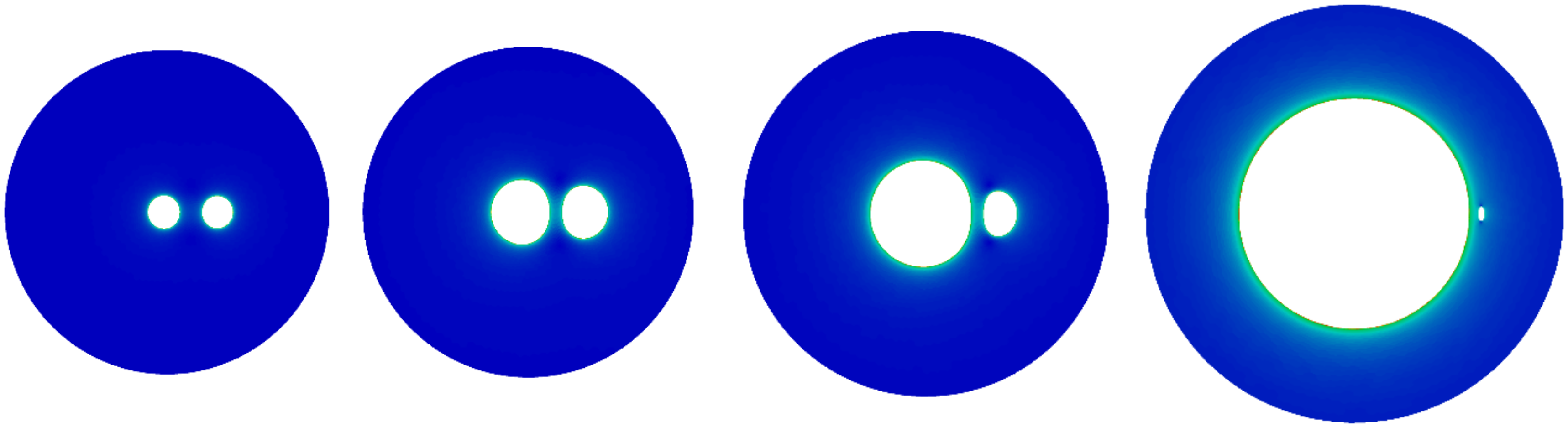
# Polygonal Finite Elements for Finite Elasticity



1% Stretch 3% Stretch 5% Stretch 20% Stretch

Heng Chi, Cameron Talischi,

Oscar Lopez-Pamies, Glaucio H. Paulino

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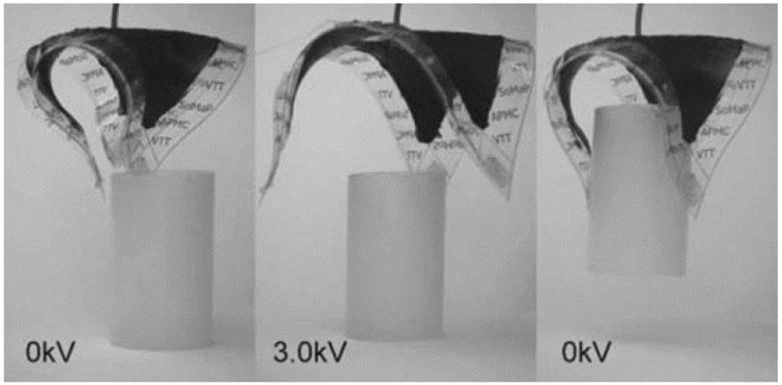
University of Illinois at Urbana-Champaign

# Soft Materials

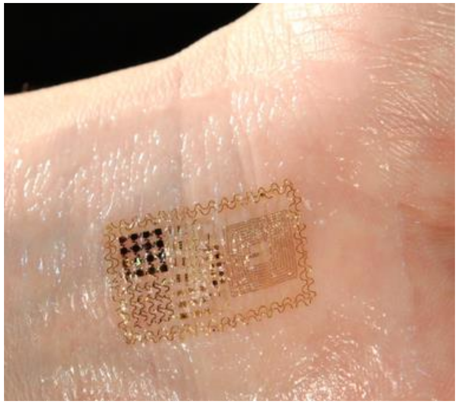
- Soft materials, such as electro-and magneto- active elastomers, hold great potential for engineering applications:


Yakacki, imechanica.org

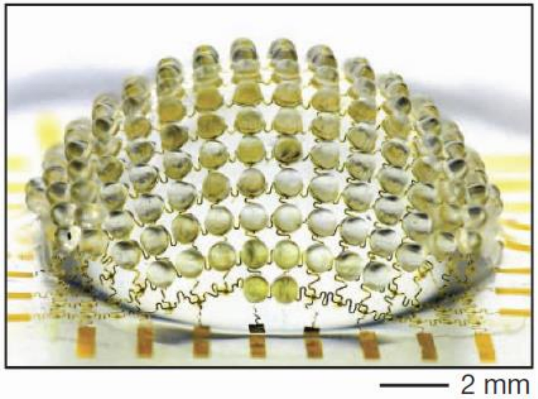



Kofod, Wirges,Paajanen Bauer, 2007

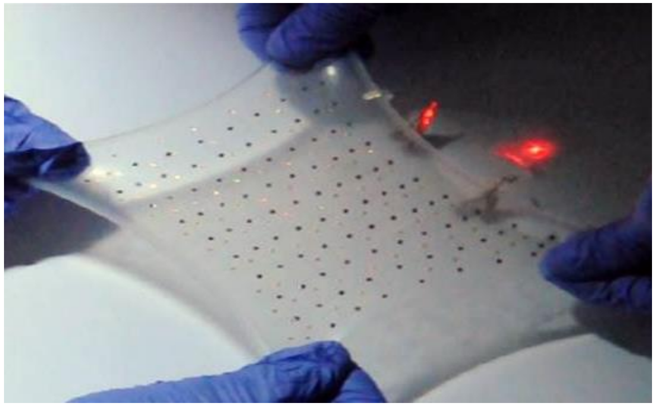
Bar-Cohen, 2004



Yeo et al, 2013



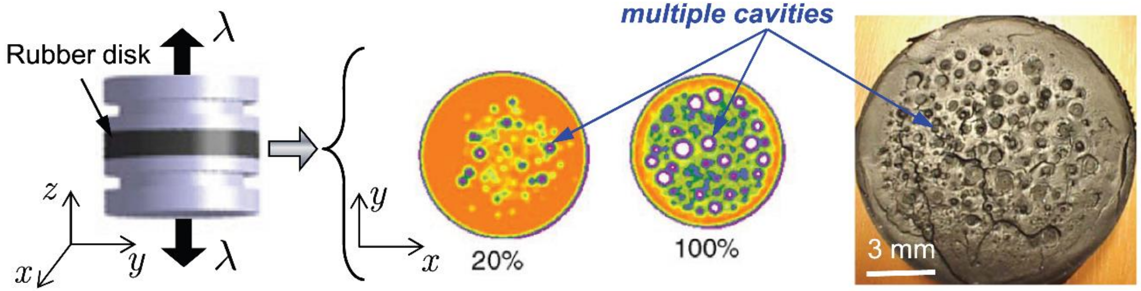

Song et al, 2013



Xu et al, 2013

# Soft Materials

- Soft materials characterized by several distinct features:
  - ❖ Large reversible deformation

