RESEARCH

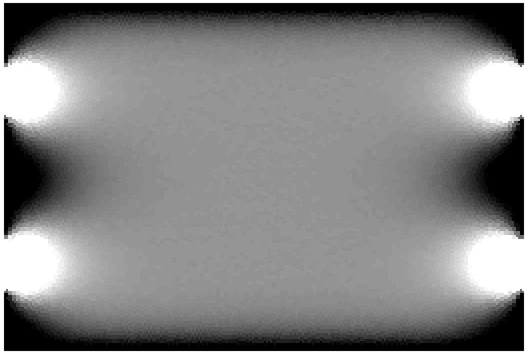
- Topology optimization of fluids in Stokes flow

Double pipe problem





Optimal double pipes for *d*=1



Optimal double pipes for *d*=1.5

CODE EFFICIENCY

Comparison with the 88-line code*

Mesh Size	90x30	150x50	300x100	600x200
PolyTop	11.9	31.5	135.5	764.1
88-line	10.9	33.0	252.2	3092.9
88-line†	**9.7**	**24.3**	**119.7**	**708.8**

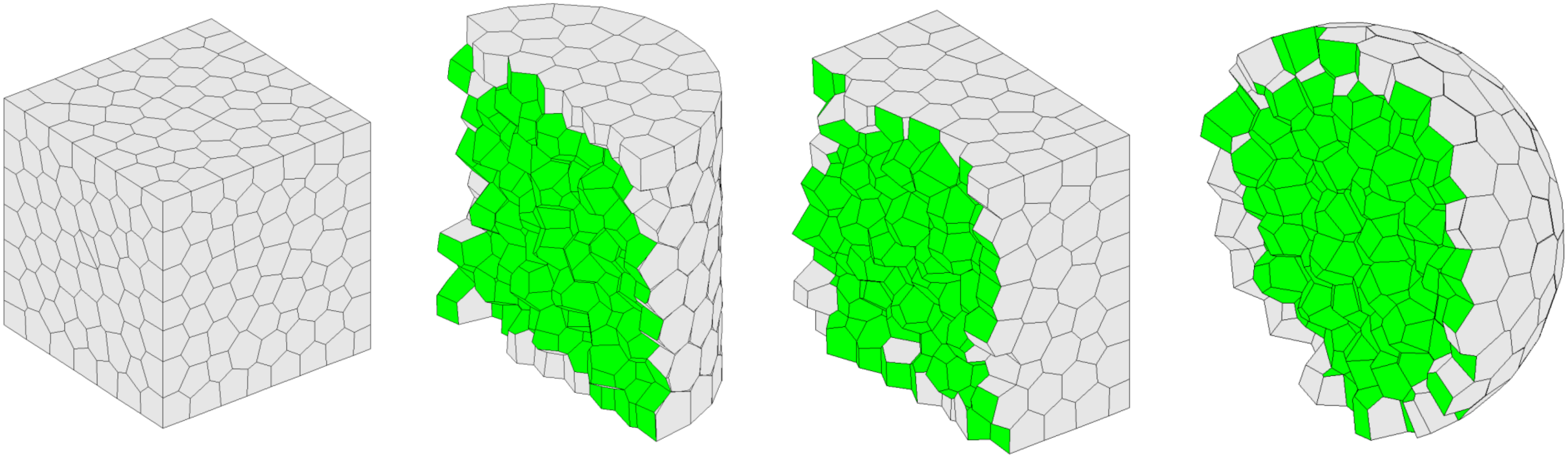
(times in seconds for 200 optimization iterations)

† Linearized Volume Constrained: $V(\mathbf{z}) = \sum_{\ell=1}^{N} (\mathbf{Pz})_\ell = \mathbf{1}^T (\mathbf{Pz}) = \left(\mathbf{1}^T\mathbf{P}\right)\mathbf{z} = \left(\mathbf{P}^T\mathbf{1}\right)^T \mathbf{z}$

