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Error estimation using hypersingular integrals in boundary element methods for linear elasticity

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Abstract

A natural measure of the error in the boundary element method rests on the use of both the standard boundary integral equation (BIE) and the hypersingular BIE (HBIE). An approximate (numerical) solution can be obtained using either one of the BIEs. One expects that the residual, obtained when such an approximate solution is substituted to the other BIE is related to the error in the solution. The present work is developed for vector field problems of linear elasticity. In this context, suitable 'hypersingular residuals' are shown, under certain special circumstances, to be globally related to the error. Further, heuristic arguments are given for general mixed boundary value problems. The calculated residuals are used to compute element error indicators, and these error indicators are shown to compare well with actual errors in several numerical examples, for which exact errors are known. Conclusions are drawn and potential extensions of the present error estimation method are discussed. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Hypersingular integrals; Boundary element methods; Linear elasticity

1. Introduction

A natural measure of the error in a boundary integral method, first proposed by Paulino [1] and Paulino et al. [2], rests on the simultaneous use of both the standard boundary integral equation (BIE) and the hypersingular BIE (HBIE). This, so called 'two level' iteration scheme, is a novel idea that is the basis for this paper. Suppose that the approximate (numerical) solution, using the standard BIE, has been obtained. Then one might expect that the residual (called the hypersingular residual), obtained when this approximate solution is substituted into the HBIE, is related to the error. The procedure is symmetric in the sense that the above approach can be reversed and one can obtain the singular residual, which is also expected to be related to the error. Numerical experiments have corroborated this idea [1,2]. It has been proved in Menon et al. [3] (see, also, Menon [4]) that, under certain favorable conditions (e.g. no cracks or corners), the actual error is related to the hypersingular residual. Numerical results presented in [3] are most encouraging. The contribution of the present work is an extension of the work in potential theory [3] to linear elasticity. This time, proofs of relationships of the actual error to the hypersingular residual are presented for Dirichlet and Neumann (i.e. displacement prescribed and traction prescribed) boundary value problems (BVPs), while heuristic error estimates are presented for mixed BVPs in linear elasticity. Numerical results for two-dimensional (2D) linear elasticity problems, with mixed boundary conditions, and in the presence of corners, are, again, consistent with available theoretical solutions.

It is important to mention some recent related work in this context. The 'two level' iteration scheme, cited above, has been employed successfully for error estimation and adaptive analysis in the symmetric Galerkin boundary element method (SGBEM — Paulino and Gray [5]), in the meshless boundary node method (BNM — Chati et al. [6]), and in the boundary contour method (BCM — Mukherjee and Mukherjee [7]). Thus, the idea is applicable to the standard BEM as well as to closely related methods.

1.1. Hypersingular boundary element method

It is possible to formulate two distinct boundary integral equations — the standard BIE and the HBIE — to represent

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the same BVP. A unified derivation of both integral equations is available in Wendland [8]. HBIEs are derived from a properly differentiated version of the usual BIEs. HBIEs have diverse important applications and are the subject of considerable current research (see, for example, Krishnasamy et al. [9], Tanaka et al. [10], Paulino [1] and Chen and Hong [11] for recent surveys of the field). HBIEs, for example, have been employed for the evaluation of boundary stresses (e.g. Guiggiani [12], Wilde and Aliabadi [13], Zhao and Lan [14], Chati and Mukherjee [15]), in wave scattering (e.g. Krishnasamy et al. [16]), in fracture mechanics (e.g. Gray et al. [17], Lutz et al. [18], Paulino [1], Gray and Paulino [19], Chan et al. [20], Mukherjee [21]), to obtain symmetric Galerkin boundary element formulations (e.g. Bonnet [22], Gray et al. [23], Gray and Paulino [24,25]), to obtain the hypersingular boundary contour method (Phan et al. [26], Mukherjee and Mukherjee [27]), and for error analysis (Paulino et al. [2], Menon [4], Menon et al. [3], Chati et al. [6], Mukherjee and Mukherjee [7], Liang et al. [28], Denda and Dong [29]). Finally, Mukherjee [30] has recently presented some new results on finite parts of hypersingular integrals.

1.2. Error estimation

A particular strength of the finite element method (FEM) is the well developed theory of error estimation, and its use in adaptive methods (see, for example, Ciarlet [31], Eriksson et al. [32]). In contrast, error estimation in the boundary element method (BEM) is a subject that has attracted attention mainly over the past decade, and much work remains to be done. For recent surveys on error estimation and adaptivity in the BEM, see Sloan [33], Kita and Kamiya [34], Liapis [35] and Paulino et al. [36].