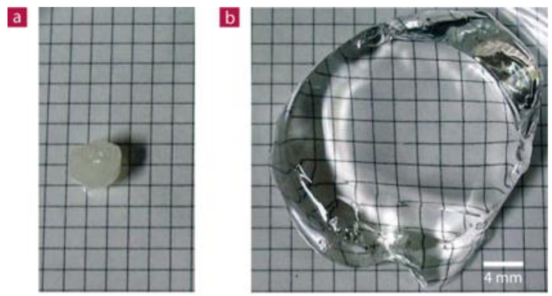



"Poker chip" test

In-situ X-ray images at various applied stretches $\lambda$

Image after a stretch of 100%

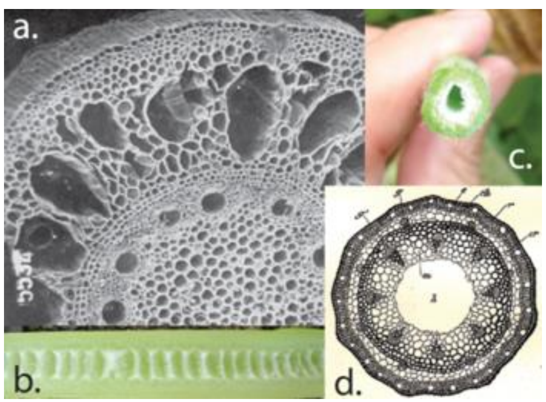
Bayraktar,Bessri,Bathias, 2007



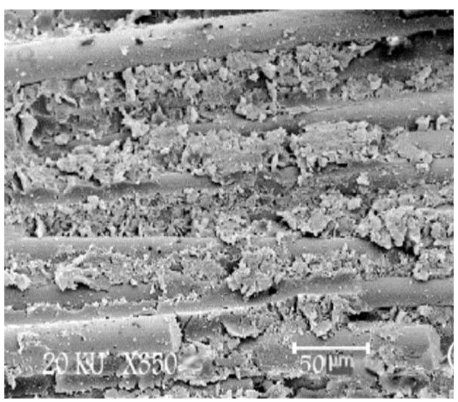

Swelling of lipophilic polyelectrolyte gel

Ono et al, 2007

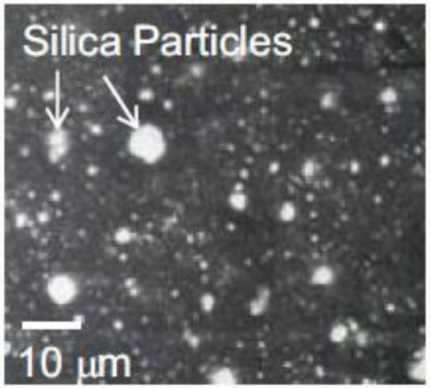
  - ❖ Complex micro-structures




Goriely,Moulton,Vandiver, 2010




Bagherpour, 2012




Ramier, 2004

  - ❖ Incompressible or nearly incompressible

# Theoretical Considerations

- Soft materials are considered hyperelastic, whose deformations can be described by a:
  - Displacement based formulation:

$$\Pi(\mathbf{u}) = \min_{\mathbf{v}\in\mathcal{K}} \left[\int_\Omega W(\mathbf{X},\mathbf{F}(\mathbf{v}))\,\mathrm{d}\mathbf{X} - \int_\Omega \mathbf{f}\cdot\mathbf{v}\mathrm{d}\mathbf{X} - \int_{\Gamma_\mathbf{t}} \mathbf{t}^0\cdot\mathbf{v}\mathrm{d}S\right]$$

  - Two-field mixed formulation:

$$\widehat{\Pi}(\mathbf{u},\widehat{p}) = \min_{\mathbf{v}\in\mathcal{K}}\max_{\widehat{q}\in\mathcal{Q}} \left[\int_\Omega \left\{-\widehat{W}^*(\mathbf{X},\mathbf{F}(\mathbf{v}),\widehat{q}) + \widehat{q}[\det\mathbf{F}(\mathbf{v})-1]\right\}\mathrm{d}\mathbf{X} - \int_\Omega \mathbf{f}\cdot\mathbf{v}\mathrm{d}\mathbf{X} - \int_{\Gamma_\mathbf{t}} \mathbf{t}^0\cdot\mathbf{v}\mathrm{d}S\right]$$

$$\widehat{W}^*(\mathbf{X},\mathbf{F},\widehat{q}) = \max_J \left\{\widehat{q}(J-1) - \widehat{W}(\mathbf{X},\mathbf{F},J)\right\}$$

