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System Reliability Based Topology Optimization of Structures under Stochastic Excitations

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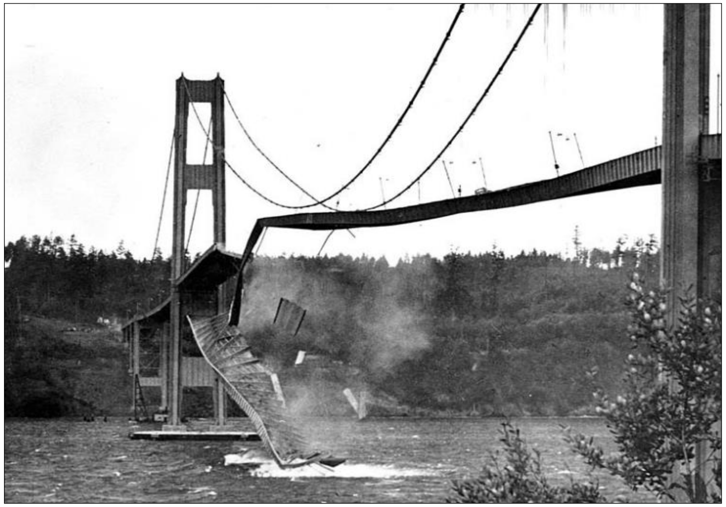
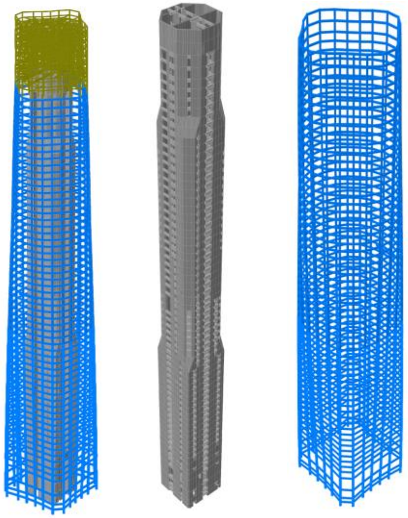
Structural engineering under Natural hazards and risks

San Francisco Earthquake, 1907 http://www.documentingreality.com



Tacoma bridge, 1940 http://failuremag.com

Courtesy of Skidmore, Owing and Merrill, LLP



Structural system

Motivation

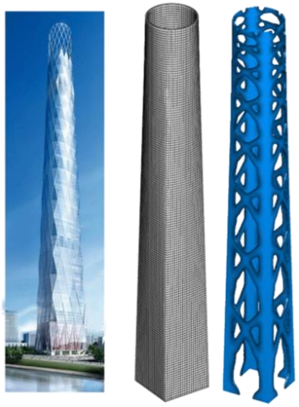
Research aims to find **optimal structural system** under **stochastic excitations**

John Hancock Center
http://en.wikipedia.org/wiki/John_Hancock_Center

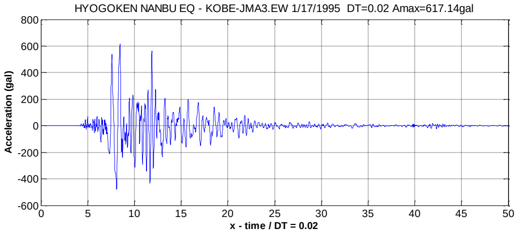
Ssiger International Plaza
Courtesy of Skidmore, Owing and Merrill, LLP

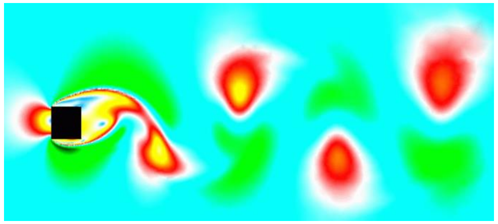
Incorporation of **random vibration** theories into **topology optimization**

- Topology optimization
- Stochastic excitation

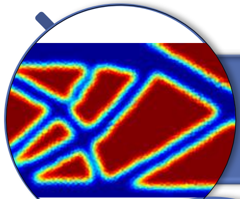


Stromberg *et al.* (2011)

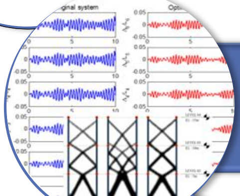




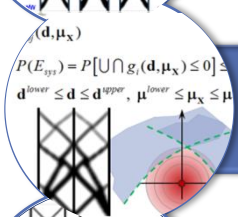
Contents



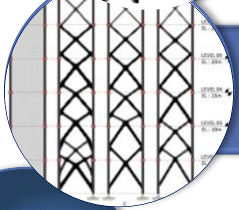
Topology optimization



Topology optimization under stochastic excitations



System reliability-based topology optimization under stochastic excitations

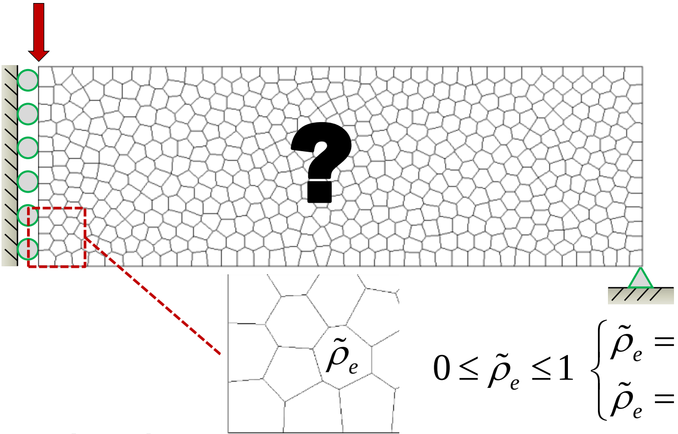


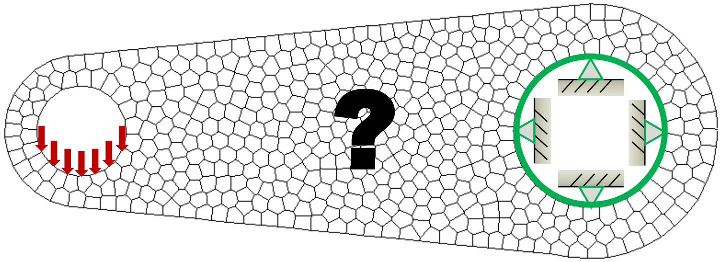
Numerical example: multi-story building structure

Conclusions & future research

Topology Optimization

Topology optimization **aims** to **identify optimal material layouts** of problems through mathematical programming while **fulfilling given design constraints**





$$0 \le \tilde{\rho}_e \le 1 \begin{cases} \tilde{\rho}_e = 0 & \text{void} \\ \tilde{\rho}_e = 1 & \text{solid} \end{cases}$$

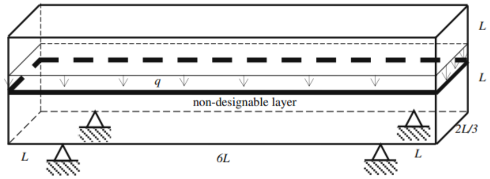
$$\mathbf{K}^e(\tilde{\rho}_e) = \tilde{\rho}_e{}^p \int_{\Omega_e} \mathbf{B}^{\mathrm{T}}\mathbf{D}_0\mathbf{B}\ \Omega_e$$

$$\mathbf{M}_e(\tilde{\rho}_e) = \tilde{\rho}_e{}^q \int_{\Omega_e} \mathbf{N}^{\mathrm{T}}\rho\mathbf{N}\ d\Omega_e$$

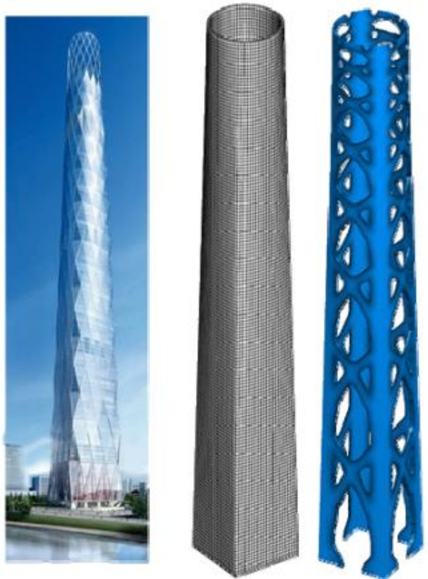
$$\boxed{\begin{array}{ll} \min\limits_{\mathbf{d}} & \text{compliance} \\ s.t & \text{volume constraint} \end{array}}$$

Design variable : Element density

- Applications



Nguyen *at el.* (2010)



