Polygonal Finite Elements for Finite Elasticity

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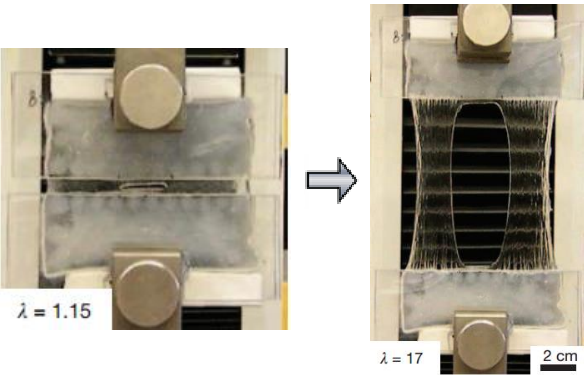
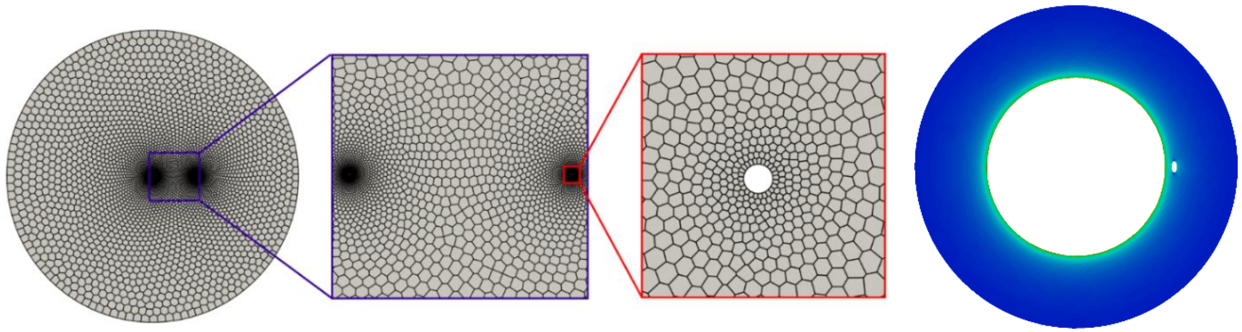
Engineering Mechanics Institute Conference 2016 / Probabilistic Mechanics & Reliability Conference 2016

Motivation and Introduction

- The large deformation impose challenge on the computational simulation of soft materials

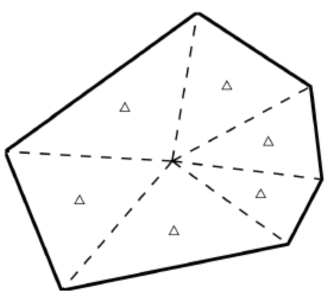


Jeong-Yun Sun et al, 2007

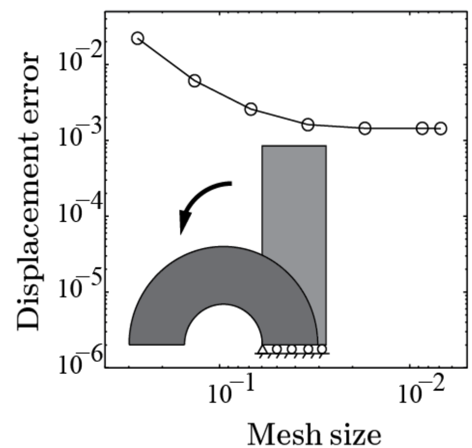


Chi et al, 2015

- However, efficient quadrature rules for polygonal elements are difficult to construct in practice.



1st quadrature



Variational Formulation of Finite Elasticity

- Soft materials are considered as hyperelastic, whose deformations can be described by:

❖ **Two field mixed formulation:**

$$\widehat{\Pi}\left(\mathbf{u}^{\circ}, \widehat{p}^{\circ}\right) = \min_{\mathbf{u}\in\mathcal{K}} \max_{\widehat{p}\in\mathcal{Q}} \int_{\Omega} \left\{-\widehat{W}^{*}\left(\mathbf{X}, \mathbf{F}\left(\mathbf{u}\right), \widehat{p}\right) + \widehat{p}\left[\det \mathbf{F}\left(\mathbf{u}\right) - 1\right]\right\} \mathrm{d}\mathbf{X} - \int_{\Omega} \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{X} - \int_{\partial\Omega^{\sigma}} \mathbf{t}\cdot\mathbf{u}\,\mathrm{d}S$$

where

$$\widehat{W}^{*}\left(\mathbf{X}, \mathbf{F}, \widehat{q}\right) = \max_{J} \left\{\widehat{q}\left(J-1\right) - W\left(\mathbf{X}, \mathbf{F}, J\right)\right\}$$

