Topology Optimization Considering the Drucker–Prager Criterion Using a Surrogate Nonlinear Elastic Constitutive Model

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Motivation

- **The flow theory of plasticity** [Hill, 1950; Bell, 1968; Lubliner, 1992]:

The behavior of the plastic materials is characterized by NONLINEARITY and PATH-DEPENDENT physical processes (i.e., energy dissipation & unloading).

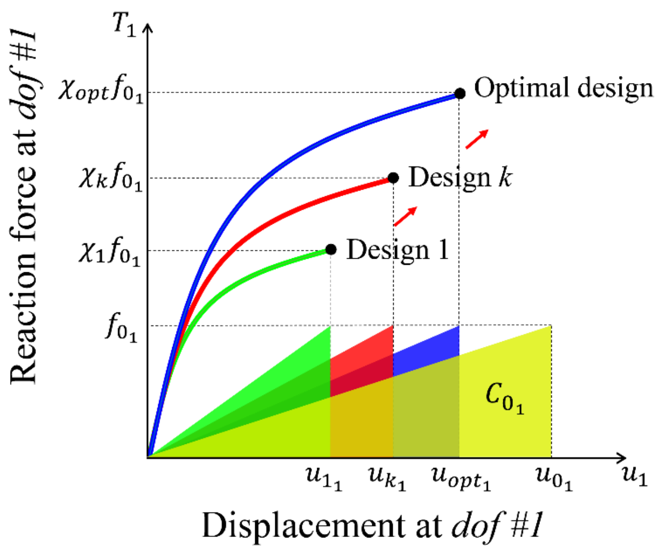
- **Generalized J_2 deformation theory of plasticity** [Hencky, 1924; Drucker and Prager, 1952; Chen and Han 1988; Sonato et al., 2015]:

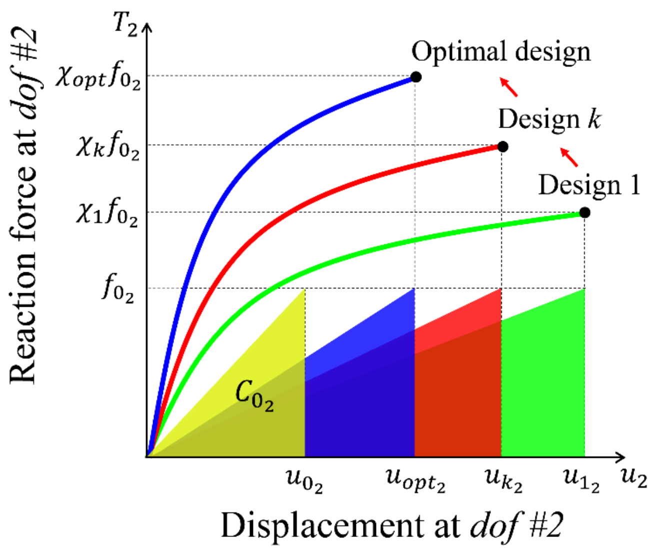
A surrogate nonlinear elastic model:

$$\boldsymbol{\sigma} = \lambda\big(J_1(\boldsymbol{\varepsilon}), J_2(\boldsymbol{\varepsilon}_d)\big)\, J_1(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\big(J_1(\boldsymbol{\varepsilon}), J_2(\boldsymbol{\varepsilon}_d)\big)\, \boldsymbol{\varepsilon}$$

- **Assuming small deformation, the nonlinear elastic solution is equivalent to the plastic solution under proportional loading.**

- **Nonlinear FEM analysis: Energy control approach**





Formulations and Sensitivity analyses

Max. structural strain energy

$$
\begin{aligned}
&\max_{\boldsymbol{\rho}} \quad J_U(\boldsymbol{\rho}) = U\big(\boldsymbol{\rho}, \boldsymbol{u}(\boldsymbol{\rho})\big) \\
&s.t. \quad V \le V_0 \\
&\qquad\quad 0 \le \rho_i \le 1 \\
&\text{with} \\
&\begin{cases} \boldsymbol{T}\left(\boldsymbol{\rho}, \boldsymbol{u}(\boldsymbol{\rho})\right) = \chi\big(\boldsymbol{\rho}, \boldsymbol{u}(\boldsymbol{\rho})\big)\boldsymbol{f}_0 \\ \boldsymbol{f}_0^{\mathrm{T}} \boldsymbol{u}(\boldsymbol{\rho}) = 2C_0 \end{cases} \\
&\begin{cases} \min_{\boldsymbol{u}} \quad U(\boldsymbol{u}) \\ s.t. \quad \boldsymbol{f}_0^{\mathrm{T}} \boldsymbol{u} = 2C_0 \end{cases}
\end{aligned}
$$