$\Omega$ = Undeformed configuration

$\mathbf{f}$ = Body force per unit undeformed volume

$W(\mathbf{X},\mathbf{F},J)$ = Stored-energy function

$\mathbf{t}^0$ = Surface traction per unit undeformed area

$\mathcal{K},\mathcal{Q}$ = Admissible set for displacement and pressure field

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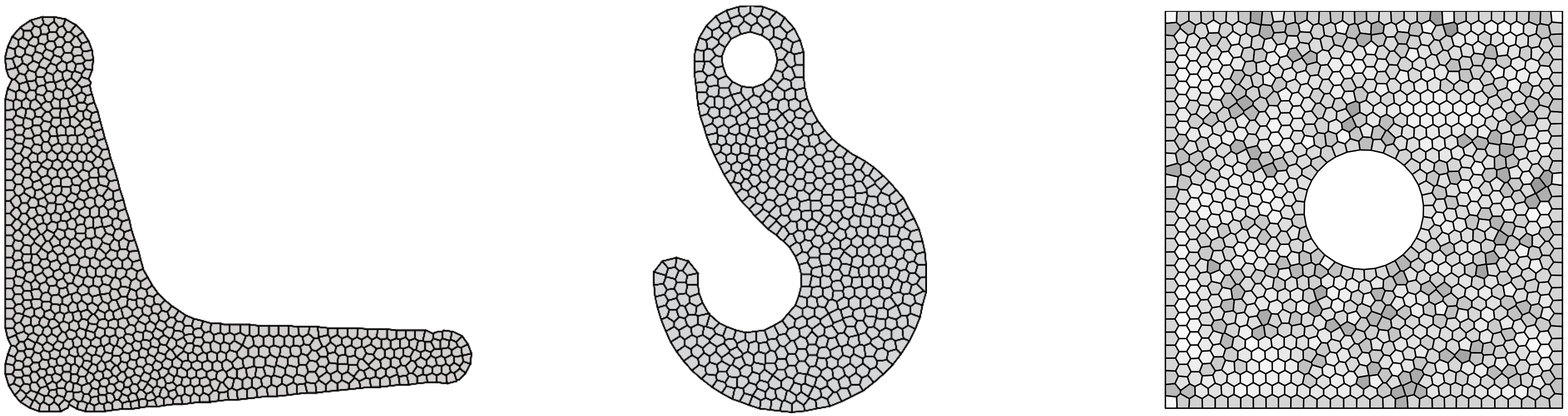
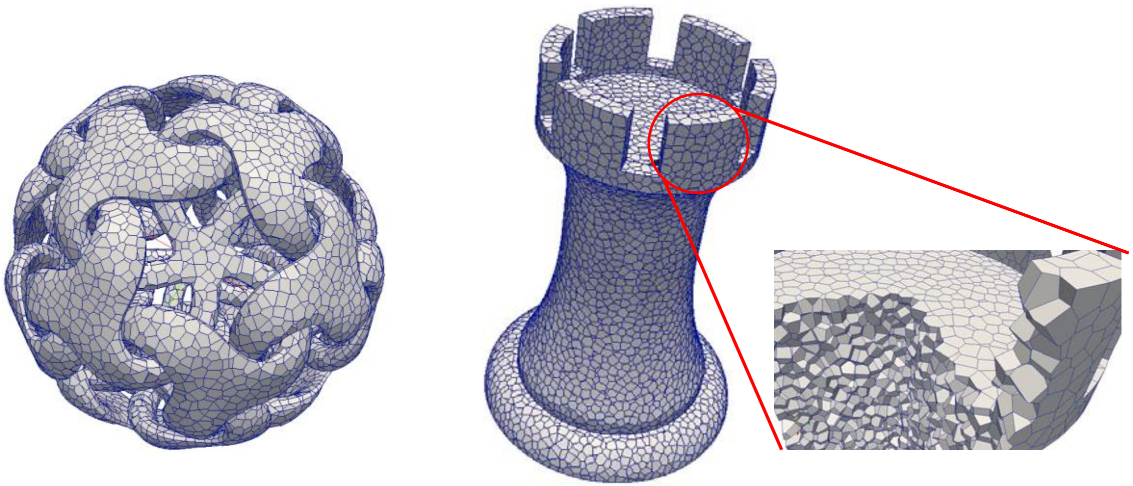
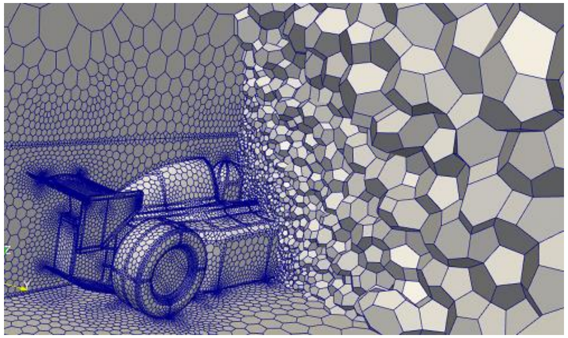
# Polygonal and Polyhedral Discretizations



Talischi, Paulino, Pereira, Menezes, 2012



Ebeida, Mitchell, 2012



www.symscape.com

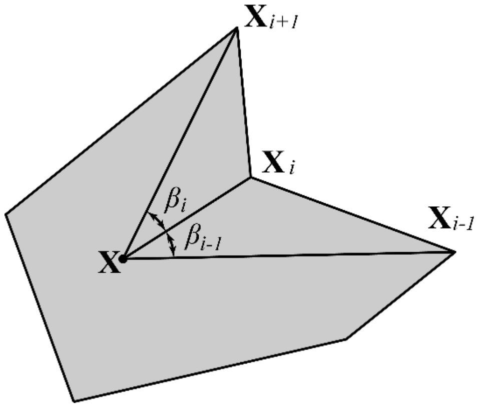
# Polygonal Finite Element Approximations

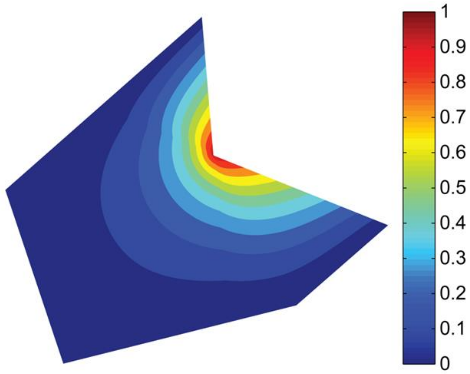
- The displacement space

$$\mathcal{K}_h = \left\{ \mathbf{v}_h \in \left[ C^0(\Omega) \right]^2 \cap \mathcal{K} : \mathbf{v}_h|_E \in \left[ \mathcal{V}(E) \right]^2, \forall E \in \mathcal{T}_h \right\}$$

$$\varphi_i(\mathbf{X}) = \frac{w_i(\mathbf{X})}{\sum_{j=1}^{n} w_j(\mathbf{X})}$$

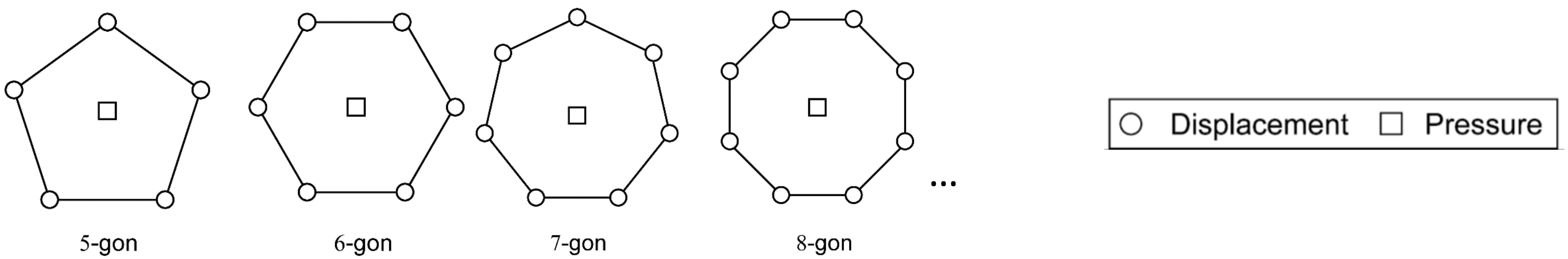
$$w_i(\mathbf{X}) = \frac{\tan\left[\frac{\beta_{i-1}(\mathbf{X})}{2}\right] + \tan\left[\frac{\beta_i(\mathbf{X})}{2}\right]}{\left|\left|\mathbf{X} - \mathbf{X}_i\right|\right|}$$