* Andreassen E., Clausen A., Schevenels M., Lazarov B., Sigmund O., "*Efficient topology optimization in MATLAB using 88 lines of code*", **JSMO**, 43(1):1–16, **2011.** doi:10.1007/s00158-010-0594-7

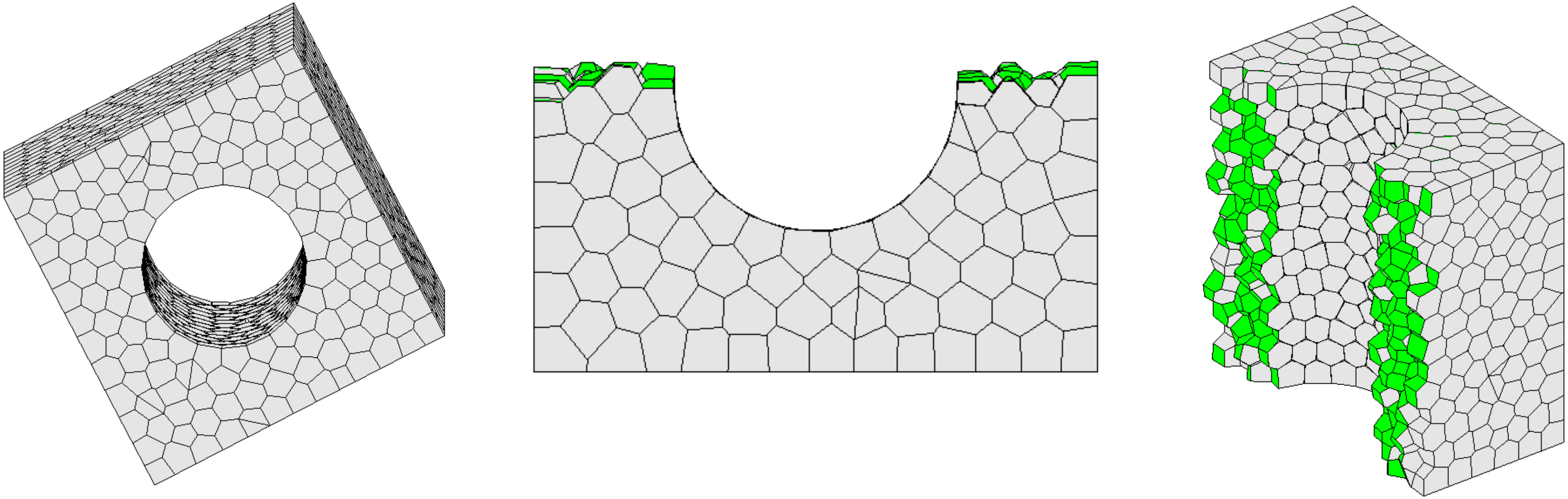
3D MESHES

500 Elements – 200 Lloyd's Iterations



3D MESHES

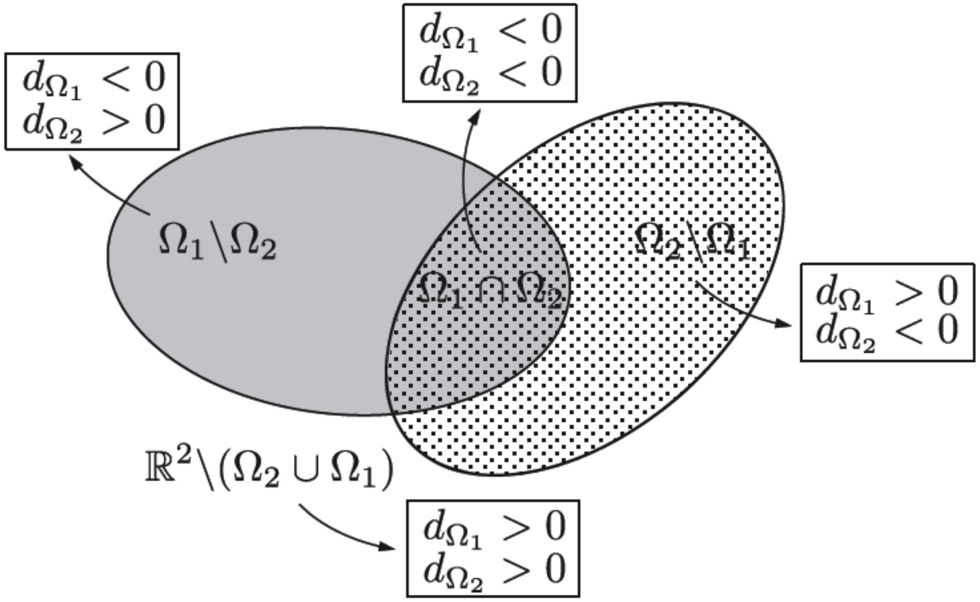
2000 Elements – 200 Lloyd's Iterations



MESH GENERATION – BASIC IDEAS

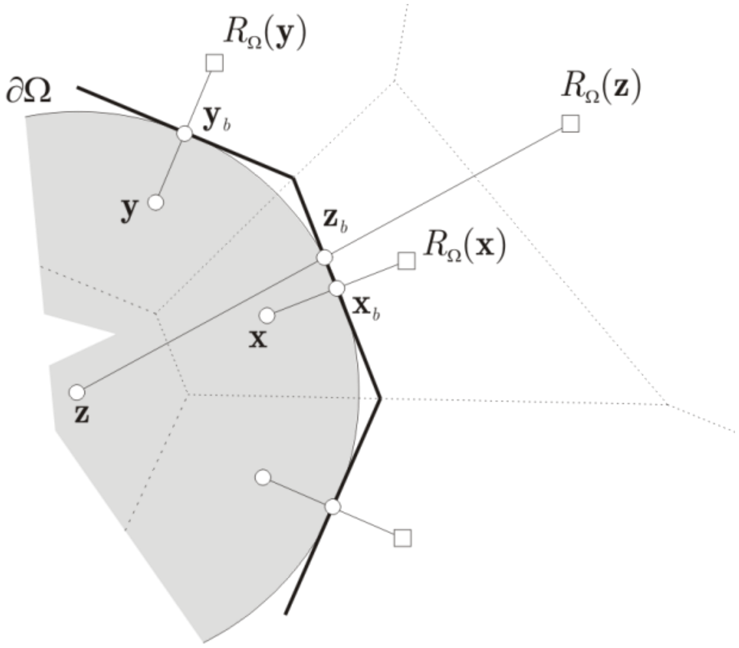
Implicit Representation

- Moreover, set operations such as **union**, **intersection**, and **difference** can be used to combine different geometries:



MESH GENERATION – ALGORITHM

A polygonal discretization can be obtained from the Voronoi diagram of a given set of seeds and their reflections:



- We first reflect each point in **P** about the *closest* boundary point of Ω and denote the resulting set of points by $R_\Omega(\mathbf{P})$.
- We then construct the Voronoi diagram of the plane by including the original point set as well as its reflection.
- Finally we incorporate Lloyd's iterations to obtain a point set **P** that produces a CVT.