$$
\frac{dJ_U(\boldsymbol{\rho})}{d\rho_i} = \frac{\partial U\big(\boldsymbol{\rho}, \boldsymbol{u}(\boldsymbol{\rho})\big)}{\partial \rho_i}
$$

Max. load factor

$$
\begin{aligned}
&\max_{\boldsymbol{\rho}} \quad J_\chi(\boldsymbol{\rho}) = \chi\big(\boldsymbol{\rho}, \boldsymbol{u}(\boldsymbol{\rho})\big) \\
&s.t. \quad V \le V_0 \\
&\qquad\quad 0 \le \rho_i \le 1 \\
&\text{with} \\
&\begin{cases} \boldsymbol{T}\left(\boldsymbol{\rho}, \boldsymbol{u}(\boldsymbol{\rho})\right) = \chi\big(\boldsymbol{\rho}, \boldsymbol{u}(\boldsymbol{\rho})\big)\boldsymbol{f}_0 \\ \boldsymbol{f}_0^{\mathrm{T}} \boldsymbol{u}(\boldsymbol{\rho}) = 2C_0 \end{cases} \\
&\begin{cases} \min_{\boldsymbol{u}} \quad U(\boldsymbol{u}) \\ s.t. \quad \boldsymbol{f}_0^{\mathrm{T}} \boldsymbol{u} = 2C_0 \end{cases}
\end{aligned}
$$

$$
\frac{dJ_\chi(\boldsymbol{\rho})}{d\rho_i} = \frac{\left(\frac{\partial \boldsymbol{T}}{\partial \rho_i}\right)^{\mathrm{T}} (\boldsymbol{K}_T)^{-1} \boldsymbol{f}_0}{\boldsymbol{f}_0^{\mathrm{T}} (\boldsymbol{K}_T)^{-1} \boldsymbol{f}_0}
$$

Niu F, Xu S, Cheng G. (2011). A general formulation of structural topology optimization for maximizing structural stiffness. *Struct Multidiscip Optim*, 43(4):561–572.

Klarbring A, Strömberg N. (2013). Topology optimization of hyperelastic bodies including nonzero prescribed displacements. *Struct Multidiscip Optim*, 47(1):37–48.

Zhao T, Ramos Jr. AS, Paulino GH. (2019). Material Nonlinear Topology Optimization Considering the von Mises Criterion through an Asymptotic Approach: Max Strain Energy and Max Load Factor Formulations. *Int J Numer Methods Eng*, 118:804-828.

A surrogate nonlinear elastic model

- **Drucker-Prager yield function**

$$f(\boldsymbol{\sigma}) = \beta J_1(\boldsymbol{\sigma}) + \sqrt{J_2(\boldsymbol{s}(\boldsymbol{\sigma}))} - k_s \quad \text{where} \quad \beta = \frac{1}{2\sqrt{3}}\frac{(1-2\upsilon^p)}{(1+\upsilon^p)} \ \& \ k_s = \left(\beta + \frac{1}{\sqrt{3}}\right)\sigma_y$$

- **Nonlinear elastic constitutive relationship**

$$\boldsymbol{\sigma} = \lambda J_1(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$$