- The pressure space

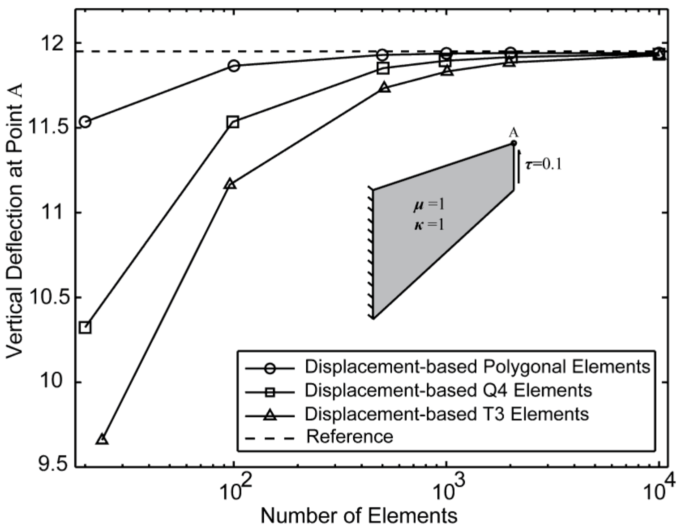
$$\mathcal{Q}_h = \left\{ \widehat{q}_h \in \mathcal{Q} : \widehat{q}_h|_E = \text{constant}, \forall E \in \mathcal{T}_h \right\}$$

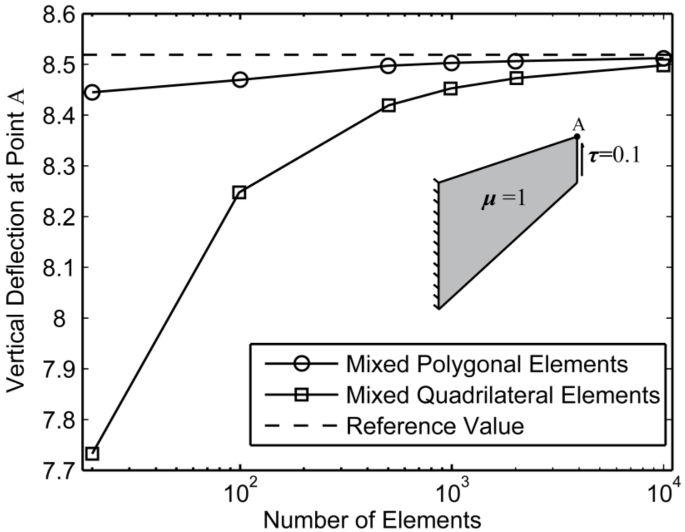




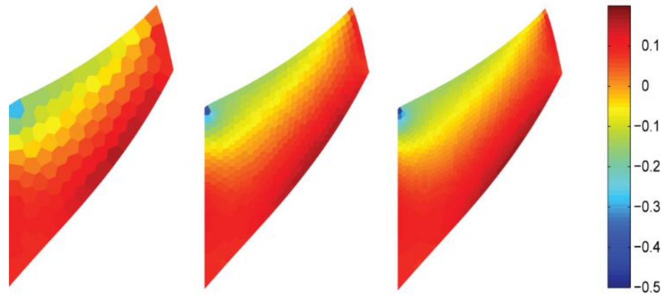

Floater MS. Mean value coordinates. *Computer Aided Geometric Design* 2003; 20(1):19–27.

# Polygonal Finite Element Methods

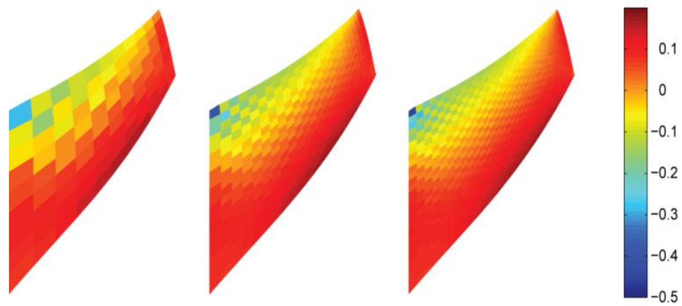


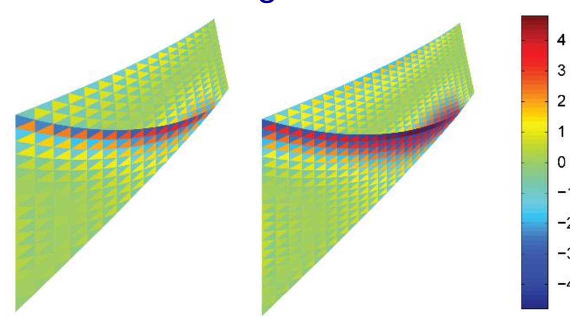

Centroidal voronoi tessellation (CVT) meshes




Linear quadrilateral meshes




Linear triangular meshes



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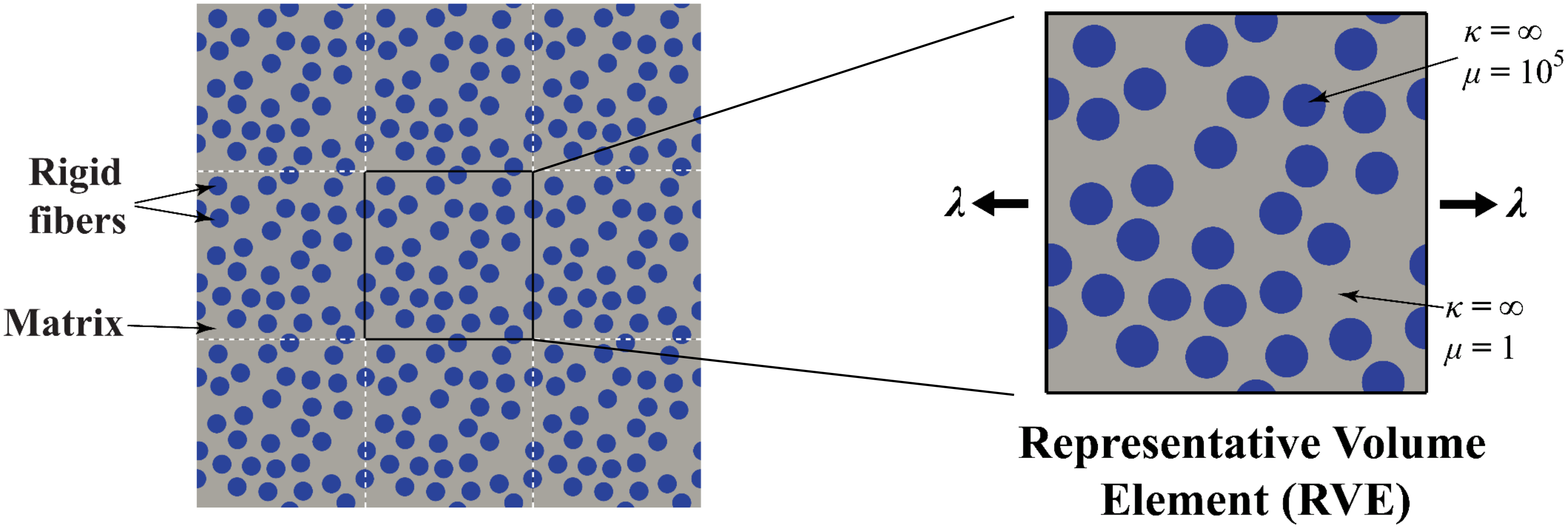
# Application: Filled Elastomers