Stromberg *et al.* (2011)

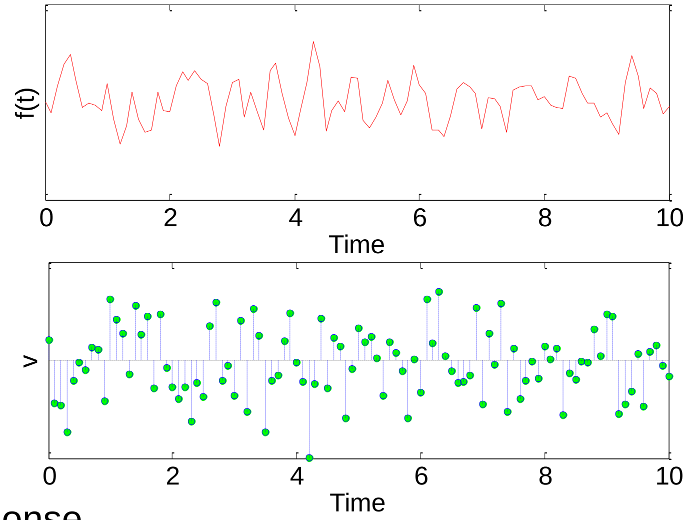
Discrete Representation Method

(Der Kiureghian, 2000)

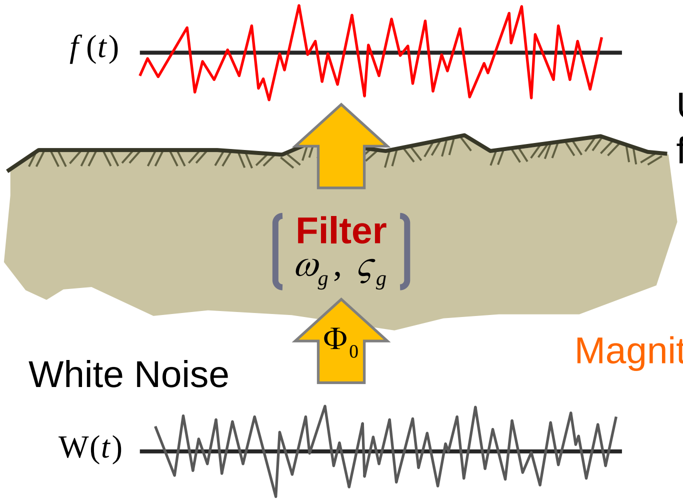
- Discretization of Random process

$$f(t) = \mu(t) + \sum_{i=1}^{n} v_i s_i(t) = \mu(t) + \mathbf{s}(t)^T \mathbf{v}$$

  - v: uncorrelated standard normal **random variables**
  - s(*t*): **deterministic basis functions** based on the spectral characteristics of the process



- Example : Filtered white noise (earthquake)



Unit-impulse response function of the filter

$$f(t) = \int_0^t v(\tau) s(t-\tau) d\tau$$

$$\cong \sum_{i=1}^{n} v_i s_i(t) = \sum_{i=1}^{n} W_i \cdot h_f(t-t_i)\Delta t$$

Magnitude of WN impulse

$$= \sum_{i=1}^{n} \sqrt{2\pi\Phi_0 / \Delta t} \cdot v_i \cdot h_f(t-t_i)\Delta t \;=\; \mathbf{s}(t)^{\mathrm{T}} \mathbf{v}$$

Der Kiureghian, A. (2000). The geometry of random vibrations and solutions by FORM and SORM. *Probabilistic Engineering Mechanics*, 15(1),: 81-90.

Response of Linear System to Stochastic Excitation

Linear system + Gaussian process

- Duhamel's Integral

$$u(t)=\int_{0}^{t} f(\tau)h_{s}(t-\tau)d\tau$$

  ○ $h_s(t)$: the unit-impulse response function of the system

- Response

$$u(t)=\int_{0}^{t}\sum_{i=1}^{n} v_{i}s_{i}(\tau)h_{s}(t-\tau)d\tau=\sum_{i=1}^{n} v_{i}a_{i}(t)=\mathbf{a}(t)^{\mathrm{T}}\mathbf{v}$$

$$a_{i}(t)=\int_{0}^{t} s_{i}(\tau)h_{s}(t-\tau)d\tau,\quad i=1,\ldots,n$$

Deterministic, time-dependent
- filter + structure

Random, time-independent

- **a**(*t*) in FEM settings

$$e.g.\ \mathbf{M}(\tilde{\boldsymbol{\rho}})\ddot{\mathbf{u}}(t,\tilde{\boldsymbol{\rho}})+\mathbf{C}(\tilde{\boldsymbol{\rho}})\dot{\mathbf{u}}(t,\tilde{\boldsymbol{\rho}})+\mathbf{K}(\tilde{\boldsymbol{\rho}})\mathbf{u}(t,\tilde{\boldsymbol{\rho}})=\mathbf{f}(t,\tilde{\boldsymbol{\rho}})$$

$$\begin{pmatrix} u(t_1) \\ u(t_2) \\ \vdots \\ u(t_{n-1}) \\ u(t_n) \end{pmatrix}=\begin{pmatrix} u(\Delta t) \\ u(2\Delta t) \\ \vdots \\ u(t_0-\Delta t) \\ u(t_0) \end{pmatrix}=\begin{bmatrix} 0 & 0 & \cdots & 0 & v_1 \\ 0 & 0 & \cdots & v_1 & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & v_1 & \cdots & v_{n-2} & v_{n-1} \\ v_1 & v_2 & \cdots & v_{n-1} & v_n \end{bmatrix}\begin{pmatrix} a_1(t_0) \\ a_2(t_0) \\ \vdots \\ a_{n-1}(t_0) \\ a_n(t_0) \end{pmatrix}$$

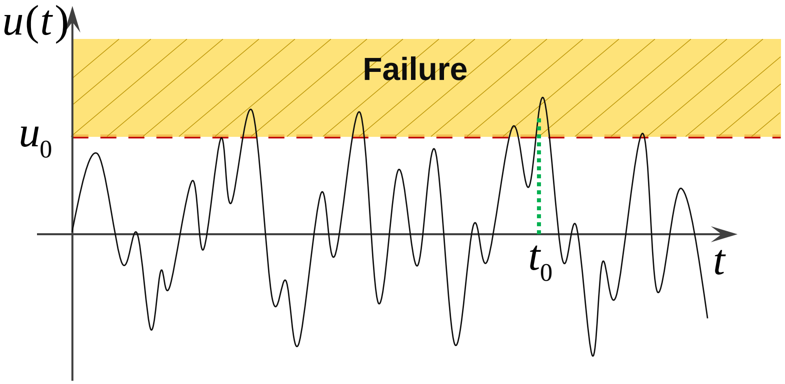
Numerical time integration

Inversely computed

Instantaneous Failure Probability

- ‘Instantaneous’ failure events of a linear system

$$E_f = \{u(t_0) \geq u_0\} = \{\mathbf{a}(t_0)^{\mathrm{T}} \mathbf{v} \geq u_0\}$$