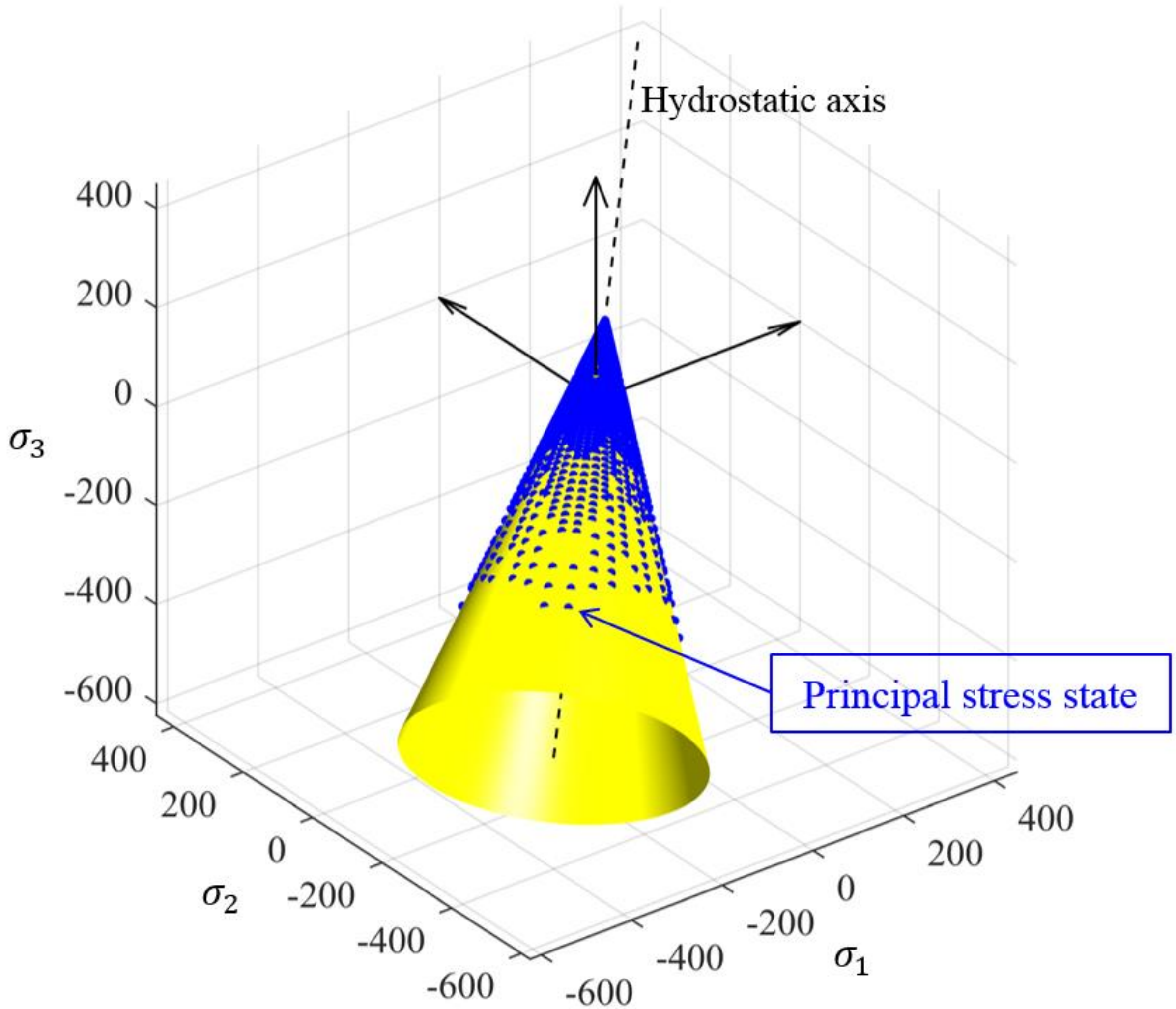
$$\lambda(\phi_1,\phi_2) = \frac{3\upsilon + (\phi_2 - \phi_1)E}{3(1+\upsilon+\phi_2 E)(1-2\upsilon+\phi_1 E)}E \qquad \mu(\phi_2) = \frac{E}{2(1+\upsilon+\phi_2 E)}$$

$$\phi_1(J_1(\boldsymbol{\varepsilon}),J_2(\boldsymbol{\varepsilon}_d)) = \frac{6\beta\left\{3\beta(1+\upsilon)EJ_1(\boldsymbol{\varepsilon}) + (1-2\upsilon)\left[3E\sqrt{J_2(\boldsymbol{\varepsilon}_d)} - \left(3\beta+\sqrt{3}\right)(1+\upsilon)\sigma_y\right]\right\}}{E\left[3E\left(J_1(\boldsymbol{\varepsilon}) - 6\beta\sqrt{J_2(\boldsymbol{\varepsilon}_d)}\right) + 6\beta\left(3\beta+\sqrt{3}\right)(1+\upsilon)\sigma_y\right]}$$