Ω = Undeformed configuration	$\mathbf{f}$ = Body force per unit undeformed volume
$W\left(\mathbf{X}, \mathbf{F}, J\right)$ = Stored-energy function	$\mathbf{t}$ = Surface traction per unit undeformed area
$\mathcal{K}, \mathcal{Q}$ = Admissible sets for displacement field and pressure field	

Chi, H., Talischi, C., Lopez-Pamies, O., Paulino, G.H. 2015. Polygonal finite elements for finite elasticity. *International Journal for Numerical Methods in Engineering,* 101, 305–328

Polygonal Finite Element Spaces

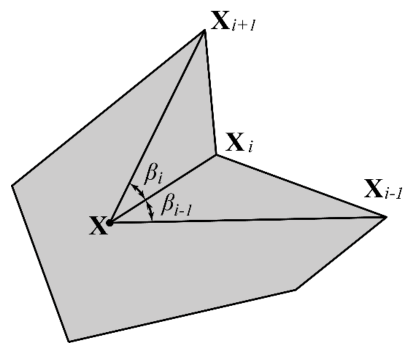
- The displacement space

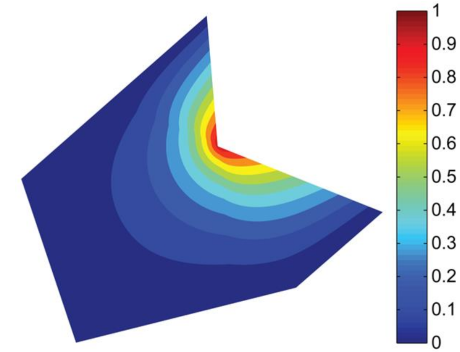
$$\mathcal{K}_{h,k} = \left\{ \mathbf{v}_h \in \left[C^0(\Omega)\right]^2 \cap \mathcal{K} : \mathbf{v}_h|_E \in \left[\mathcal{M}_k(E)\right]^2, \forall E \in \Omega_h \right\}$$

❖ k=1, Mean Value coordinates

$$\varphi_i^1(\mathbf{X}) = \frac{w_i(\mathbf{X})}{\sum_{j=1}^{n} w_j(\mathbf{X})} \quad \text{with:}$$

$$w_i(\mathbf{X}) = \frac{\tan\left[\frac{\beta_{i-1}(\mathbf{X})}{2}\right] + \tan\left[\frac{\beta_i(\mathbf{X})}{2}\right]}{\|\mathbf{X} - \mathbf{X}_i\|}$$

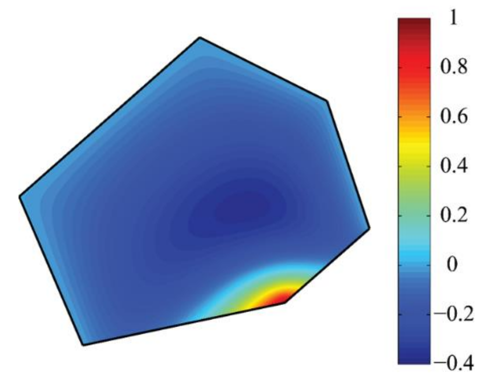




❖ k=2, linear combination of pairwise products of φ_i^1

$$\varphi_i^2(\mathbf{X}) = \sum_{j=1}^{n} \sum_{l=1}^{n} c_{jl}^i \varphi_j^1(\mathbf{X}) \varphi_l^1(\mathbf{X}) \quad \text{such that:}$$

$$p(\mathbf{X}) = \sum_{i=1}^{n} \left[p(\mathbf{X}_i) \varphi_i^2(\mathbf{X}) + p\left(\widehat{\mathbf{X}}_i\right) \varphi_{i+n}^2(\mathbf{X}) \right], \quad \forall p \in \mathcal{P}_2(E)$$



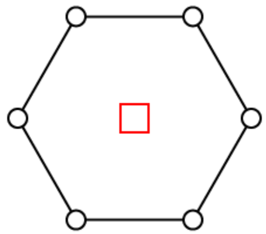
Floater, M. S. 2003. Mean value coordinates. *Computer aided geometric design*, 20(1), 19-27.
Rand, A., Gillette, A., and Bajaj, C. 2014. Quadratic serendipity finite elements on polygons using generalized barycentric coordinates. *Mathematics of computation*, 83(290), 2691-2716.