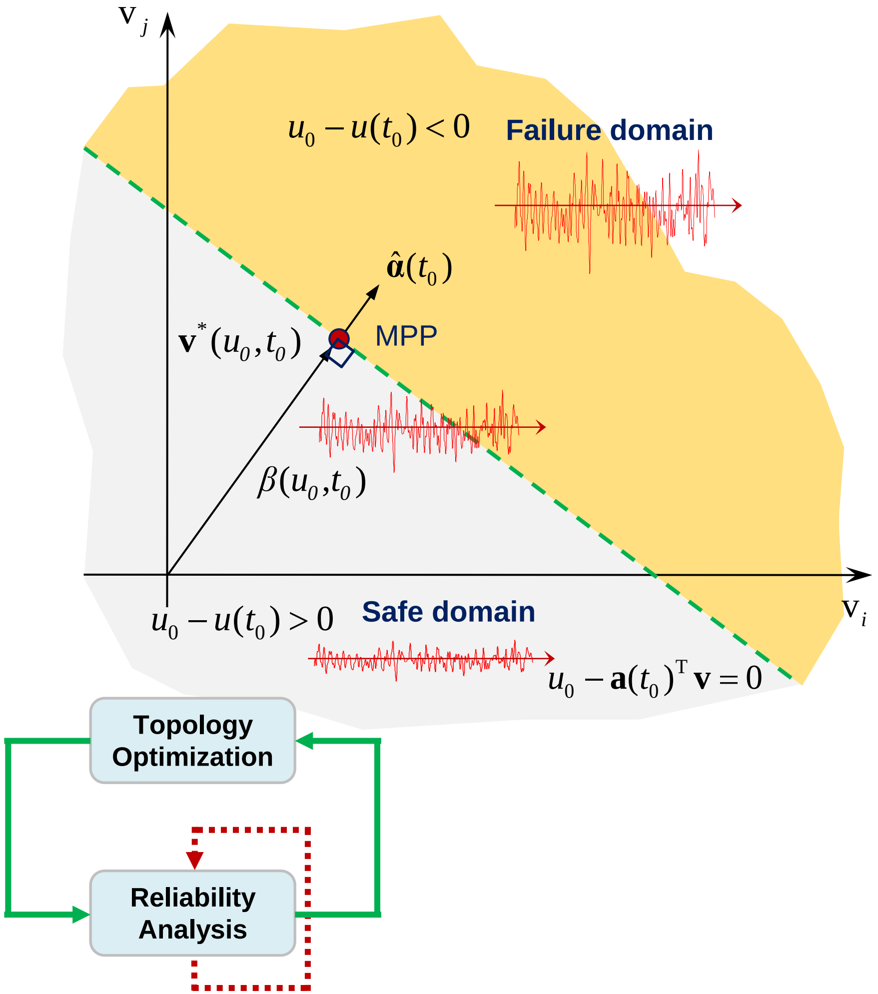


- Failure Probability, $P(E_f)$

$$P\left(u(t_0) \geq u_0\right) = P\left(u_0 - \mathbf{a}^T(t_0)\mathbf{v} \leq 0\right) = P\left(g(u) \leq 0\right)$$

$$g(u) = \frac{u_0}{\|\mathbf{a}(t_0)\|} - \frac{\mathbf{a}^T(t_0)}{\|\mathbf{a}(t_0)\|}\mathbf{v} = \beta(u_0, t_0) - \hat{\boldsymbol{\alpha}}(t_0) \cdot \mathbf{v}$$

$$\boxed{\begin{aligned} &P\left(u(t_0) \geq u_0\right) = \Phi[-\beta(u_0, t_0)] \\ &\beta(u_0, t_0) = u_0 / \|\mathbf{a}(t_0)\| = \hat{\boldsymbol{\alpha}}(t_0) \cdot \mathbf{v}^* \end{aligned}}$$



Topology Optimization of Structure under Stochastic Excitations

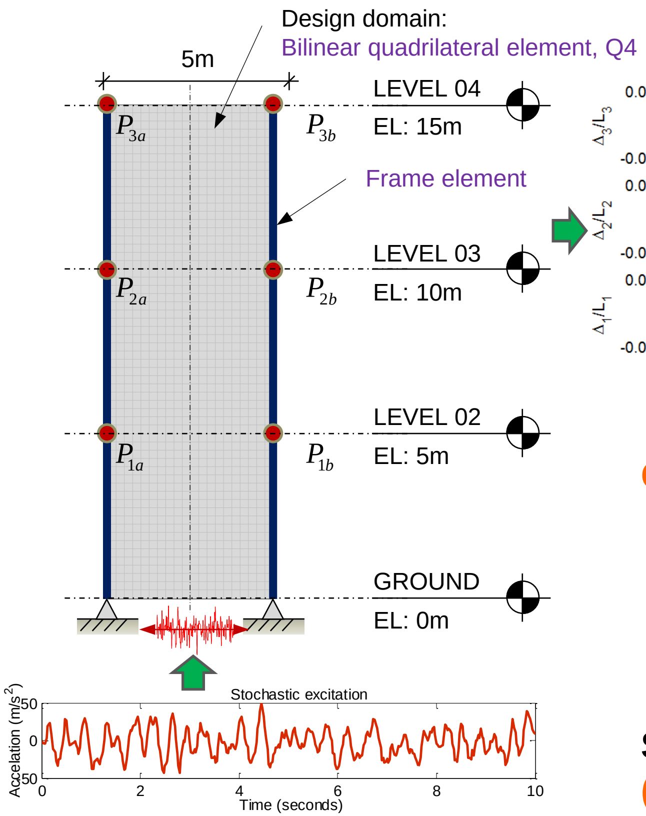
- Stochastic topology optimization framework (Chun, Song, Paulino 2012)

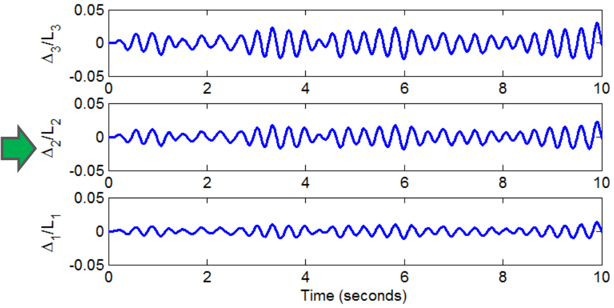
$$
\begin{aligned}
&\min_{\mathbf{d}} \quad f(\tilde{\boldsymbol{\rho}}(\mathbf{d})) \\
&s.t \quad P(E_i) \le P_{fi}^{\text{target}} \quad i = 1,\ldots,n \\
&with \; \mathbf{M}(\tilde{\boldsymbol{\rho}})\ddot{\mathbf{u}}(t,\tilde{\boldsymbol{\rho}}) + \mathbf{C}(\tilde{\boldsymbol{\rho}})\dot{\mathbf{u}}(t,\tilde{\boldsymbol{\rho}}) + \mathbf{K}(\tilde{\boldsymbol{\rho}})\mathbf{u}(t,\tilde{\boldsymbol{\rho}}) = -\mathbf{M}\mathbf{l}f(t))
\end{aligned}
$$

Objective function

Probabilistic constraints on responses of a structure (**Instantaneous** failure probability)

Stochastic excitation (filtered Gaussian process)

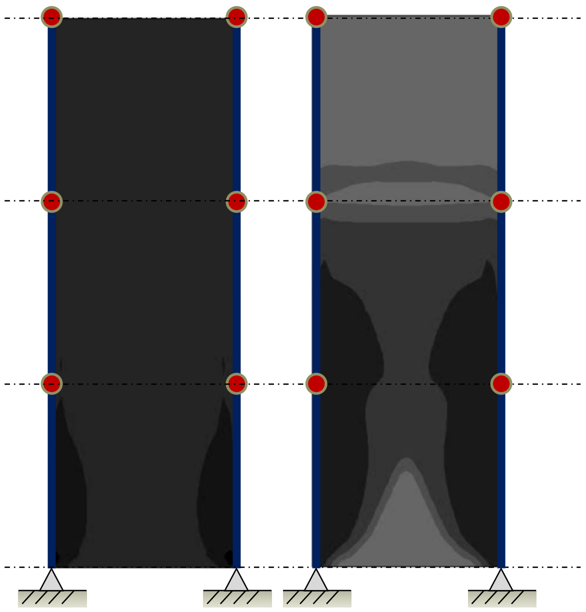




Component failure events, E_i

$$P(E_i) \le P_{fi}^{\text{target}} \quad i = 1,\ldots,n$$

Solve optimization problem (CRBTO)



	A	B
β	1.3	2.2
P_f	9.68%	1.39%

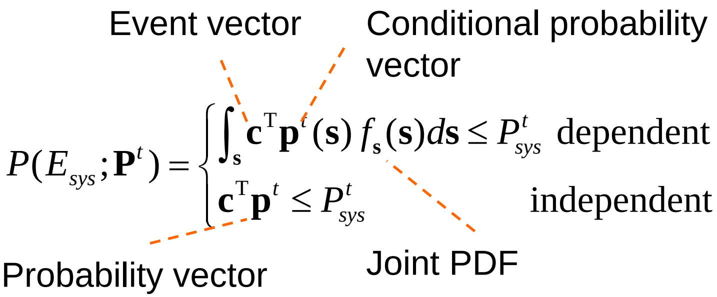
SYSTEM RBTO of Building Structures under Stochastic Excitations

SRBTO : System Reliability Based Topology Optimization

- Motivation : further extension of the stochastic topology optimization framework aforementioned is expected to provide more realistic means for the prediction of a given building's response to stochastic excitations
  - **Time**: the first passage probability $P(E_{sys})=P(x<\max_{0<t<t_n}|X(t)|)=P\left(\bigcup_{i=1}^{n}|X(t_i)|>x\right)$
  - **Location**
  - **Failure modes**: different types of design constraints such as a target tip displacement, a target natural frequency of a structure

Matrix-based System Reliability Method (MSR)(Song and Kang 2009, Kang *et al*. 2012)

- General system
- Statistical dependency
- Parameter sensitivity



Event vector

Conditional probability vector

$$P(E_{sys};\mathbf{P}^t)=\begin{cases}\int_{\mathbf{s}}\mathbf{c}^{\mathrm{T}}\mathbf{p}^t(\mathbf{s})\,f_{\mathbf{s}}(\mathbf{s})d\mathbf{s}\le P_{sys}^t & \text{dependent}\\ \mathbf{c}^{\mathrm{T}}\mathbf{p}^t\le P_{sys}^t & \text{independent}\end{cases}$$

Probability vector

Joint PDF

Song, J., and W.-H. Kang (2009). System reliability and sensitivity under statistical dependence by matrix-based system reliability method. Structural Safety, Vol. 31(2), 148-156.
Kang, W.-H., Y.-J. Lee, J. Song, and B. Gencturk (2012). Further development of matrix-based system reliability method and applications to structural systems. Structure and Infrastructure Engineering: Maintenance, Management, Life-cycle Design and Performance. Vol. 8(5), 441-457.