- **Neo-Hookean matrix reinforced with 30% of rigid fibers**



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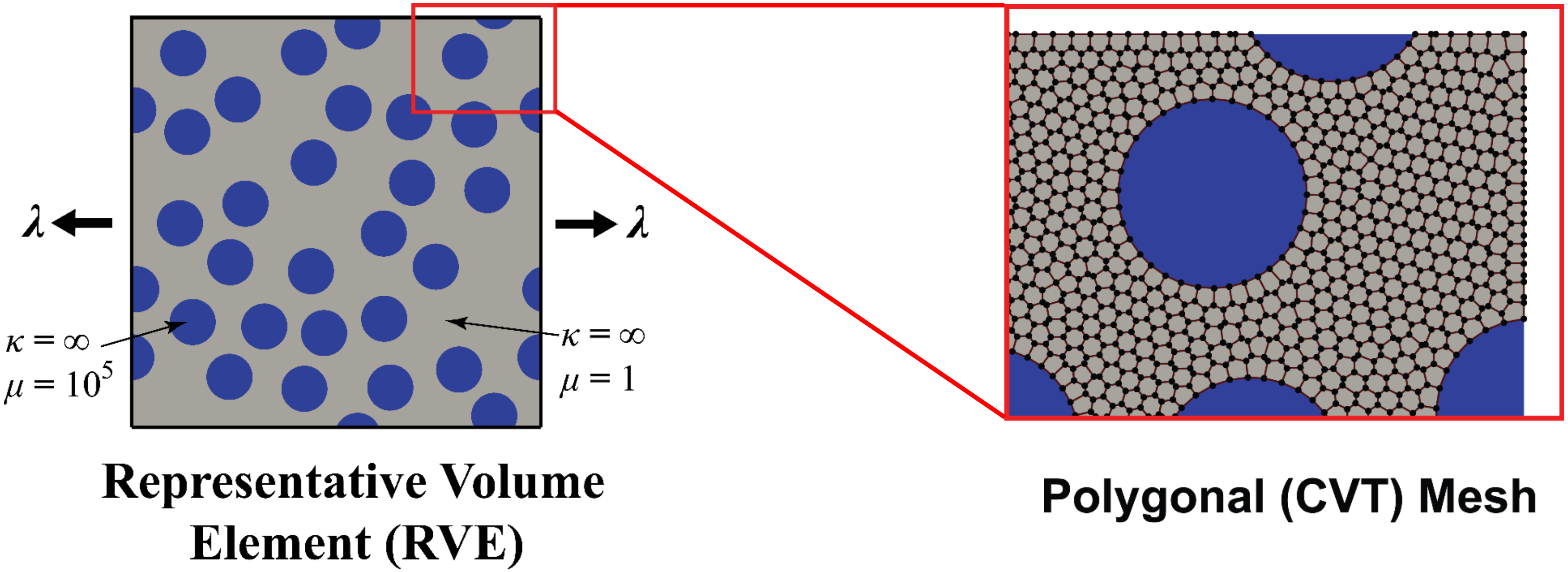
# Application: Filled Elastomers

- **Polygonal discretization:**



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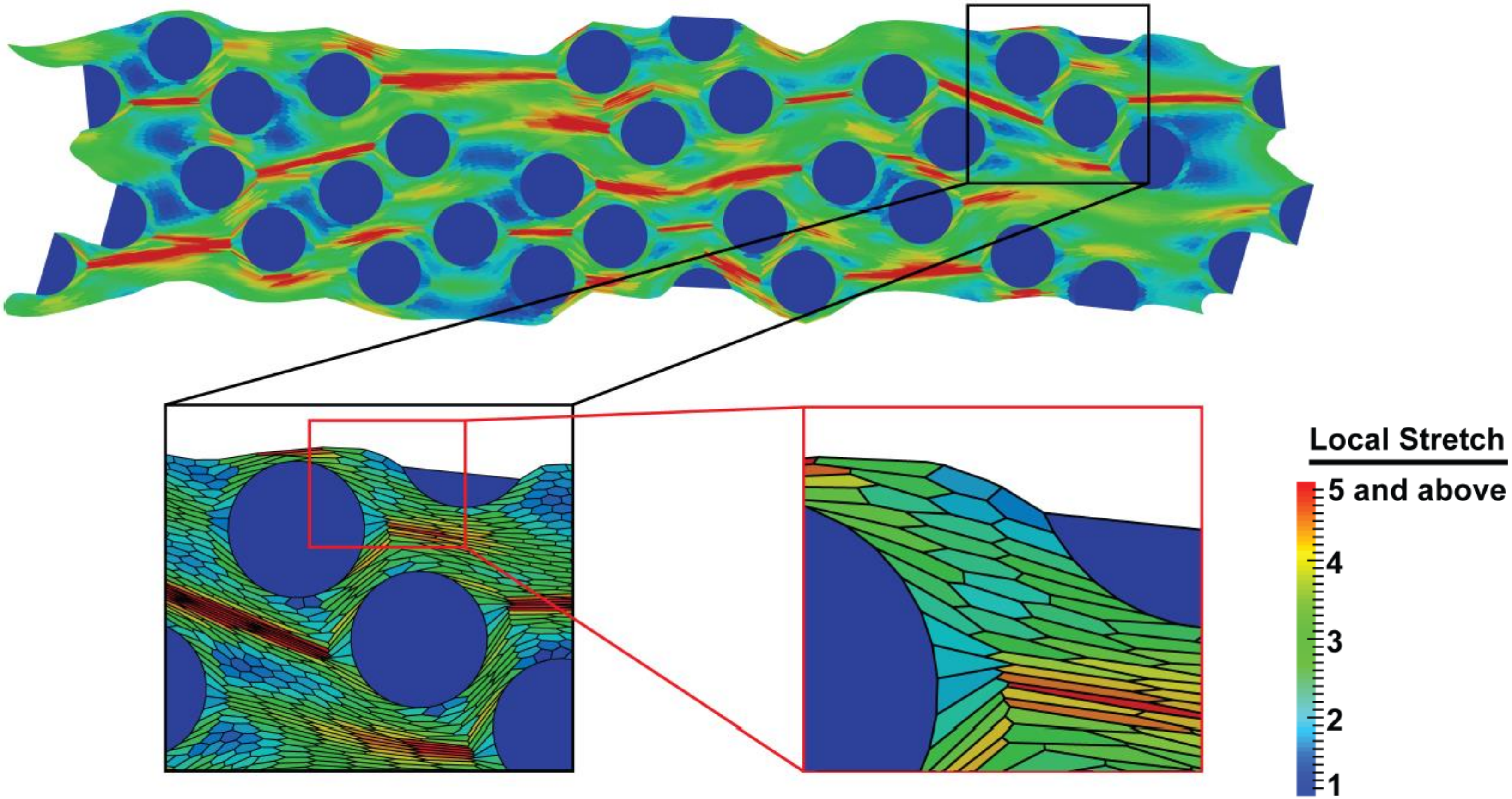
# Application: Filled Elastomers