Polygonal Finite Element Approximations

- The pressure spaces:
  - **Discontinuous** across the element edges:

$$\mathcal{Q}^{D}_{h,k-1} = \{q_h \in \mathcal{Q} : q_h|_E \in \mathcal{P}_{k-1}(E), \forall E \in \Omega_h\}$$

$k = 1$



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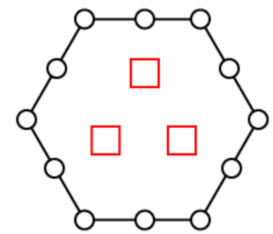
$\mathcal{M}_1 - \mathcal{P}^D_0$ elements

$k = 2$



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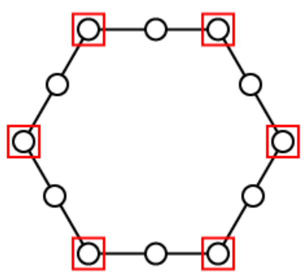
$\mathcal{M}_2 - \mathcal{P}^D_1$ elements

  - **Continuous** across the element edges:

$$\mathcal{Q}^{C}_{h,k-1} = \{q_h \in C^0(\Omega) : q_h|_E \in \mathcal{M}_{k-1}(E), \forall E \in \Omega_h\}$$

$k = 2$



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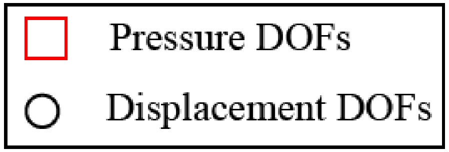
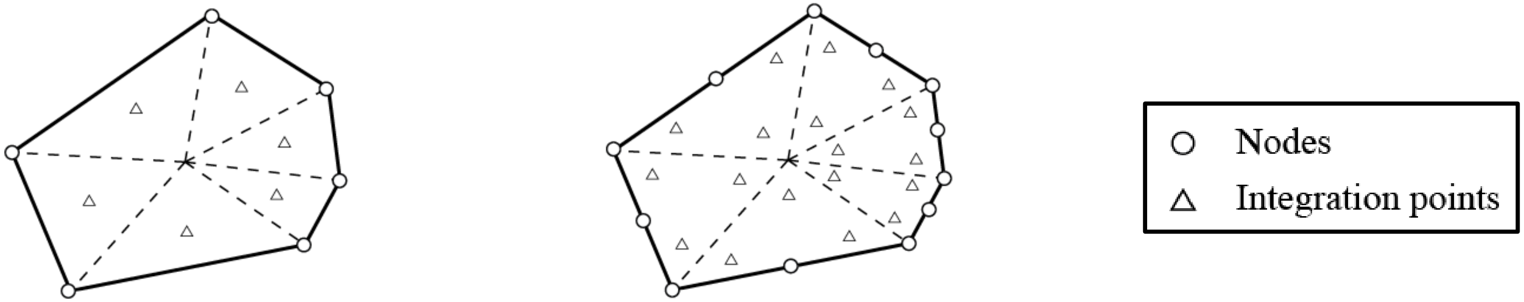


$\mathcal{M}_2 - \mathcal{M}_1$ elements

Integration Scheme and Gradient Correction

- We use the triangulation scheme over element E , denoted as $\oint_E$



$k = 1$ $\qquad$ $k = 2$

- The **definition** of the correction to the gradient is as follows:

 For any given $v \in \mathcal{M}_k(E)$, the corrected gradient $\nabla_{E,k} v$ is the one satisfies:

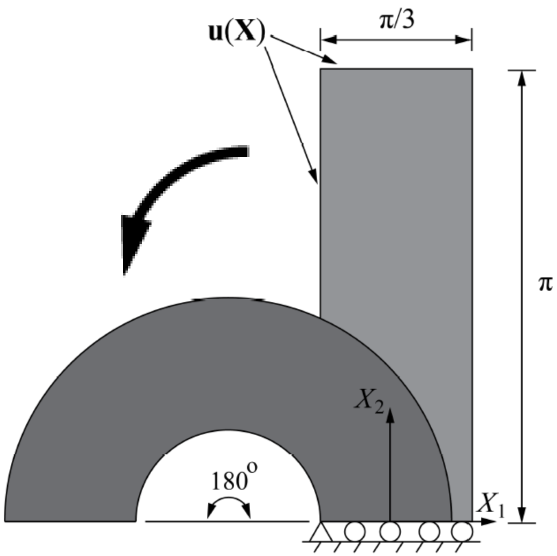
❖ $\nabla_{E,k} v - \nabla v \in [\mathcal{P}_{k-1}(E)]^2$