SYSTEM RBTO of Building Structures under Stochastic Excitations

- Formulation

$$\min_{\mathbf{d},\,P_i^t} \; f(\tilde{\boldsymbol{\rho}}(\mathbf{d}))$$

$$s.t\; g_{P_i^t} = g_i(t_0, u_0, P_i^t, \tilde{\boldsymbol{\rho}}) \geq 0, \quad i = 1, \ldots, n$$

P_i^t quantile of the *i*-th limit-state function

$$P(E_{sys}; \mathbf{P}^t) = \begin{cases} \int_{\mathbf{s}} \mathbf{c}^T \mathbf{p}^t(\mathbf{s}) \, f_{\mathbf{s}}(\mathbf{s}) d\mathbf{s} \leq P_{sys}^t & \text{dependent} \\ \mathbf{c}^T \mathbf{p}^t \leq P_{sys}^t & \text{independent} \end{cases}$$

Probabilistic constraints solved by **MSR** method

$$\text{with} \;\; \mathbf{M}(\tilde{\boldsymbol{\rho}})\ddot{\mathbf{u}}(t, \tilde{\boldsymbol{\rho}}) + \mathbf{C}(\tilde{\boldsymbol{\rho}})\dot{\mathbf{u}}(t, \tilde{\boldsymbol{\rho}}) + \mathbf{K}(\tilde{\boldsymbol{\rho}})\mathbf{u}(t, \tilde{\boldsymbol{\rho}}) = \mathbf{f}(t, \tilde{\boldsymbol{\rho}})$$

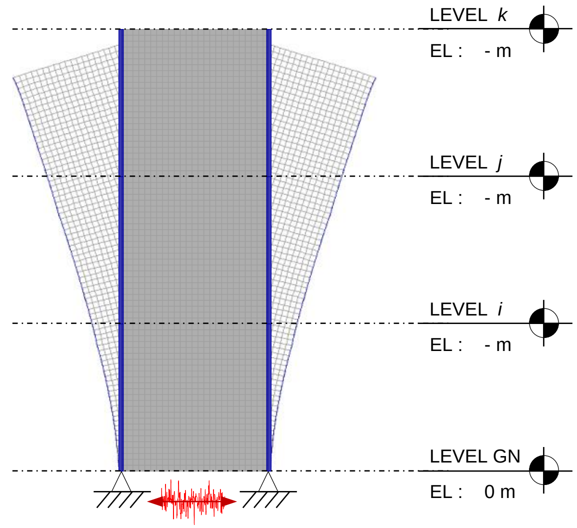
System equation

$\tilde{\boldsymbol{\rho}}$: Filtered density

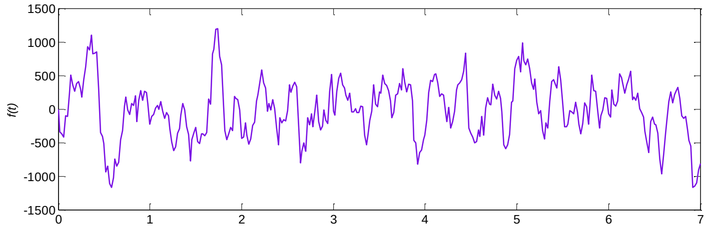
M: Mass matrix, **C**: Damping matrix, **K**: Stiffness matrix

$f(t)$: Stochastic process (e.g., for earthquake loading $\mathbf{f}(t) = -\mathbf{MI}\ddot{u}_g(t)) = -\mathbf{MI}f(t))$

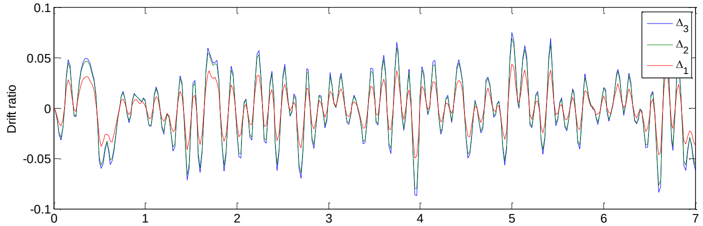
E_{sys}: System failure event, P_{sys}^{target}: Target failure probability



Stochastic excitations



Responses



Sensitivity Calculation

Calculation of **sensitivity** of the terms in the constraints with respect to design variables is an essential procedure for efficient **gradient-based optimization**

- Formulation
- Design variables: d, $P_i^{\,t}$

$$\min_{\mathbf{d},\,P_i^t} \; f(\tilde{\boldsymbol{\rho}}(\mathbf{d}))$$

$$s.t \; g_{P_i^t} = g_i(t_0, u_0, P_i^t, \tilde{\boldsymbol{\rho}}) \geq 0, \quad i = 1, \ldots, n$$

$$P(E_{sys}; \mathbf{P}^t) = \begin{cases} \int_{\mathbf{s}} \mathbf{c}^T \mathbf{p}^t(\mathbf{s}) f_{\mathbf{s}}(\mathbf{s}) d\mathbf{s} \leq P_{sys}^t & \text{dependent} \\ \mathbf{c}^T \mathbf{p}^t \leq P_{sys}^t & \text{independent} \end{cases}$$

$$\text{with } \mathbf{M}(\tilde{\boldsymbol{\rho}})\ddot{\mathbf{u}}(t,\tilde{\boldsymbol{\rho}}) + \mathbf{C}(\tilde{\boldsymbol{\rho}})\dot{\mathbf{u}}(t,\tilde{\boldsymbol{\rho}}) + \mathbf{K}(\tilde{\boldsymbol{\rho}})\mathbf{u}(t,\tilde{\boldsymbol{\rho}}) = \mathbf{f}(t,\tilde{\boldsymbol{\rho}})$$

$$\partial g_{P_i^t} / \partial d_e, \quad \partial g_{P_i^t} / \partial P_i^t$$

$$\partial P_i(\mathbf{s}) / \partial d_e, \quad \partial P_i(\mathbf{s}) / \partial P_i^t$$