Many error estimators in the BEM are essentially heuristic and, unlike for the FEM, theoretical work in this field has been quite limited. Rank [37] proposed error indicators and an adaptive algorithm of the BEM using techniques similar to those used in the FEM. Most notable is the work of Yu and Wendland [38–41], who have presented local error estimates based on a linear error-residual relation that is very effective in the FEM. More recently, Carstensen et al. [42–45] have presented error estimates for the BEM analogous to the approach of Eriksson [32] for the FEM. There are numerous stumbling blocks in the development of a satisfactory theoretical analysis of a generic boundary value problem (BVP). First, theoretical analyses are easiest for Galerkin schemes, but most engineering codes, to date, use collocation-based methods (see, for example, Banerjee [46]). Though one can view collocation schemes as variants of Petrov–Galerkin methods, and, in fact, numerous theoretical analyses exist for collocation methods (see, for example, references in [33]), the mathematical analysis for this class of problems is difficult. Theoretical analyses for mixed boundary conditions are limited and involved (Wendland et al. [47]) and the presence of corners and cracks has been a source of challenging problems for many years (Sloan [33], Costabel and Stephan [48], Costabel et al. [49]). Of course, problems with corners and mixed boundary conditions are the ones of most practical interest, and for such situations one has to rely mostly on numerical experiments.

During the past few years, there has been a marked interest, among mathematicians in the field, in extending analyses for the BEM with singular integrals to hypersingular integrals [43,44,50,51]. For instance, Feistauer et al. [51] have studied the solution of the exterior Neumann problem for the Helmholtz equation formulated as an HBIE. Their paper contains a rigorous analysis of hypersingular integral equations and addresses the problem of noncompatibility of the residual norm, where additional hypotheses are needed to design a practical error estimate. These authors use residuals to estimate the error, but, unlike the present work, they do not use the BIE and the HBIE simultaneously. Finally, Goldberg and Bowman [52] have used superconvergence of the Sloan interate [53,54] to show the asymptotic equivalence of the error and the residual. They have used Galerkin methods, an iteration scheme that uses the same integral equation for the approximation and for the iterates, and usual residuals in their work.

Thus, the 'two level' iteration scheme is a novel idea that merits further consideration. The error estimator presented in this paper relies on this idea and is elaborated upon in the rest of the present work.

1.3. Outline of this paper

This paper is organized as follows. A brief literature review, of the hypersingular boundary element method (HBEM) and error estimation in the FEM and the BEM, appears above in this Section. Section 2 presents a quick summary of the BIE and HBIE for linear elasticity. Section 3 is concerned with 'pointwise' (which simply means that the quantity of interest is evaluated at selected points) residual-based error estimates for Dirichlet, Neumann and mixed BVPs in linear elasticity. Interesting relationships between the actual error and the hypersingular residuals are proved for the first two classes of problems, while heuristic error estimators are presented for mixed BVPs. Element-based error indicators, relying on the pointwise error measures presented in Section 3, are proposed in Section 4. Numerical results for two mixed BVPs in 2D linear elasticity appear in Section 5. Concluding remarks are given in Section 6. Appendix A completes the paper.

2. Boundary integral equations

The basic notation and boundary integral formulation for planar, isotropic, homogeneous linear elasticity are presented in this section. Linear operators are defined, and the BIEs are rewritten in compact notation. Some properties of these linear operators are also discussed. A thorough account of BIE methods can be found in the recent book by Constanda [55].

2.1. Governing equations

The starting point is the well known BIE due to Rizzo [56], for linear elasticity, at an internal point p in a body B

$$u_i(P) = \int_{\partial B} [U_{ij}(p,Q)t_j(Q) - T_{ij}(p,Q)u_j(Q)]\,\mathrm{d}s_Q \tag{1}$$

in which ∂B is the bounding surface of a body B with infinitesimal surface area $\mathbf{d}s = \mathrm{d}s\mathbf{n}$, where n is the unit outward normal to ∂B at a point Q on it. The traction vector is $\mathbf{t}$ and the displacement vector is $\mathbf{u}$. A source point is denoted as p (or P) and a field point as q (or Q). Upper case letters denote points on ∂B while lower case letters denote points inside B. The BEM Kelvin kernels $\mathbf{U}$ and $\mathbf{T}$ are available in many references, and are also given in Appendix A of this paper for completeness.