$$R_\Omega(\mathbf{x}) = \mathbf{x} - 2d_\Omega(\mathbf{x})\nabla d_\Omega(\mathbf{x}) \qquad R_\Omega(\mathbf{P}) := \{R_\Omega(\mathbf{y}) : \mathbf{y} \in \mathbf{P}\}$$

CODE STRUCTURE

PolyTop (Main Loop)

```
function [z,V,fem] = PolyTop(fem,opt)
Iter=0; Tol=opt.Tol*(opt.zMax-opt.zMin); Change=2*Tol; z=opt.zIni; P=opt.P;
[E,dEdy,V,dVdy] = opt.MatIntFnc(P*z);
[FigHandle,FigData] = InitialPlot(fem,V);
while (Iter<opt.MaxIter) && (Change>Tol)
  Iter = Iter + 1;
  %Compute cost functionals and analysis sensitivities
  [f,dfdE,dfdV,fem] = ObjectiveFnc(fem,E,V);
  [g,dgdE,dgdV,fem] = ConstraintFnc(fem,E,V,opt.VolFrac);
  %Compute design sensitivities
  dfdz = P'*(dEdy.*dfdE + dVdy.*dfdV);
  dgdz = P'*(dEdy.*dgdE + dVdy.*dgdV);
  %Update design variable and analysis parameters
  [z,Change] = UpdateScheme(dfdz,g,dgdz,z,opt);
  [E,dEdy,V,dVdy] = opt.MatIntFnc(P*z);
  %Output results
  fprintf('It: %i \t Objective: %1.3f\tChange: %1.3f\n',Iter,f,Change);
  set(FigHandle,'FaceColor','flat','CData',1-V(FigData)); drawnow
end
```