- Analytical derivation

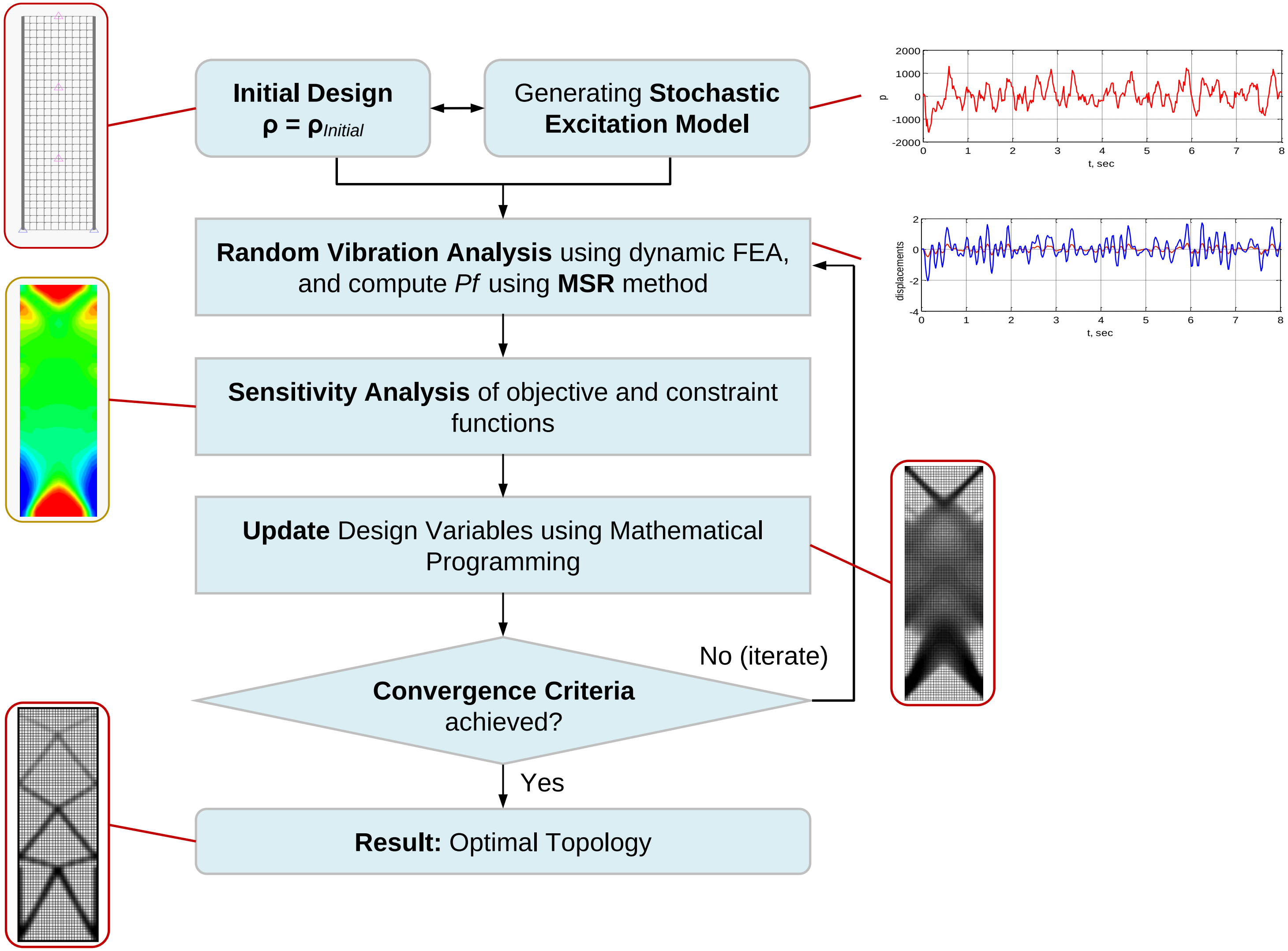
$$\frac{\partial g_{P_i^t}}{\partial d_e} = \sum_{j=1}^{n_e} \frac{\partial g_{P_i^t}}{\partial \tilde{\rho}_j} \cdot \frac{\partial \tilde{\rho}_j}{\partial d_e}$$
$$= \frac{\Phi^{-1}(P_i^t)}{\left\| \mathbf{a}^{\mathrm{T}}(t_0, \tilde{\boldsymbol{\rho}}) \right\|} \cdot \sum_{j=1}^{n_e} \sum_{i=1}^{n} \left(\frac{\partial a_i(t_0, \tilde{\boldsymbol{\rho}})}{\partial \tilde{\rho}_j} \cdot a_i(t_0, \tilde{\boldsymbol{\rho}}) \right) \cdot \frac{\partial \tilde{\rho}_j}{\partial d_e}$$
$$= \sum_{j=1}^{n_e} \sum_{i=1}^{n} \left(c_i(t_0, \tilde{\boldsymbol{\rho}}, P_i^t) \cdot \frac{\partial a_i(t_0, \tilde{\boldsymbol{\rho}})}{\partial \tilde{\rho}_e} \right) \cdot \frac{\partial \tilde{\rho}_j}{\partial d_e}$$

$$\frac{\partial P_i(\mathbf{s})}{\partial d_e} = 0$$

$$\frac{\partial g_{P_i^t}}{\partial P_i^t} = \frac{\partial g_{P_i^t}}{\partial \beta_i^t} \cdot \frac{\partial \beta_i^t}{\partial P_i^t} = -\frac{\partial g_{P_i^t}}{\partial \beta_i^t} \cdot \frac{1}{\varphi(-\beta_i^t)}$$
$$= -\frac{\partial \left(u_0 - \beta_i^t \left\| \mathbf{a}^{\mathrm{T}}(t_0, \tilde{\boldsymbol{\rho}}) \right\| \right)}{\partial \beta_i^t} \cdot \frac{1}{\varphi(-\beta_i^t)}$$
$$= \left\| \mathbf{a}^{\mathrm{T}}(t_0, \tilde{\boldsymbol{\rho}}) \right\| \cdot \frac{1}{\varphi(-\beta_i^t)}$$

$$\frac{\partial P_i(\mathbf{s})}{\partial P_i^t} = \frac{\partial P_i(\mathbf{s})}{\partial \beta_i^t} \cdot \frac{\partial \beta_i^t}{\partial P_i^t} = -\frac{\partial P_i(\mathbf{s})}{\partial \beta_i^t} \cdot \frac{1}{\varphi(-\beta_i^t)}$$
$$= -\frac{\partial}{\partial \beta_i^t} \Phi\left(\frac{-\beta_i^t - \sum_{k=1}^{m} (r_{ik} s_k)}{\sqrt{1 - \sum_{k=1}^{m} r_{ik}^2}} \right) \cdot \frac{1}{\varphi(-\beta_i^t)}$$
$$= \frac{1}{\sqrt{1 - \sum_{k=1}^{m} r_{ik}^2}} \varphi\left(\frac{-\beta_i^t - \sum_{k=1}^{m} (r_{ik} s_k)}{\sqrt{1 - \sum_{k=1}^{m} r_{ik}^2}} \right) \cdot \frac{1}{\varphi(-\beta_i^t)}$$

Optimization Procedure



Initial Design
$\rho = \rho_{Initial}$

Generating **Stochastic Excitation Model**

Random Vibration Analysis using dynamic FEA, and compute *Pf* using **MSR** method

Sensitivity Analysis of objective and constraint functions

Update Design Variables using Mathematical Programming

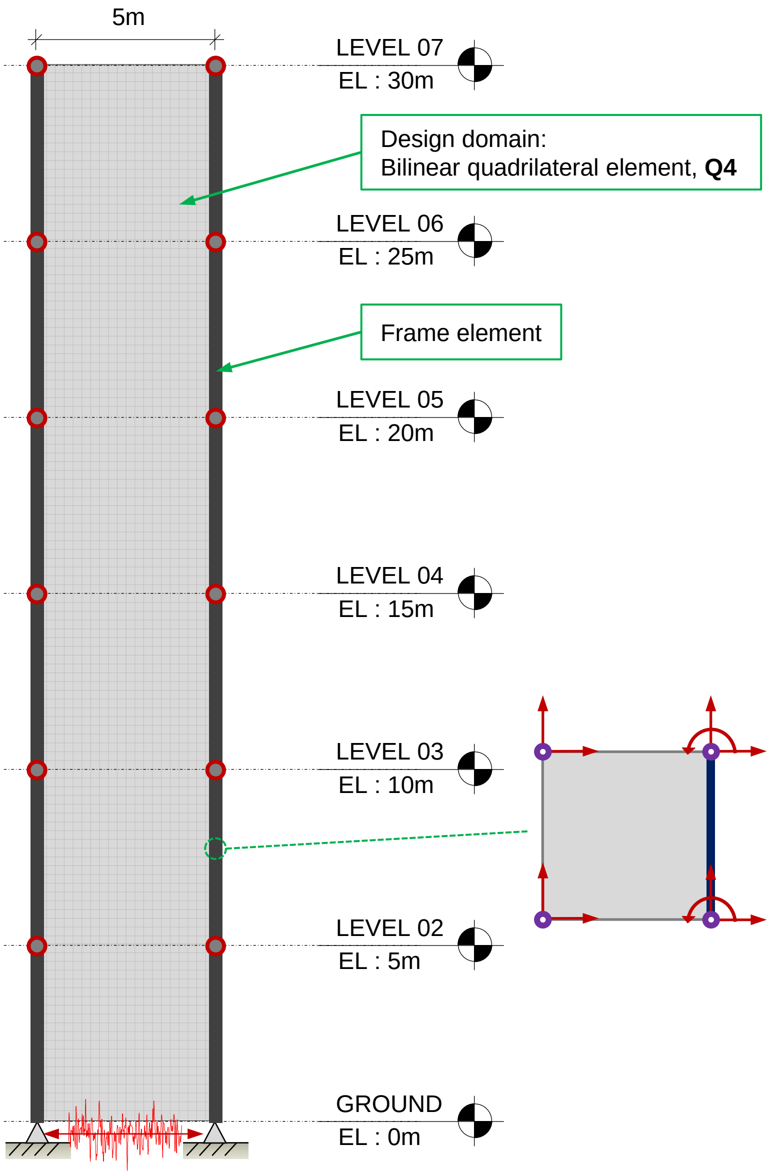
Convergence Criteria achieved?