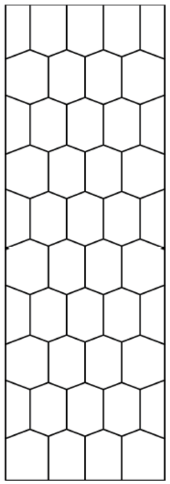
❖ $\oint_E \mathbf{p} \cdot \nabla_{E,k} v \mathrm{d}\mathbf{X} = \int_{\partial E} (\mathbf{p} \cdot \mathbf{N})\, v \mathrm{d}S - \oint_E v\, \mathrm{Div}\mathbf{p} \mathrm{d}\mathbf{X}, \quad \forall \mathbf{p} \in [\mathcal{P}_{k-1}(E)]^2$

Talischi, C., Pereira, A., Menezes, I.F.M., Paulino., G.H. 2015. Gradient correction for polygonal and polyhedral finite elements. *International Journal for Numerical Methods in Engineering.* 102, 728-747.

Convergence of Mixed Polygonal FEM

- Consider a boundary value problem with displacement boundary condition applied as follows:





$$u_1 = r(X_1)\cos(X_2) - r\left(-\frac{\pi}{6}\right) - \frac{\pi}{6} - X_1$$

$$u_2 = r(X_1)\sin(X_2) - X_2$$

$$r(X_1) = \sqrt{2X_1 + \beta} \quad \text{with} \quad \beta = \sqrt{\frac{4\pi^2}{9} + 4}$$

$$\widehat{p} = -\frac{\mu}{2}\left[\frac{1}{r(X_1)^2} - r(X_1)^2 + \mu\beta\right]$$

Displacement error: $\epsilon_{1,\mathbf{u}} = \|\nabla\mathbf{u} - \nabla\mathbf{u}_h\|$

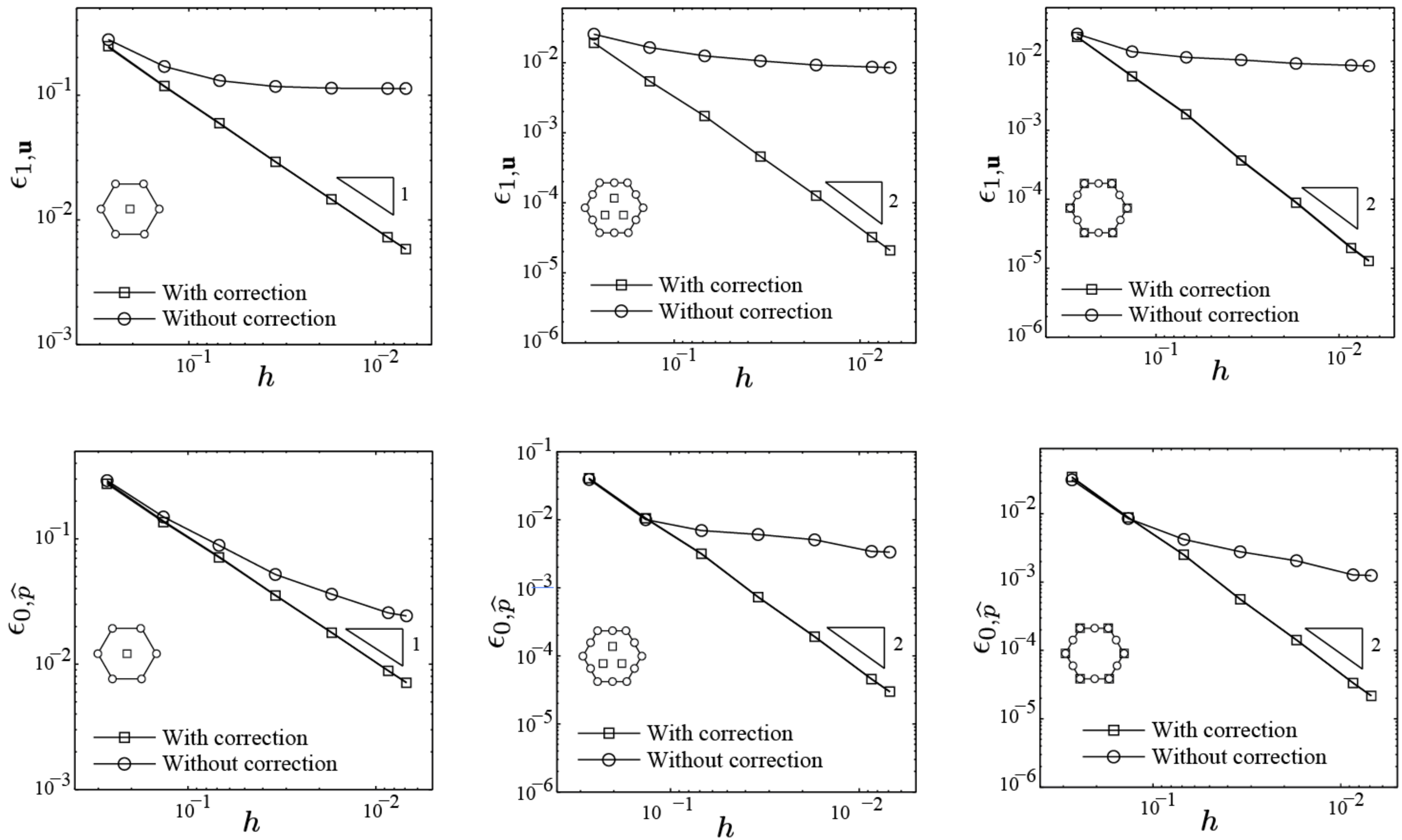
Pressure error: $\epsilon_{0,\widehat{p}} = \|\widehat{p} - \widehat{p}_h\|$