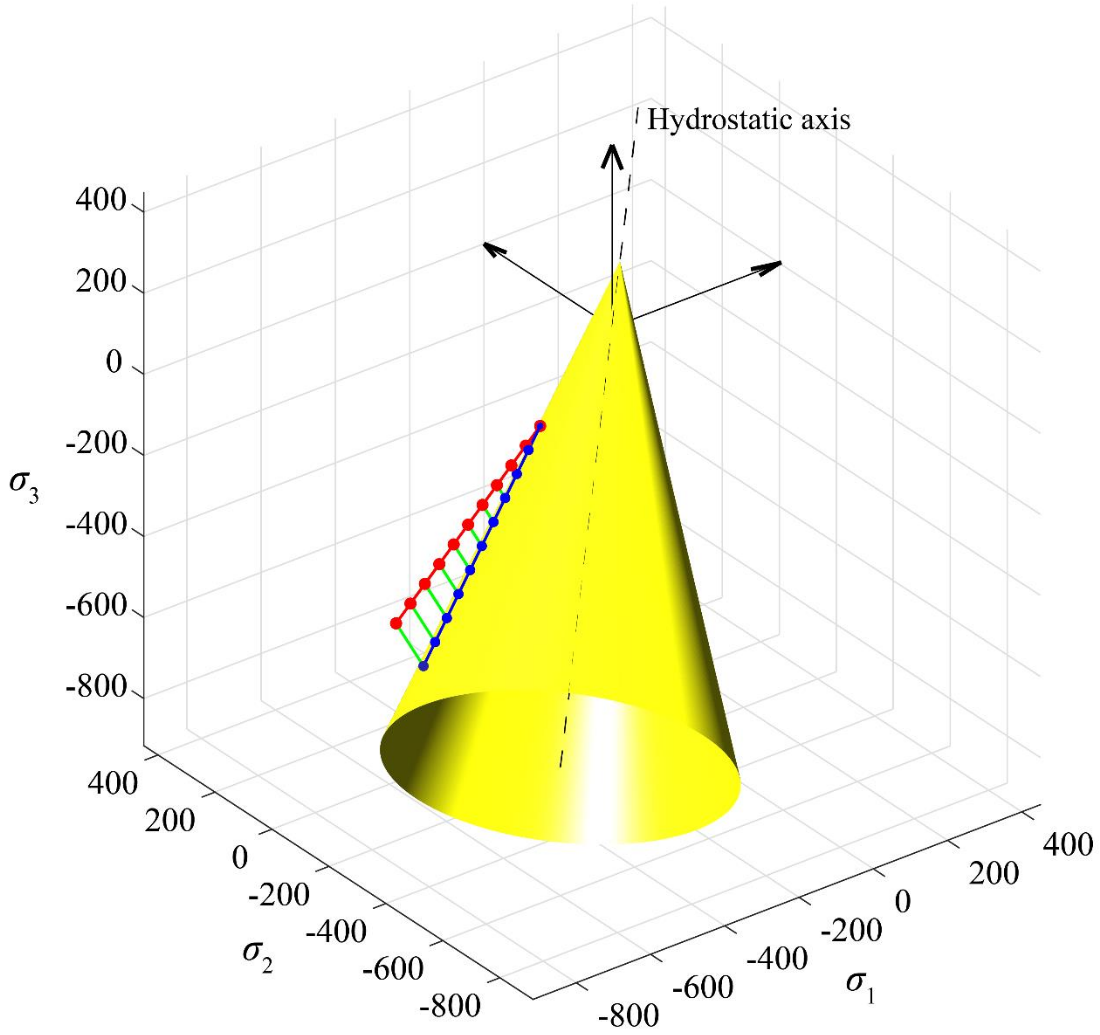
$$\phi_2(J_1(\boldsymbol{\varepsilon}),J_2(\boldsymbol{\varepsilon}_d)) = \frac{3\beta(1+\upsilon)EJ_1(\boldsymbol{\varepsilon}) + (1-2\upsilon)\left[3E\sqrt{J_2(\boldsymbol{\varepsilon}_d)} - \left(3\beta+\sqrt{3}\right)(1+\upsilon)\sigma_y\right]}{E\left\{18\beta^2 E\sqrt{J_2(\boldsymbol{\varepsilon}_d)} - 3\beta\left[EJ_1(\boldsymbol{\varepsilon}) - (1-2\upsilon)\sigma_y\right] + \sqrt{3}(1-2\upsilon)\sigma_y\right\}}$$