- **Polygonal discretization:**




**Final deformation state reached at $\lambda = 2.1$**



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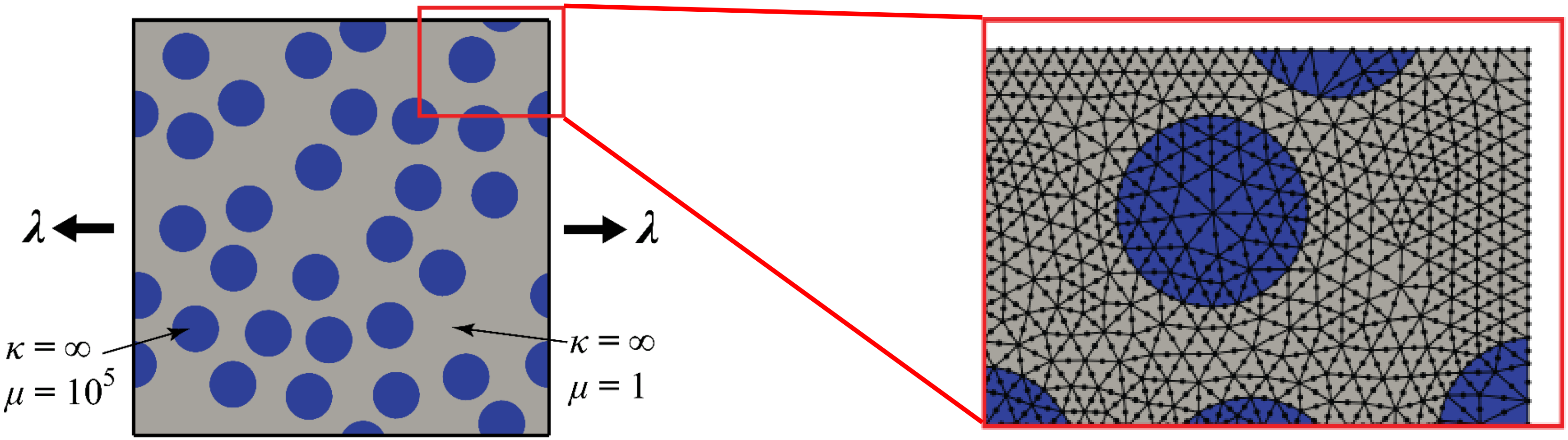
# Application: Filled Elastomers

- **Triangular discretization**




**Representative Volume Element (RVE)**

**Quadratic Triangular Mesh**

**Final deformation state reached at $\lambda = 1.46$**

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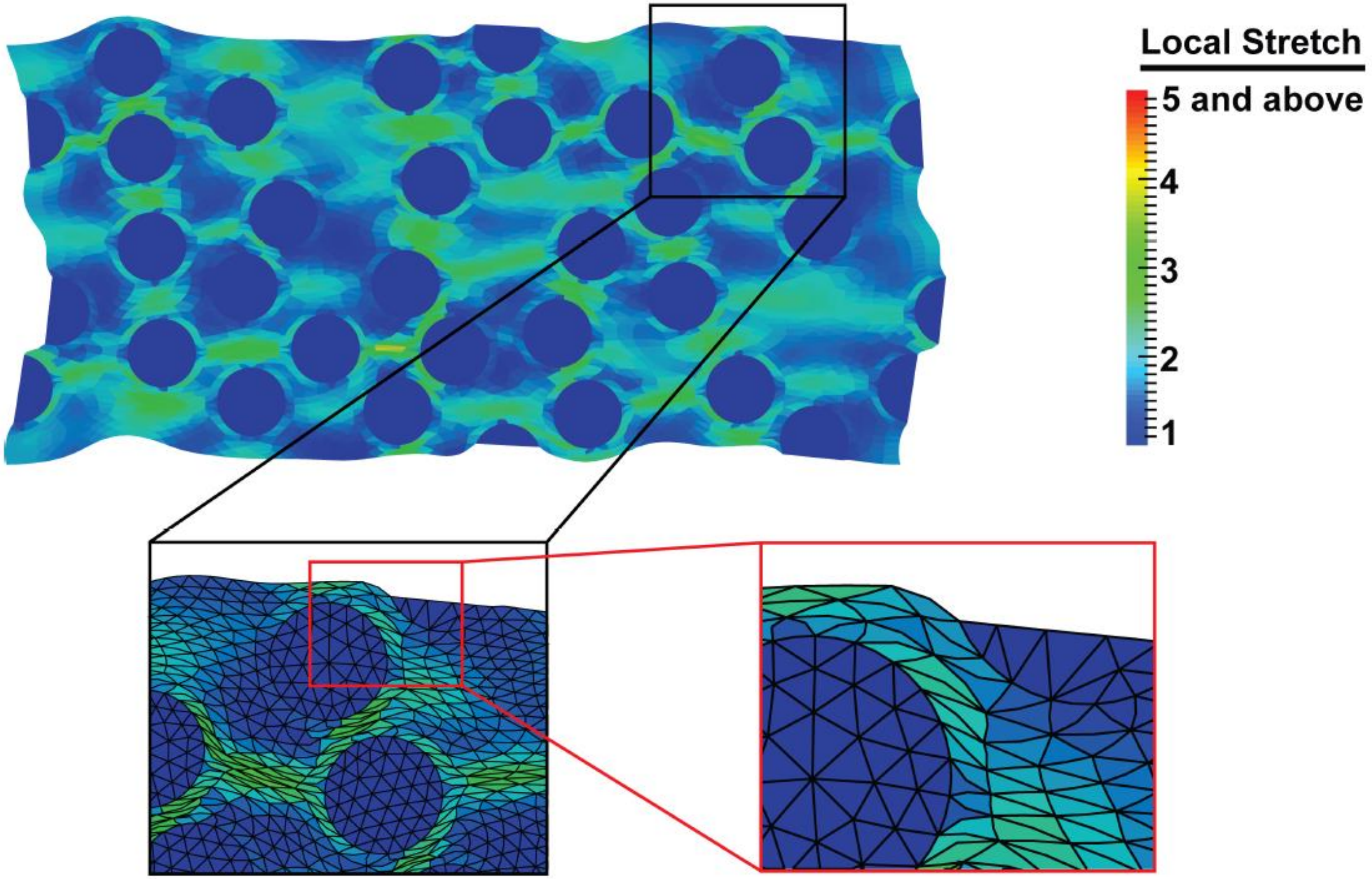
# Application: Filled Elastomers