Convergence of Mixed Polygonal FEM

$\mathcal{M}_1 - \mathcal{P}_0^D$ elements | $\mathcal{M}_2 - \mathcal{P}_1^D$ elements | $\mathcal{M}_2 - \mathcal{M}_1$ elements

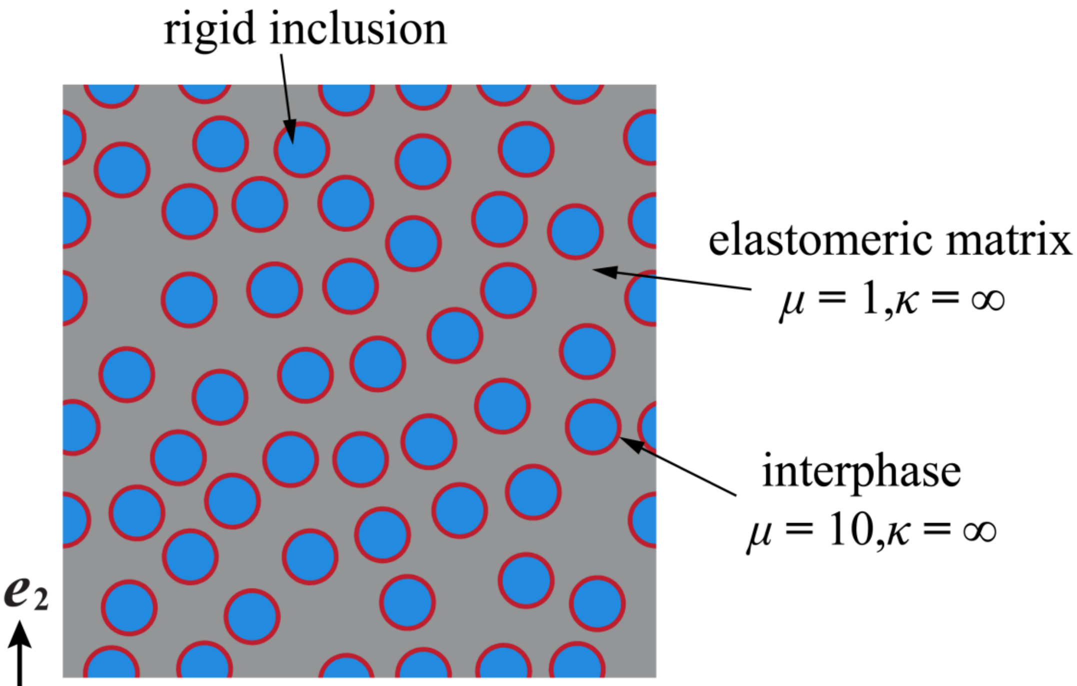
Displacement error

Pressure error



Filled Elastomers

- The microstructure and the unit cell considered:



Unit Cell

❖ Incompressible Neo-Hookean Material

$$W(\mathbf{X}, \mathbf{F}) = \begin{cases} \frac{\mu}{2}\left[\mathbf{F} : \mathbf{F} - 3\right] & \text{if} \quad \det \mathbf{F} = 1 \\ +\infty & \text{otherwise} \end{cases}$$

❖ $N = 50$ particles, with a volume fraction of the particle: $c_p = 25\%$.

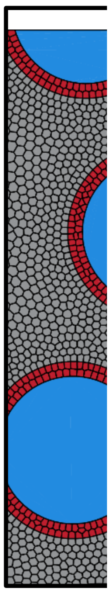
❖ The thickness of the interphase: $t = 0.2R_p$, where R_p is the radius of the particle.