No (iterate)

Yes

Result: Optimal Topology

Numerical example : Six-story building



Design domain:
Bilinear quadrilateral element, **Q4**

Frame element

$$\min_{\mathbf{d},\, P_i^t} \quad V(\tilde{\boldsymbol{\rho}}(\mathbf{d}))$$

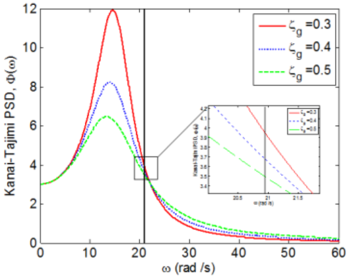
$$s.t\ g_{P_i^t} = g_i(t_0, u_0, P_i^t, \tilde{\boldsymbol{\rho}}) \geq 0, \quad i = 1, \ldots, n$$

$$P(E_{sys};\mathbf{P}^t) = \begin{cases} \int_{\mathbf{s}} \mathbf{c}^T \mathbf{p}^t(\mathbf{s}) f_{\mathbf{s}}(\mathbf{s}) d\mathbf{s} \leq P_{sys}^t & \text{dependent} \\ \mathbf{c}^T \mathbf{p}^t \leq P_{sys}^t & \text{independent} \end{cases}$$

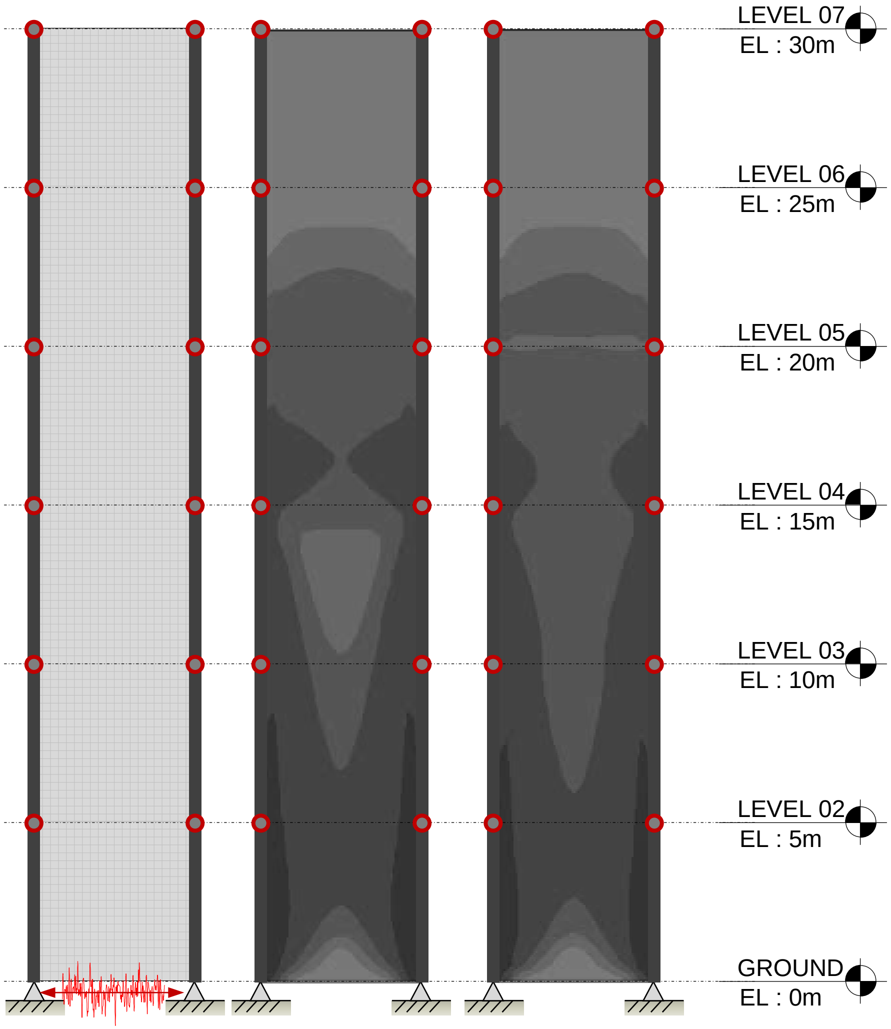
$$\text{with } \mathbf{M}(\tilde{\boldsymbol{\rho}})\ddot{\mathbf{u}}(t,\tilde{\boldsymbol{\rho}}) + \mathbf{C}(\tilde{\boldsymbol{\rho}})\dot{\mathbf{u}}(t,\tilde{\boldsymbol{\rho}}) + \mathbf{K}(\tilde{\boldsymbol{\rho}})\mathbf{u}(t,\tilde{\boldsymbol{\rho}}) = \mathbf{f}(t,\tilde{\boldsymbol{\rho}})$$

- Failure event: **Inter-story drift ratio**
  It occurs when inter-story drift ratio **exceeds** a prescribed threshold value
- System Failure event: $E_{sys} = \bigcup_{i=2}^{7} E_{f_i}$
  It occurs when **at least one of** component events happens
- Filtered Gaussian process using **Kanai-Tajimi filter**

$$h_f(t) = \exp(-\zeta_f \omega_f t)\frac{(2\zeta_f^2 - 1)\omega_f}{\sqrt{1-\zeta_f^2}} \sin(\omega_f \sqrt{1-\zeta_f^2} \cdot t)$$
$$-\exp(-\zeta_f \omega_f t) 2\zeta_f \omega_f \cos(\omega_f \sqrt{1-\zeta_f^2} \cdot t)$$



Numerical example : Six-story building



CRBTO *:*
β^t_i=1.7 (*P*^{t}*=4.46%)*
volfrac=28.5%
(P$^t_{sys}$=6.67%)

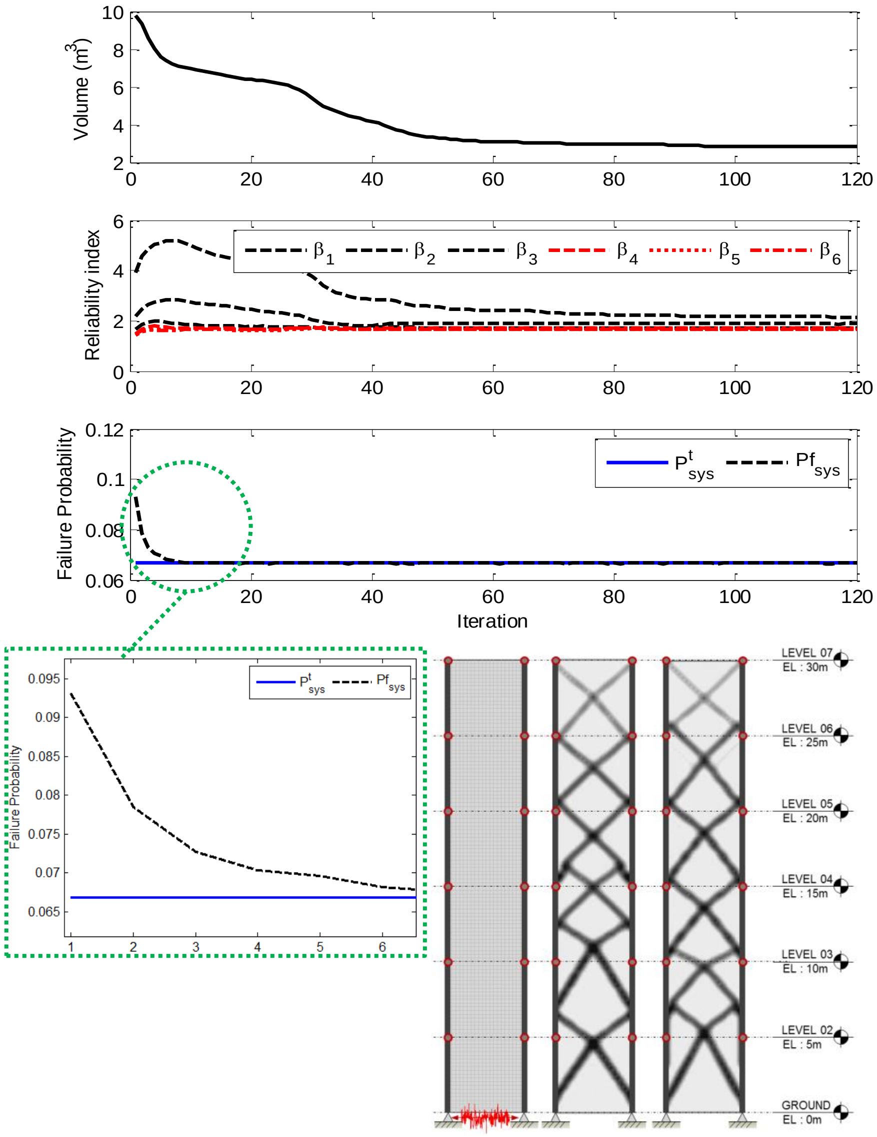
SRBTO *:*
β^t_{sys}=1.502 (*P*$^t_{sys}$*=6.67%)*
volfrac=28.45%;