For planar elasticity, the Kelvin kernels $\mathbf{U}$ and $\mathbf{T}$ are O (ln r) and $O(1/r)$ singular, respectively, as a source point approaches a field point. Here r is the Euclidean distance between these points. Taking the limit of Eq. (1) as $p \to P$, a Cauchy singular integral equation results

$$u_i(P) = \lim_{p\to P} \int_{\partial B} [U_{ij}(p,Q)t_j(Q) - T_{ij}(p,Q)u_j(Q)]\,\mathrm{d}s_Q. \tag{2}$$

It can be shown that the resulting integral equation is well defined under suitable assumptions on the smoothness of the boundary and of the displacement function.

The non-singular Eq. (1) can be differentiated under the integral sign to obtain the displacement gradients $u_{i,j}(p)$. Now, using Hooke's law

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) \tag{3}$$

(where $\boldsymbol{\sigma}$ is the stress, λ and μ are the Lamé constants for the material and $\boldsymbol{\delta}$ is the Kronecker delta), one can obtain the stress tensor at an internal point. This equation is

$$\sigma_{ij}(p) = \int_{\partial B} [D_{ijk}(p,Q)t_k(Q) - S_{ijk}(p,Q)u_k(Q)]\,\mathrm{d}s_Q. \tag{4}$$

The new kernels $\mathbf{D}$ and $\mathbf{S}$ are $O(1/r)$ and $O(1/r^2)$ singular, respectively, as $p \to P$. They are also given in Appendix A.

Again, taking the appropriate limit of Eq. (4) as $p \to P$ results in

$$\sigma_{ij}(P) = \lim_{p\to P} \int_{\partial B} [D_{ijk}(p,Q)t_k(Q) - S_{ijk}(p,Q)u_k(Q)]\,\mathrm{d}s_Q. \tag{5}$$

Finally, one can take the dot product of both sides of Eq. (5) with the unit normal $\mathbf{N}(P)$ to ∂B at a source point P. The result is:

$$t_i(P) = \lim_{p\to P}\left[\left[\int_{\partial B} (D_{ijk}(p,Q)t_k(Q) - S_{ijk}(p,Q)u_k(Q))\,\mathrm{d}s_Q\right]N_j(P).\right. \tag{6}$$

In view of the nature of the singularity of the kernel $\mathbf{S}$ above, Eq. (6) is called hypersingular (hypersingular BIE or HBIE). Again, for Eq. (6) to be well defined, it is necessary that the displacement, stress and the boundary ∂B satisfy certain smoothness requirements (see, for example, Martin et al. [57] and Mukherjee and Mukherjee [7]). In this work, Eqs. (2) and (6) are defined in the *limit to the boundary (LTB) sense* (see, for example, Gray and Manne [58], Paulino [1] and Mukherjee [30]).

The boundary variables in an elasticity problem are the displacement $\mathbf{u}$ and the traction $\mathbf{t}$. *It is important to note here that either the BIE (2) or the HBIE (6) can be used to solve a well-posed boundary value problem in linear elasticity.*

2.2. Linear operators

Boundary integral equations can be analyzed by viewing them as linear equations in a Hilbert space. A very readable account of this topic is available in Kress [59]. Following Sloan [33], it is assumed here that the boundary ∂B is a C^1 continuous closed Jordan curve given by the mapping

$$z : [0,1] \to \partial B, z \in C^1, |z'| \neq 0$$

where $z \in \mathbf{C}$, the space of complex numbers. The present analysis excludes domains with corners. It is also assumed that any integrable function v on ∂B may be represented in a Fourier series

$$v \sim \sum_{k=-\infty}^{\infty} \hat{v}(k)\,\mathrm{e}^{2\pi\,ikx_1} = a_0 + \sum_{k=1}^{\infty} [a_k \cos(2\pi k x_1) + b_k \sin(2\pi k x_1)], \tag{7}$$

where $i \equiv \sqrt{-1}$ and

$$\hat{v}(k) = \int_0^1 \mathrm{e}^{-2\pi ikx_1} v(x_1)\,\mathrm{d}x_1, \qquad k \in \mathbf{Z} \tag{8}$$

in which $\mathbf{Z}$ denotes the space of integers.

The following Lemma is very useful for the work presented in this paper.

Lemma 1. *If $\mathscr{A}: B_1 \to B_2$ is a continuous linear operator that has a continuous inverse, and $\mathscr{A}x = y$, then there exist real positive constants C_1 and C_2, such that*

$$C_1\|y\|_{B_2} \leq \|x\|_{B_1} \leq C_2\|y\|_{B_2},$$

where $\|\cdot\|_{B_i}$ denotes a suitable norm of the appropriate function (in the Banach space B_i).