❖ The effective volume fraction is $c = c_p + c_i = 36\%$

Chi, H., Lopez-Pamies, O., Paulino, G.H. 2015. A paradigm for higher-order polygonal elements in finite elasticity using a gradient correction scheme, *Computer Methods in Applied Mechanics and Engineering*, Vol. 306, pp. 216-251. 2016.

Filled Elastomers

- Similar number of nodes in the matrix



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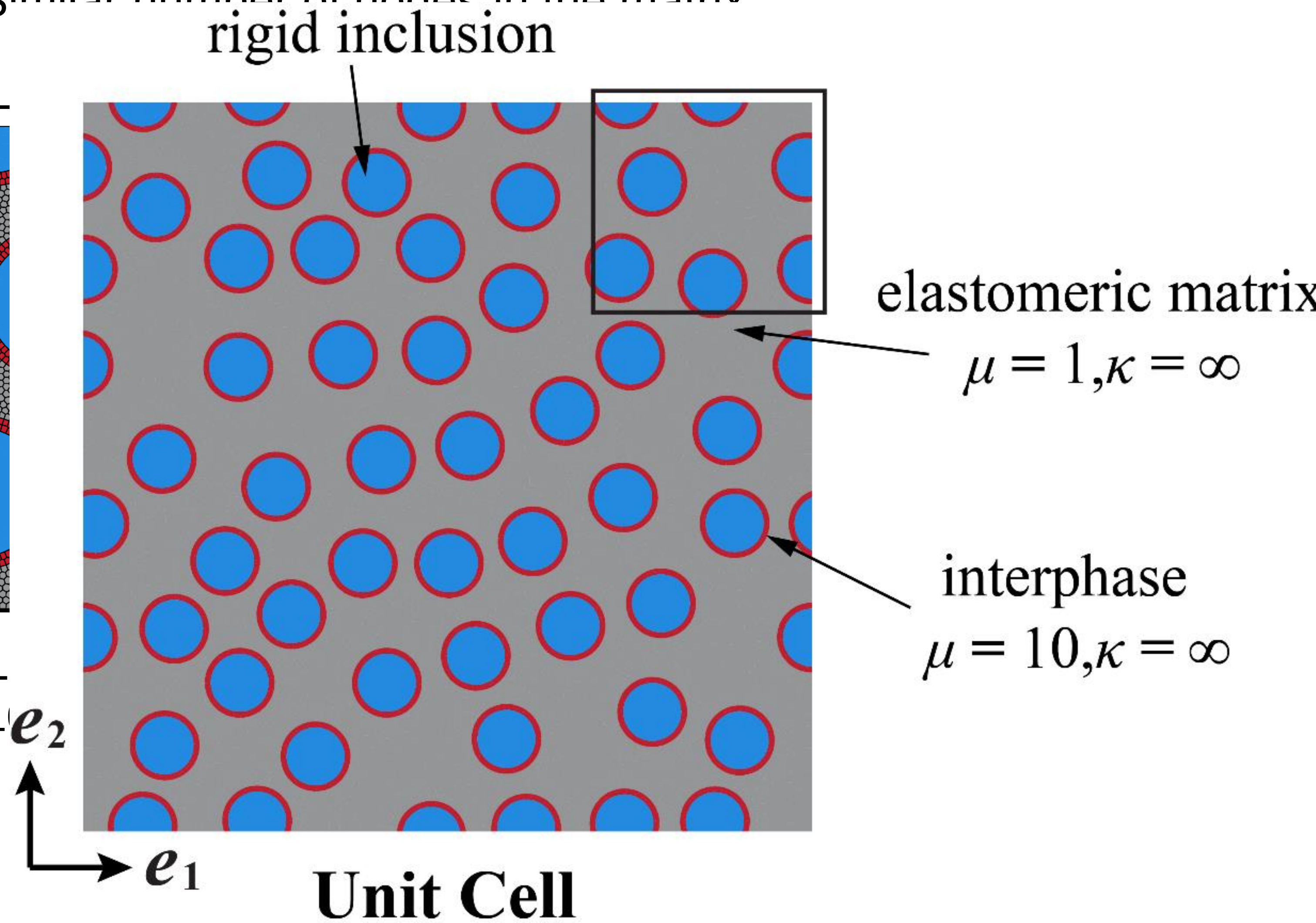
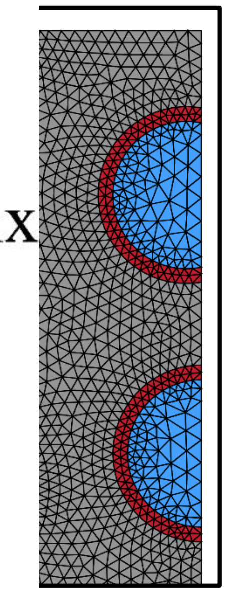
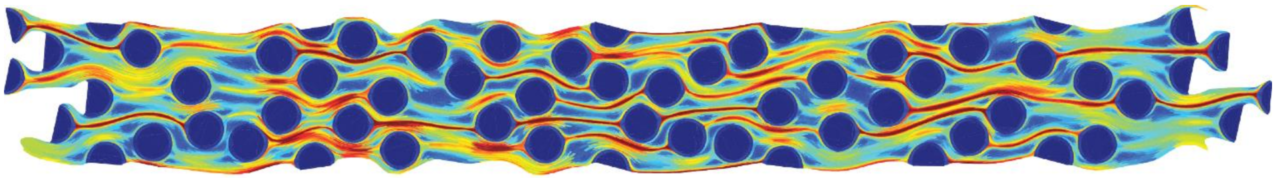