Sonato M, Piccolroaz A, Miszuris W, Mishuris G. (2015). General transmission conditions for thin elasto-plastic pressure-dependent interphase between dissimilar materials. *Int J Solids Struct*, 64–65:9–21.

Principal stress state of the nonlinear elastic model



Projection of the trial inadmissible linear elastic stress state on the Drucker – Prager surface



Analytical strain energy density function φ

Explicit expression

***Key*: The increment of the stress components on the Drucker – Prager yield surface is constant with respect to the increment of the strain components.**

$$\boldsymbol{\sigma} = \frac{\partial \varphi\big(J_1(\boldsymbol{\varepsilon}), J_2(\boldsymbol{\varepsilon}_d)\big)}{\partial \boldsymbol{\varepsilon}}$$

$$\varphi = \frac{1}{2}\boldsymbol{\sigma}^{\mathrm{I}} : \boldsymbol{\varepsilon}^{\mathrm{I}} + \left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{I}}\right)\int_0^1 \boldsymbol{\sigma}' : \boldsymbol{\varepsilon}' \, dt$$

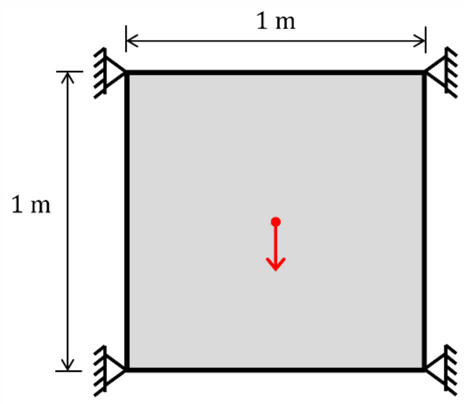
$$\boldsymbol{\varepsilon}' = (1-t)\boldsymbol{\varepsilon}^{\mathrm{I}} + t\boldsymbol{\varepsilon}$$

$$\boldsymbol{\sigma}' = (1-t)\boldsymbol{\sigma}^{\mathrm{I}} + t\boldsymbol{\sigma}$$

$$\varphi = \frac{1}{2}\left(\boldsymbol{\sigma}^{\mathrm{I}} : \boldsymbol{\varepsilon} + \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^{\mathrm{I}}\right)$$

$$U\big(\boldsymbol{\rho}, \boldsymbol{u}(\boldsymbol{\rho})\big) = \sum_{e=1}^{n} \int_{V_e} \varphi_e(\boldsymbol{u}) \, dV$$

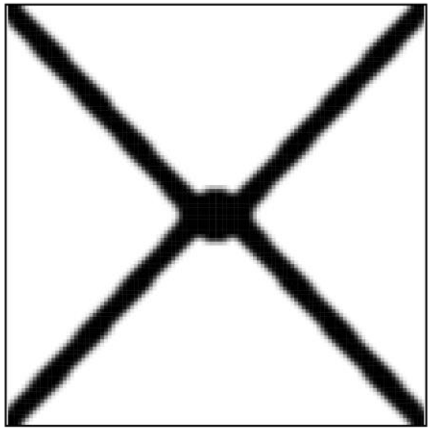
Corner-supported square



Young's modulus	$E = 1 \times 10^5$ MPa
Poisson's ratio	$\upsilon = 0.3$
Tensile strength	$\sigma_t = 10$ Mpa or 40 Mpa
Compressive strength	$\sigma_c = 40$ Mpa