CODE STRUCTURE

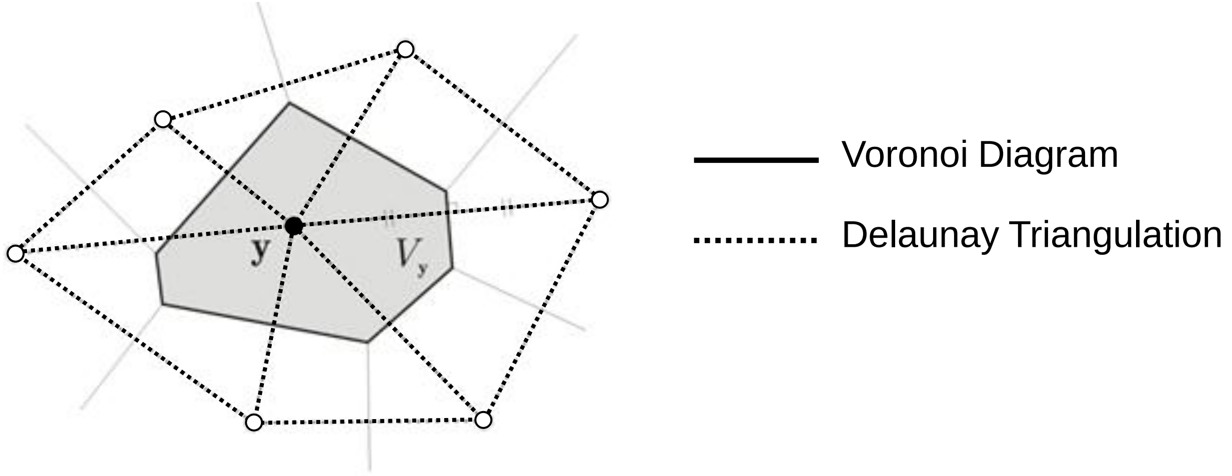
PolyMesher (Main Loop)

```
function [Node,Element,Supp,Load,P] = PolyMesher (Domain, NElem,
MaxIter,P)
if ~exist('P','var'), P=PolyMshr_RndPtSet (NElem,Domain); end
NElem = size(P,1);
Tol=5e-3; It=0; Err=1; c=1.5;
BdBox = Domain('BdBox');
Area = (BdBox(2)-BdBox(1))*(BdBox(4)-BdBox(3));
Pc = P; figure;
while(It<=MaxIter && Err>Tol)
  Alpha = c*sqrt(Area/NElem);
  P = Pc; %Lloyd's update
  R_P = PolyMshr_Rflct(P,NElem,Domain,Alpha); %Generate the reflections
  [Node,Element] = voronoin([P;R_P]); %Construct Voronoi diagram
  [Pc,A] = PolyMshr_CntrdPly(Element,Node,NElem);
  Area = sum(abs(A));
  Err = sqrt(sum((A.^2).*sum((Pc-P).*(Pc-P),2)))*NElem/Area^1.5;
  fprintf('It: %3d   Error: %1.3e\n',It,Err); It=It+1;
  if NElem<=2000, PolyMshr_PlotMsh(Node,Element,NElem); end;
end
[Node,Element] = PolyMshr_ExtrNds(NElem,Node,Element); %Extract node list
[Node,Element] = PolyMshr_CllpsEdgs(Node,Element,0.1); %Remove small
edges
[Node,Element] = PolyMshr_RsqsNds(Node,Element); %Reoder Nodes
```