Proof. The linearity and continuity of $\mathscr{A}$ and $\mathscr{A}^{-1}$ imply that $\|\mathscr{A}\|$ and $\|\mathscr{A}^{-1}\|$ are finite. From the Cauchy–Schwarz inequality, one has

$$\|y\| = \|\mathscr{A}x\| \le \|\mathscr{A}\|\,\|x\|,$$

$$\|x\| = \|\mathscr{A}^{-1}y\| \le \|\mathscr{A}^{-1}\|\,\|y\|.$$

The result now follows by choosing $C_1 = 1/\|\mathscr{A}\|$ and $C_2 = \|\mathscr{A}^{-1}\|$. □

Returning to the problem at hand, the following operators are defined

$$(\mathscr{U}_{ij}v_j)(p) := \int_{\partial B} U_{ij}(p,Q)\, v_j(Q)\, \mathrm{d}s_Q, \tag{9}$$

$$(\mathscr{T}_{ij}v_j)(p) := \int_{\partial B} T_{ij}(p,Q)v_j(Q)\, \mathrm{d}s_Q, \tag{10}$$

$$(\mathscr{D}^N_{ik}v_k)(p) := \left[\int_{\partial B} D_{ijk}(p,Q)v_k(Q)\, \mathrm{d}s_Q\right]N_j(P), \tag{11}$$

$$(\mathscr{S}^N_{ik}v_k)(p) := \left[\int_{\partial B} S_{ijk}(p,Q)v_k(Q)\, \mathrm{d}s_Q\right]N_j(P). \tag{12}$$

The operator $\mathscr{U}_{ij}$ is continuous onto the boundary, whereas $\mathscr{T}_{ij}$ and $\mathscr{D}^N_{ij}$ are not continuous (Tanaka et al. [10]) and give rise to additional bounded free terms in the limit. The hypersingular operator $\mathscr{S}^N_{ij}$ gives rise to unbounded terms that vanish when the integral is considered, for example, in the LTB sense. These terms depend on the smoothness of the boundary at the source point P. In this work, the HBIE is collocated only at regular boundary points (where the boundary is locally smooth) inside boundary elements.

Using the operators defined above, the BIE (2) and HBIE (6) become, respectively

$$\text{BIE}: \quad u_i = \mathscr{U}_{ij}t_j - \mathscr{T}_{ij}u_j, \tag{13}$$

$$\text{HBIE}: \quad t_i = \mathscr{D}^N_{ij}t_j - \mathscr{S}^N_{ij}u_j. \tag{14}$$

As in the case of potential theory (Menon et al. [3]), the LTB of the above integral operators has been used to obtain the singular integral Eqs. (13) and (14).

Remark 1. One should note, however, that key properties of the operators, such as continuity and invertibility, assume a certain regularity of the boundary (for instance no corners or cusps) [33]. These assumptions, of course, are too restrictive for the solution of practical engineering problems. Such assumptions have, nevertheless, been made here in order to obtain some mathematical understanding of the error estimation process that is described in Section 3.The numerical example problems do contain corners. The HBIE (14), however, has only been collocated at regular points on the boundary of a body.

3. Iterated HBIE and error estimation

The heuristic idea that is at the heart of the pointwise error estimation procedure described below (see also Refs [1–3]) is simple: *the amount by which an approximate solution to the BIE fails to satisfy the HBIE is a measure of the error in the approximation.* The main result of this work is that this heuristic idea, when stated formally, leads to a simple characterization of the error. In essence, the method reduces to finding a second approximation to the solution by iterating the first approximation with the HBIE. This idea is first illustrated in the context of two basic cases: the interior Dirichlet and Neumann problems. Mixed boundary conditions are considered thereafter.

3.1. Problem 1: displacement boundary conditions.

Solve the Navier–Cauchy equations

$$(\lambda + \mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} = \mathbf{0} \text{ in } B \tag{15}$$

subjected to the boundary conditions

$$\mathbf{u}=\mathbf{f} \text{ on } \partial B \tag{16}$$

This problem is analogous to the Dirichlet problem of potential theory. Under suitable restrictions on the domain, existence and uniqueness of a solution to the BVP may be shown. This is a well-known result in linear elasticity (e.g. Fung [60]). Reformulation of the problem using the integral Eqs. (13) and (14) is recalled here:

$$u_i = \mathscr{U}_{ij}t_j - \mathscr{T}_{ij}u_j,$$

$$t_i = \mathscr{D}^N_{ij}t_j - \mathscr{S}^N_{ij}u_j.$$

The first equation is the displacement BIE, and the second, the traction HBIE. Either of these may be used to formulate a method of solution for the unknown traction on the boundary. The displacement BIE leads to a system of singular integral equations of the first kind for the (unknown) traction

$$\mathscr{U}_{ij}t_j = f_i + \mathscr{T}_{ij} \qquad f_j =: g^1_i \qquad i = 1,2, \tag{17}$$

while the traction BIE gives rise to equations of the second kind (for the traction)

$$t_i - \mathscr{D}^N_{ij}t_j = -\mathscr{S}^N_{ij} \qquad f_j =: g^2_i \qquad i = 1,2. \tag{18}$$