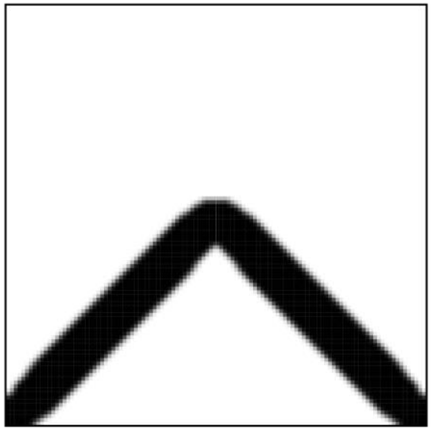
Symmetry: 6272 Q4 elements
Volume fraction: $\bar{v} = 15\%$; Density filter radius: $R = 0.018$
Constant penalty: $p = 3$

$C_0 = 2\times10^{-6}$ MJ
$\sigma_t = 10$ MPa $\sigma_c = 40$ MPa



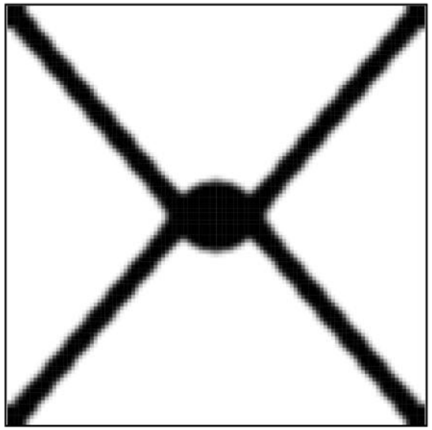
Linear solution

$C_0 = 2\times10^{-4}$ MJ
$\sigma_t = 10$ MPa $\sigma_c = 40$ MPa



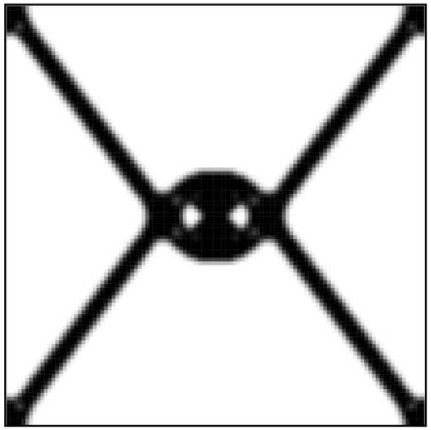
Nonlinear solution 1
(Drucker-Prager)

$C_0 = 2\times10^{-4}$ MJ
$\sigma_t = 40$ MPa $\sigma_c = 40$ MPa



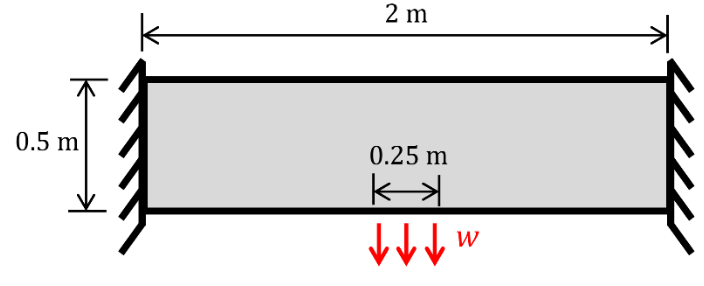
Nonlinear solution 2

$C_0 = 2\times10^{-4}$ MJ
$\sigma_t = 10$ MPa $\sigma_c = 10$ MPa



Nonlinear solution 3
(von Mises)

Clamped beam



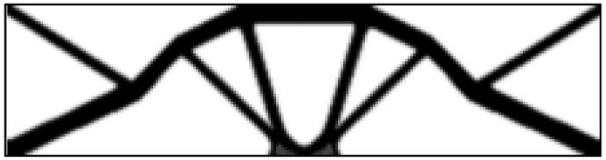
Young's modulus	$E = 1.8 \times 10^5$ MPa
Poisson's ratio	$\upsilon = 0.3$
Tensile strength	$\sigma_t = 144$ Mpa
Compressive strength	$\sigma_c = 1440$ Mpa

Symmetry: 6272 Q4 elements
Volume fraction: $\bar{v} = 25\%$; Density filter radius: $R = 0.014$
Constant penalty: $p = 3$