- **Triangular discretization**




**Final deformation state reached at $\lambda = 1.46$**



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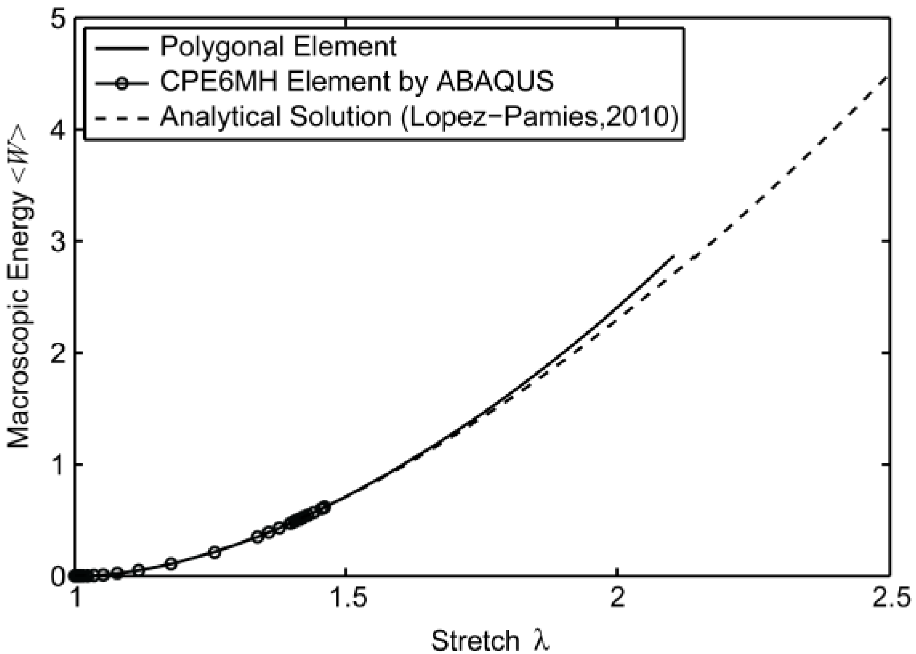
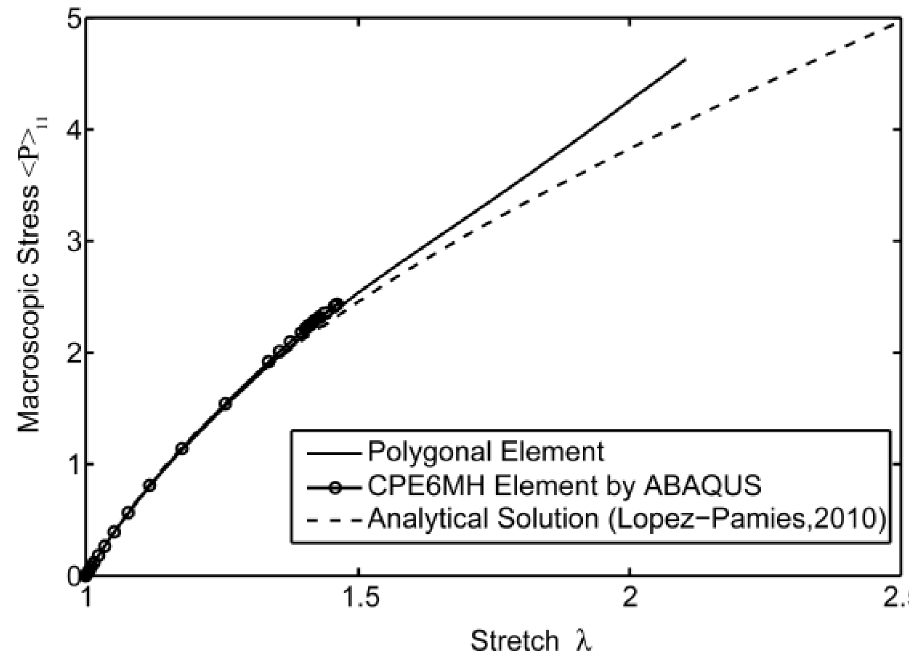
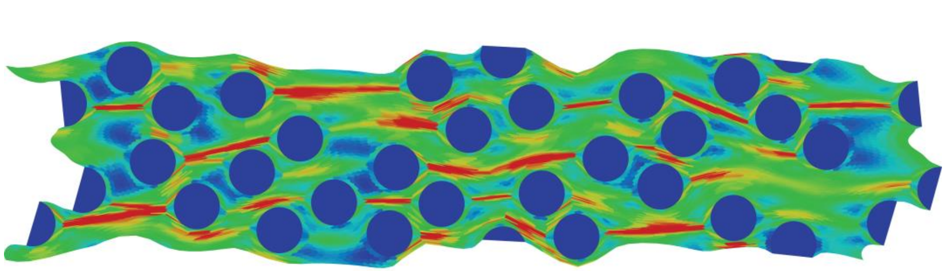
# Application: Filled Elastomers

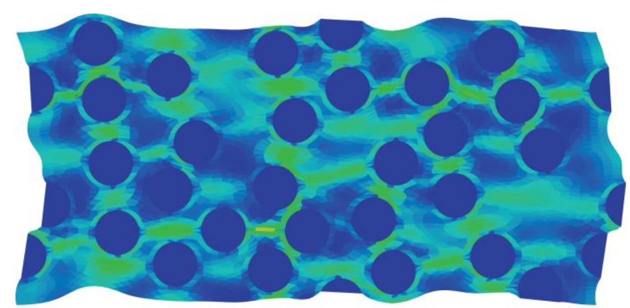
- **Comparison of macroscopic response:**