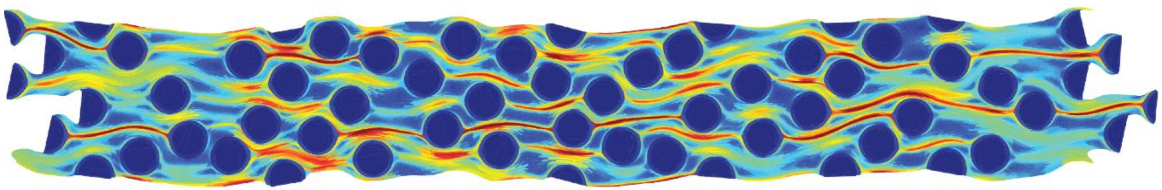
Mesh
41 Nd.)

Talischi, C., Paulino, G.H., Pereira, A., Menezes., I.F.M. "PolyMesher: A general-purpose mesh generator for polygonal elements written in Matlab." *Journal of Structural and Multidisciplinary Optimization*. 45,3, 309-328, 2012.

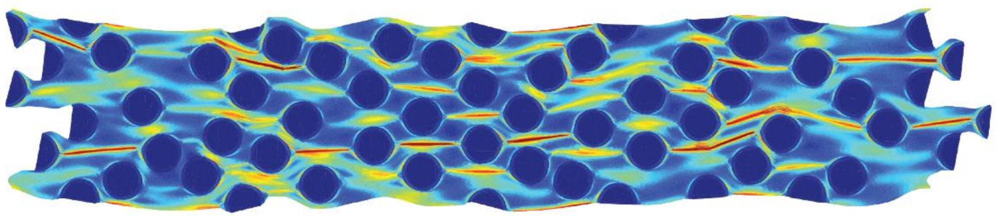
Filled Elastomers



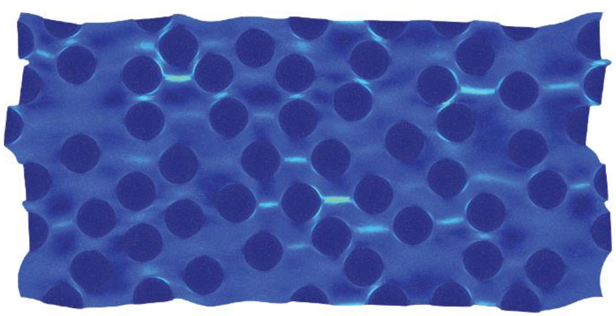
$\mathcal{M}_2 - \mathcal{M}_1$ elements $\lambda_{max} = 2.9132$



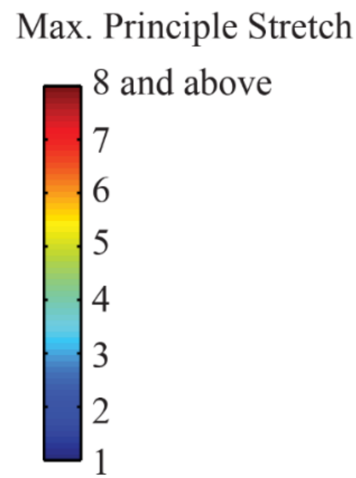
$\mathcal{M}_2 - \mathcal{P}_1^D$ elements $\lambda_{max} = 2.6456$