CRBTO:
- All target reliability index β^t_i=1.7

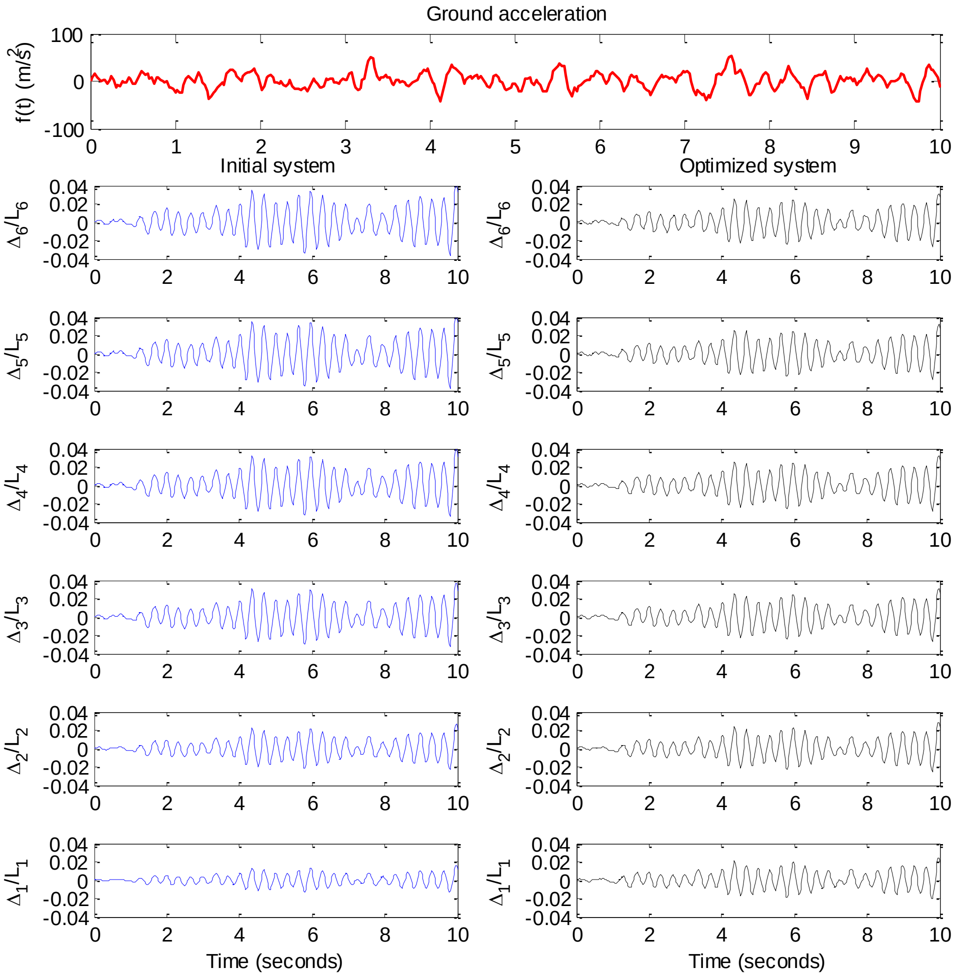
SRBTO:
- **Psys**: After **CTBTO** the probability that **at least one** of the constraints is violated (i.e. series system) is estimated by the **MSR** method

Numerical example : Six-story building

- Convergence history (**SRBTO**)



- Dynamic responses (**SRBTO**)



Conclusions & Future Research

Concluding remarks

- Development of a new framework incorporating **random vibration theories** into **topology optimization** using a **discrete representation method** for stochastic processes
- Development of **SRBTO** under stochastic excitation considering **statistical dependency** of component events using **MSR method**.
- **Application** of the proposed method to find **optimal bracing system** of the structure satisfying probabilistic constraints defined in system level.

Future Direction

- Incorporation of **first passage probability** in stochastic topology optimization
- Development of an efficient approach for computing sensitivity of system reliability with a large number of component events

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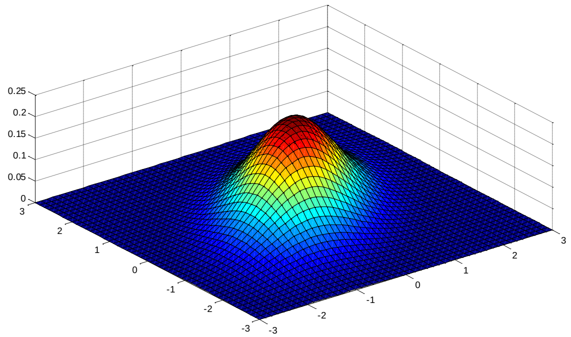
Reliability-Based Design Optimization

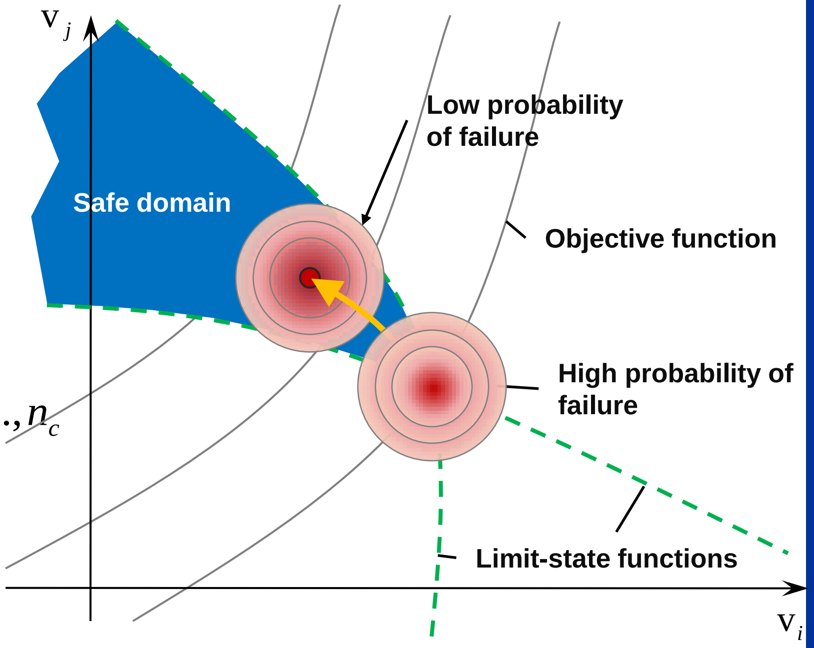
- ‘Deterministic’ design optimization (DO)

$$\begin{aligned}
&\min_{\mathbf{d}} && f_{obj}(\mathbf{d}) \\
&s.t && g_i(\mathbf{d}) > 0,\ i = 1,\dots,n_c \\
& && \mathbf{d}^{lower} \le \mathbf{d} \le \mathbf{d}^{upper}
\end{aligned}$$

- Reliability-based design optimization (RBDO)

$$\begin{aligned}
&\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} && f_{obj}(\mathbf{d},\boldsymbol{\mu}_\mathbf{X}) \\
&s.t && P(E_{sys}) = P\left[\bigcup\bigcap g_i(\mathbf{d},\boldsymbol{\mu}_\mathbf{X}) \le 0\right] \le P_{sys}{}^{\text{target}},\quad i = 1,\dots,n_c \\
& && \mathbf{d}^{lower} \le \mathbf{d} \le \mathbf{d}^{upper},\ \boldsymbol{\mu}^{lower} \le \boldsymbol{\mu}_\mathbf{X} \le \boldsymbol{\mu}^{upper}
\end{aligned}$$