$\lambda = 2.1$



$\lambda = 1.46$



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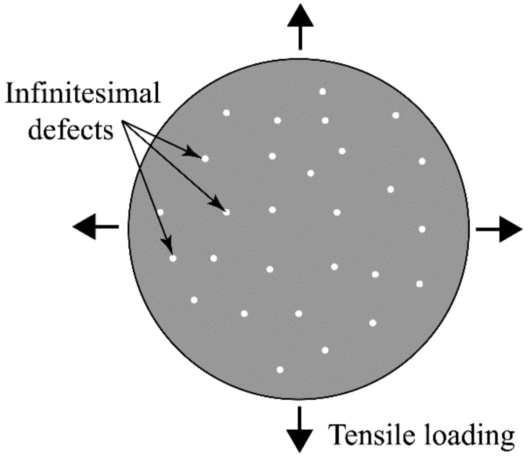
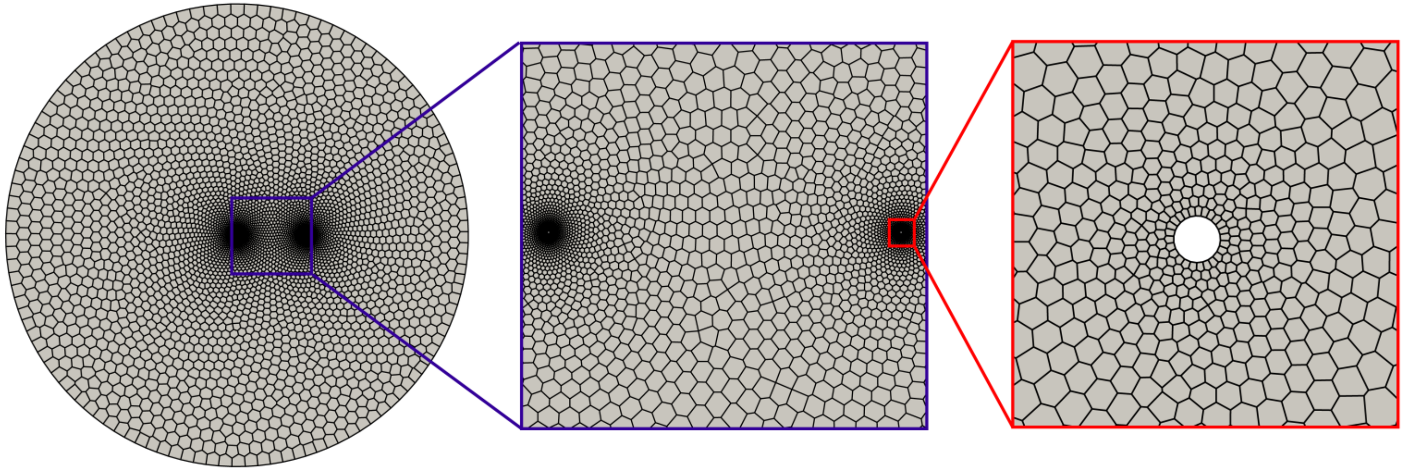
# Application: Cavitation

- **Cavitation in rubber: growth of pre-existing defects in rubber:**




- **A polygonal discretization with two pre-existing defects:**



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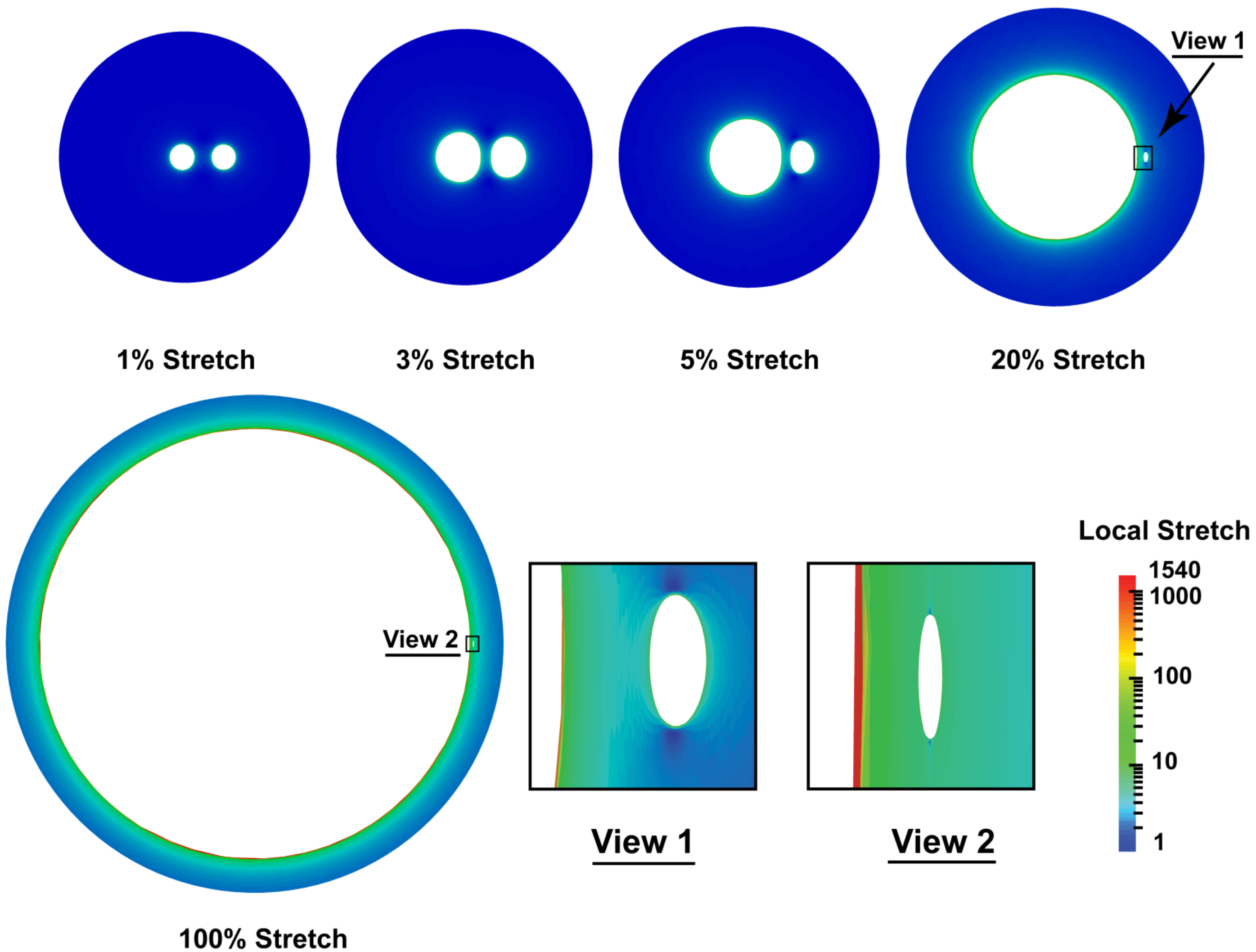
C. Talischi, G.H. Paulino, A. Pereira, I.F.M. Menezes. "PolyMesher: A general-purpose mesh generator for polygonal elements written in Matlab." *Journal of Structural and Multidisciplinary Optimization*. Vol. 45, No. 3, pp. 309-328, 2012.

# Application: Cavitation

- **Snapshots of the growth of defects at different levels of stretches**



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# Conclusions

- **Polygonal elements are numerically stable on Voronoi-type meshes without any additional treatments**
- **Polygonal discretizations provide geometric flexibility to model inclusions with arbitrary geometries and allow for easy incorporation of periodic boundary conditions, as well as bridging different length scales**
- **Polygonal elements appear to be more tolerant to large local deformations than classic triangular and quadrilateral elements**

**Thank you!**

**Questions and comments?**

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