$C_0 = 8\times10^{-4}$ MJ

Linear solution

$C_0 = 1.1\times10^{-2}$ MJ

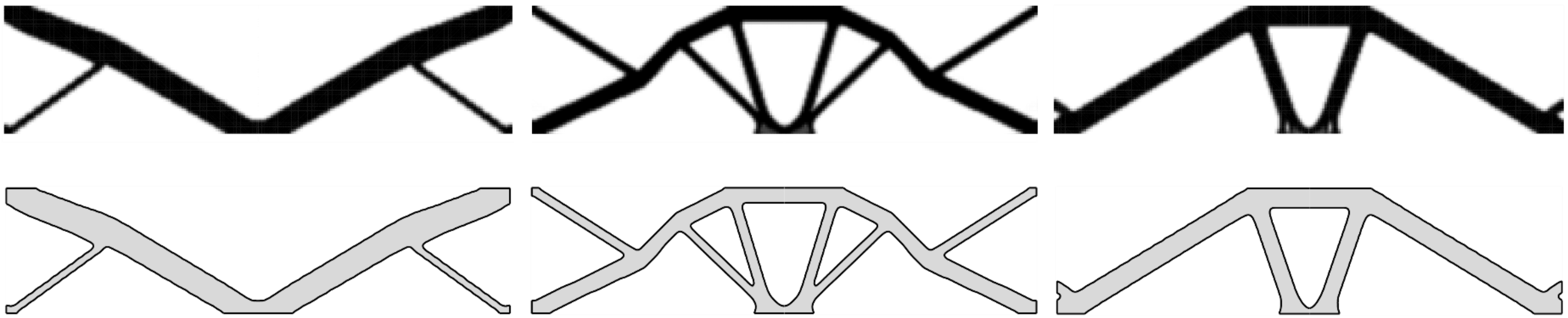


Nonlinear solution 1

$C_0 = 1.3\times10^{-2}$ MJ

Nonlinear solution 2

Convert Optimized topologies to CAD models in ABAQUS



Volume = 0.2462	Volume = 0.2466	Volume = 0.2454
6145 linear triangular elements	6106 linear triangular elements	6171 linear triangular elements

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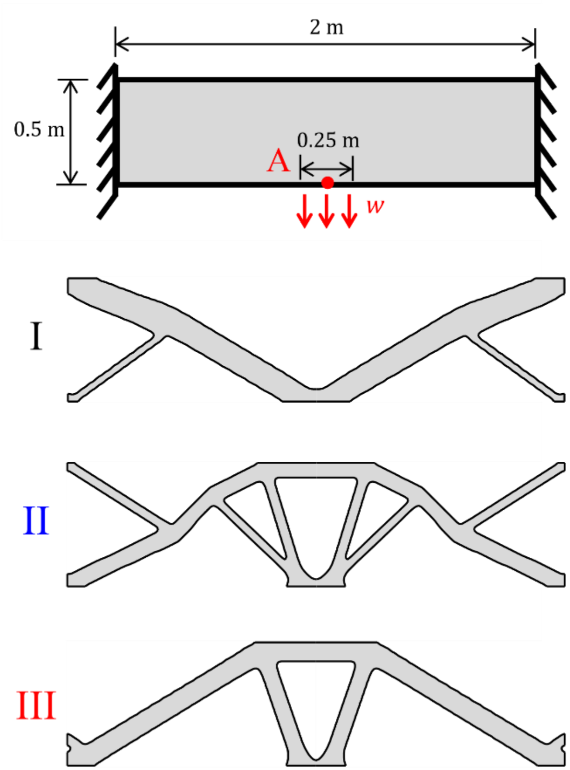
Material properties defined in ABAQUS:

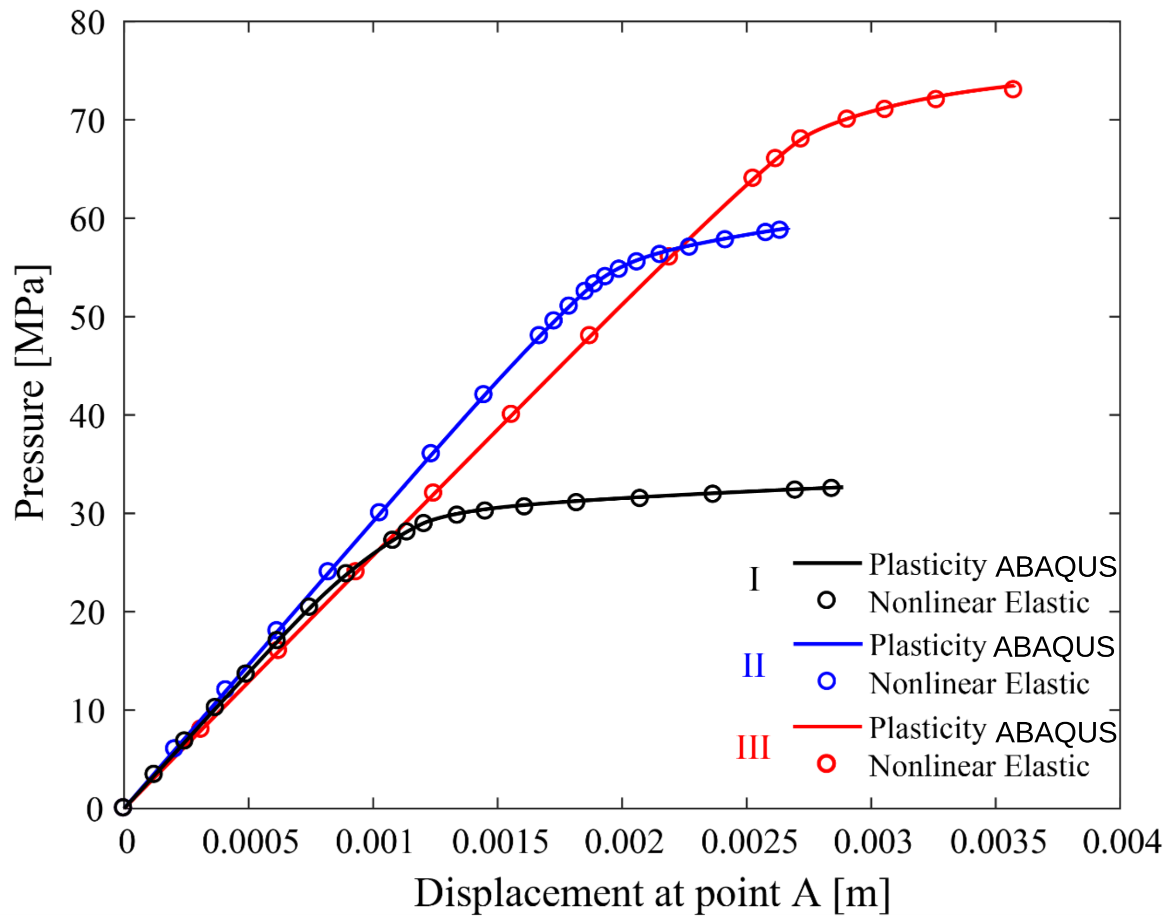
Young's modulu $\mathrm{E} = 1.8 \times 10^{5}$ MPa   Poisson's ratio $\upsilon = 0.3$

The Drucker – Prager plasticity model:
Friction angle $\psi = 67.83^{\circ}$ and cohesion stress $c = 261.82$ MPa   Standard ABAQUS

OUR surrogate nonlinear elastic model (UMAT):
$\boldsymbol{\upsilon^{p} = -0.175}$ **and** $\boldsymbol{\sigma_{y} = 144}$ **MPa**   ABAQUS UMAT

Structural performance





Concluding remarks

- **We address the material nonlinear topology optimization considering the Drucker – Prager criterion using a surrogate nonlinear elastic constitutive model.**

- **The sensitivity analysis are simple, effective and efficient for the present formulations (i.e., $\max U$ and $\max \chi$).**

- **Using the analytical strain energy density function improves the efficiency of the framework.**

- **We obtain new nonlinear solutions accounting for realistic plastic material properties, which can serve as benchmark problems.**

- **Assuming small deformation, the nonlinear elastic solution is equivalent to the plastic solution under proportional loading.**