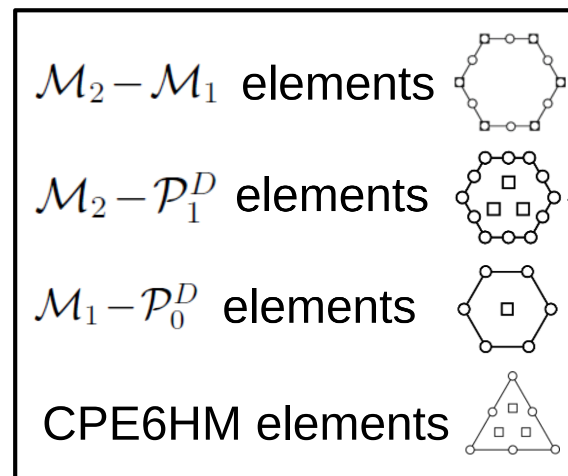


$\mathcal{M}_1 - \mathcal{P}_0^D$ elements $\lambda_{max} = 2.2615$

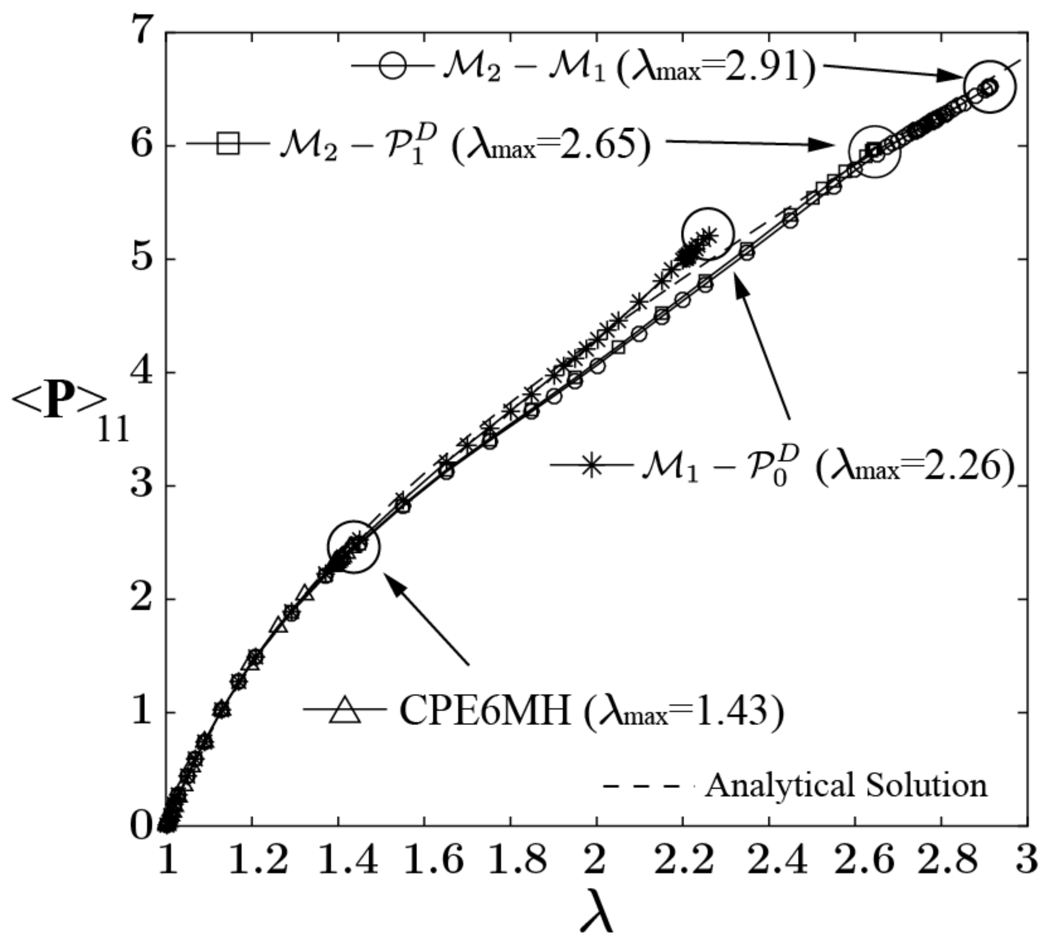


CPE6HM elements $\lambda_{max} = 1.4308$





Filled Elastomers



uniaxial tension

Goudarzi, T., Spring, D.W., Paulino, G.H., Lopez-Pamies, O. ”Filled elastomers: A theory of filler reinforcement based on hydrodynamic and interphasial effects”. *Journal of the Mechanics and Physics of Solids*. 80, 37–67, 2015

Concluding Remarks

- We presented a gradient correction scheme to develop efficient and consistent quadrature on arbitrary polygonal elements.
- The gradient correction is shown to deliver optimal convergence of the mixed FEM solutions up to quadratic order.
- Polygonal finite element appears to be able to handle larger localized deformations than the standard finite elements.

Thank you!

Questions and comments?