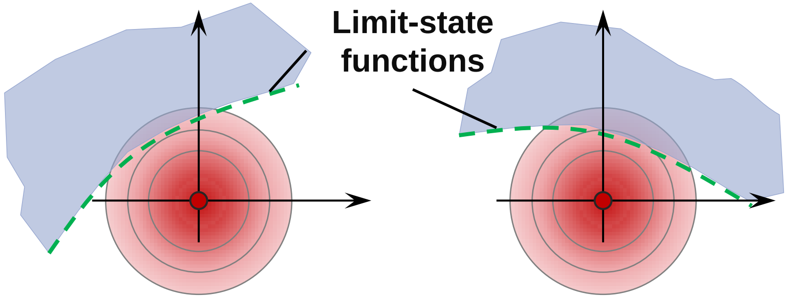




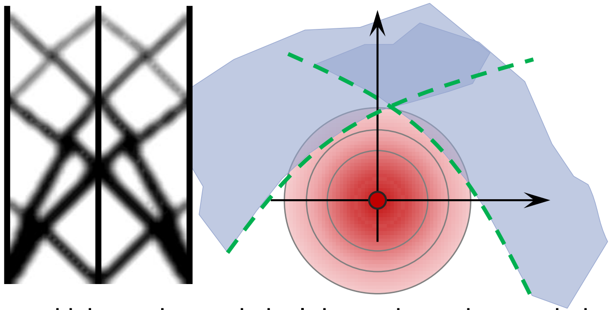
System Reliability-Based Design Optimization (SRBDO)

- Component Reliability Based Design Optimization (CRBDO)

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}} && f_{obj}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}) \\ &s.t && P\left[g_i(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}})\le 0\right]\le P_f^{\text{target}},\quad i=1,\ldots,n_c \\ & && \mathbf{d}^{lower}\le\mathbf{d}\le\mathbf{d}^{upper},\ \boldsymbol{\mu}^{lower}\le\boldsymbol{\mu}_{\mathbf{X}}\le\boldsymbol{\mu}^{upper} \end{aligned}$$



$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}} && f_{obj}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}) \\ &s.t && P(E_{sys})=P\left[\bigcup\bigcap g_i(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}})\le 0\right]\le P_{sys}^{\text{target}},\quad i=1,\ldots,n_c \\ & && \mathbf{d}^{lower}\le\mathbf{d}\le\mathbf{d}^{upper},\ \boldsymbol{\mu}^{lower}\le\boldsymbol{\mu}_{\mathbf{X}}\le\boldsymbol{\mu}^{upper} \end{aligned}$$



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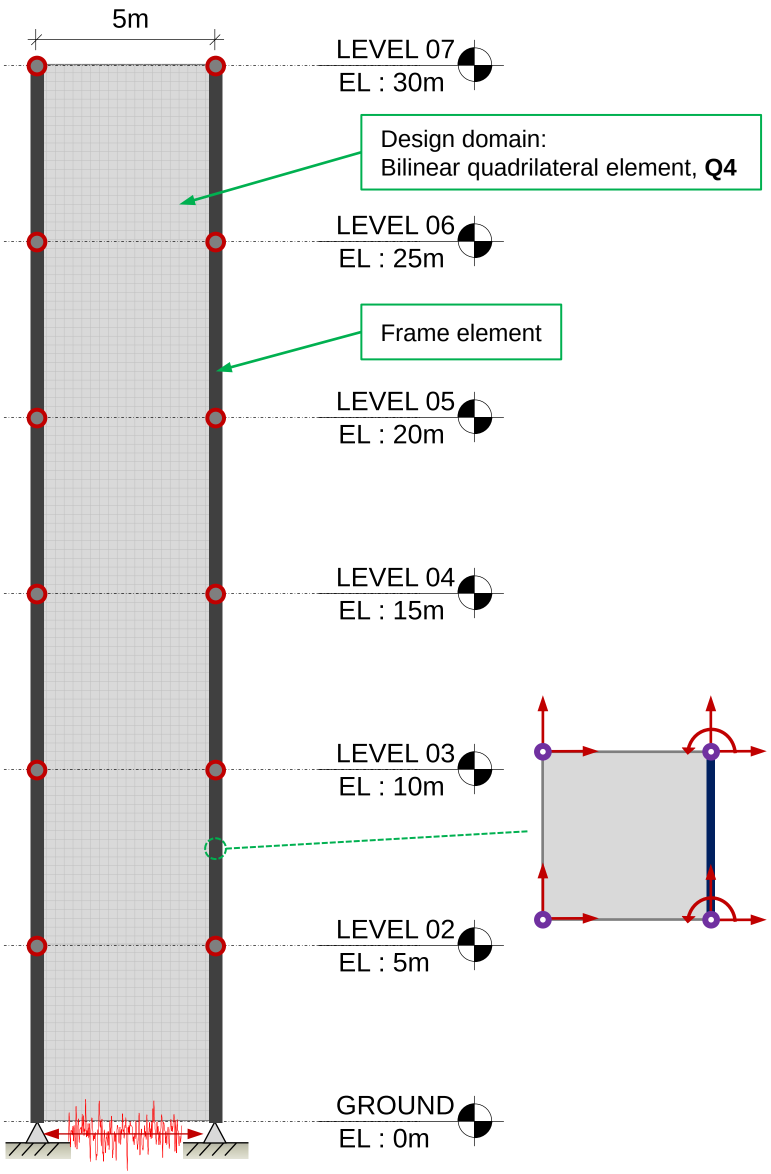
Kang, W.-H., Y.-J. Lee, J. Song, and B. Gencturk (2012). Further development of matrix-based system reliability method and applications to structural systems. Structure and Infrastructure Engineering: Maintenance, Management, Life-cycle Design and Performance. Vol. 8(5), 441-457.

Numerical example : Six-story building



Design domain:
Bilinear quadrilateral element, **Q4**

Frame element

$$\begin{aligned}&\min_{\mathbf{d},\,P_i^t} \quad V(\tilde{\boldsymbol{\rho}}(\mathbf{d}))\\ &s.t\ g_{P_i^t} = g_i(t_0, u_0, P_i^t, \tilde{\boldsymbol{\rho}}) \geq 0, \quad i = 1,\ldots,n\\ &\quad P(E_{sys};\mathbf{P}^t) = \begin{cases}\int_{\mathbf{s}} \mathbf{c}^T\mathbf{p}^t(\mathbf{s})\, f_{\mathbf{s}}(\mathbf{s})d\mathbf{s} \leq P_{sys}^t & \text{dependent}\\ \mathbf{c}^T\mathbf{p}^t \leq P_{sys}^t & \text{independent}\end{cases}\\ &\text{with}\ \ \mathbf{M}(\tilde{\boldsymbol{\rho}})\ddot{\mathbf{u}}(t,\tilde{\boldsymbol{\rho}}) + \mathbf{C}(\tilde{\boldsymbol{\rho}})\dot{\mathbf{u}}(t,\tilde{\boldsymbol{\rho}}) + \mathbf{K}(\tilde{\boldsymbol{\rho}})\mathbf{u}(t,\tilde{\boldsymbol{\rho}}) = \mathbf{f}(t,\tilde{\boldsymbol{\rho}})\end{aligned}$$

- Failure event: Inter-story drift ratio

$$E_{f_i} = \left\{u_{i0} - \left(\frac{(\mathbf{a}(t_0,\tilde{\boldsymbol{\rho}})^{\mathrm{T}}_{i,Left} + \mathbf{a}(t_0,\tilde{\boldsymbol{\rho}})^{\mathrm{T}}_{i,Right})\mathbf{v}}{2L_i}\right) \leq 0\right\} \qquad \text{for } i = 2$$

$$\left\{u_{i0} - \left(\frac{(\mathbf{a}(t_0,\tilde{\boldsymbol{\rho}})^{\mathrm{T}}_{i,Left} + \mathbf{a}(t_0,\tilde{\boldsymbol{\rho}})^{\mathrm{T}}_{i,Right})\mathbf{v}}{2L_i} - \frac{(\mathbf{a}(t_0,\tilde{\boldsymbol{\rho}})^{\mathrm{T}}_{(i-1),Left} + \mathbf{a}(t_0,\tilde{\boldsymbol{\rho}})^{\mathrm{T}}_{(i-1),Right})\mathbf{v}}{2L_i}\right) \leq 0\right\} \quad \text{for } i = 3,\ldots,7$$

- System Failure event

$$E_{sys} = \bigcup_{i=2}^{7} E_{f_i}$$

- Filtered Gaussian process using KT filter

$$h_f(t) = \exp(-\zeta_f\omega_f t)\frac{(2\zeta_f^2 - 1)\omega_f}{\sqrt{1-\zeta_f^2}}\sin(\omega_f\sqrt{1-\zeta_f^2}\cdot t)$$
$$-\exp(-\zeta_f\omega_f t)2\zeta_f\omega_f\cos(\omega_f\sqrt{1-\zeta_f^2}\cdot t)$$