The concept of Voronoi Tesselation (or diagram)

Given a set of “*n*” distinct points of seeds **P**, a Voronoi cell $V_{\mathbf{y}}$ consists of points in the plane closer to y than any other point in **P**, i.e.:

$$V_{\mathbf{y}} = \left\{ \mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{y}\| < \|\mathbf{x} - \mathbf{z}\|, \forall \mathbf{z} \in \mathbf{P} \setminus \{\mathbf{y}\} \right\}$$



MESH GENERATION – BASIC IDEAS

- The concept of Voronoi Tesselation plays a central role in our meshing algorithm

- **Centroidal Voronoi Tesselations (CVTs) enjoy a higher level of regularity which are suitable for use in Finite Element Analysis**

- Implicit representation

Centroidal Voronoi Tesselations (**CVT**s)

A Voronoi tesselation $\mathcal{T}(\mathrm{P};\Delta)$ is centroidal if, for every y $\in$ P: i.e.:

$$\mathbf{y} = \mathbf{y}_c \qquad \text{where} \qquad \mathbf{y}_c := \frac{\int_{V_{\mathbf{y}} \cap \Delta} \mathbf{x}\mu(\mathbf{x})d\mathbf{x}}{\int_{V_{\mathbf{y}} \cap \Delta} \mu(\mathbf{x})d\mathbf{x}}$$

MESH GENERATION – BASIC IDEAS

Centroidal Voronoi Tesselations (**CVT**s)

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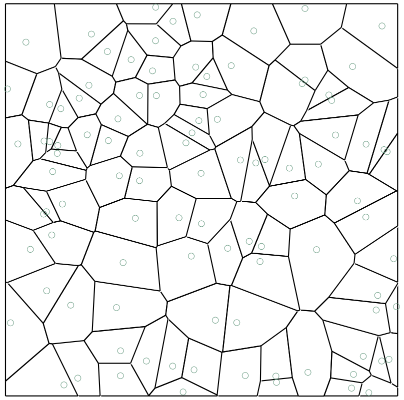
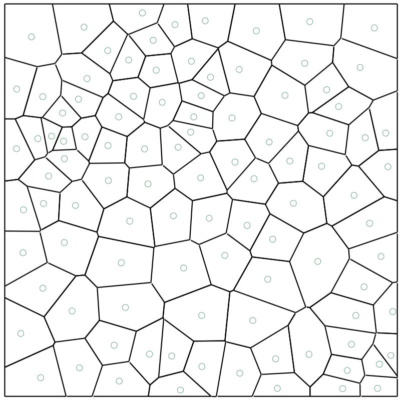
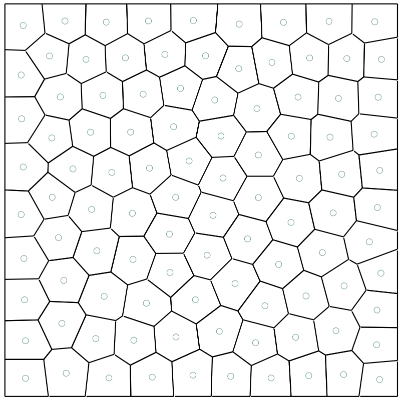
Lloyd's Iterations:



Initial random points



First iteration



After 80 iterations

MESH GENERATION – BASIC IDEAS

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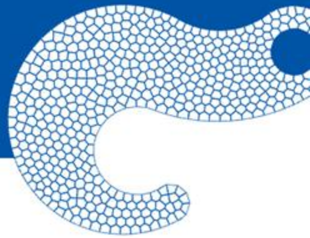
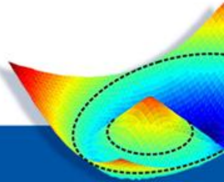
- Centroidal Voronoi Tesselations (CVTs) enjoy a higher level of regularity which are suitable for use in Finite Element Analysis

- **Implicit representation**





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Department of Mechanical Engineering / PUC-Rio

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