Recall that $\mathscr{U}_{ij}$ is log-singular, $\mathscr{T}_{ij}$ and $\mathscr{D}^N_{ij}$ are Cauchy singular, and $\mathscr{S}^N_{ij}$ is hypersingular (see Appendix A.) As in potential theory, since $\mathscr{U}_{ij}$ has a logarithmic kernel, one encounters the problem of the transfinite diameter (see, e.g. Sloan [33] and Menon et al. [3]). For instance, one may show that if the domain B is a circle of radius $\exp[(1/2)(3-4\nu)]$, then the BIE (17) does not admit a unique solution.

3.1.1. Error estimate for the primary problem

Using the two BIEs (17) and (18), one can formulate an error estimation process that is analogous to the Dirichlet problem in potential theory [3].

Step 1: Solve the displacement BIE (17) for the traction $t_i^{(1)}$

$$\mathscr{U}_{ij} t_j^{(1)} = (\mathscr{I}_{ij} + \mathscr{T}_{ij}) f_j, \tag{19}$$

where $\mathscr{I}$ is the identity operator and $\mathscr{I}_{ij}\ f_j = \delta_{ij}\ f_j = f_i$, with δ_{ij} the components of the Kronecker delta.

Step 2: Use the traction HBIE (18) to iterate the traction and obtain a second approximation $t_i^{(2)}$.

$$t_i^{(2)} = \mathscr{D}_{ij}^N t_j^{(1)} - \mathscr{S}_{ij}^N f_j. \tag{20}$$

This approximation, called the HBIE iterate, will be used for error estimation.

Let the error (in traction) in the primary solution and iterate be

$$\mathrm{e}_i^{t(1)} = t_i^{(1)} - t_i, \tag{21}$$

$$\mathrm{e}_i^{t(2)} = t_i^{(2)} - t_i, \tag{22}$$

respectively. Define the hypersingular residual to be

$$r_i^t = t_i^{(1)} - t_i^{(2)}. \tag{23}$$

One can now show that

$$r_i^t \overset{(23)}{=} t_i^{(1)} - t_i^{(2)}$$

$$\overset{(20)}{=} t_i^{(1)} - (\mathscr{D}_{ij}^N t_j^{(1)} - \mathscr{S}_{ij}^N f_j)$$

$$\overset{(21)}{=} t_i - (\mathscr{D}_{ij}^N t_j - \mathscr{S}_{ij}^N f_j) + \mathrm{e}_i^{t(1)} - \mathscr{D}_{ij}^N \mathrm{e}_j^{t(1)}$$

$$\overset{(14)}{=} (\mathscr{I}_{ij} - \mathscr{D}_{ij}^N)\mathrm{e}_j^{t(1)}$$

so that

$$r_i^t = (\mathscr{I}_{ij} - \mathscr{D}_{ij}^N)\ \mathrm{e}_j^{t(1)} \tag{24}$$

Theorem 1. *For a sufficiently smooth domain, and sufficiently smooth data and solutions (as detailed above), if the solution to the integral equations (17) and (18) is unique, then there exist real positive constants C_1 and C_2 such that*

$$C_1\|r_i^t\| \le \|\mathrm{e}_i^{t(1)}\| \le C_2\|r_i^t\|$$

Proof. The continuity of the operators is a manifestation of the elliptic nature of the partial differential equation (PDE). Uniqueness of solutions to the integral formulations implies that the operators $(\mathscr{I}_{ij} - \mathscr{D}_{ij}^N)$ and $\mathscr{U}_{ij}$ have continuous inverses [39]. Now use Lemma 1. □

3.1.2. Error estimate for the iterate

In a manner similar to the previous subsection, one can show that

$$e_i^{t(2)} \overset{(22)}{=} t_i^{(2)} - t_i$$

$$\overset{(20)}{=} \mathscr{D}_{ij}^N t_j^{(1)} - \mathscr{S}_{ij}^N f_j - t_i$$

$$\overset{(21)}{=} \mathscr{D}_{ij}^N (\mathrm{e}_j^{t(1)} + t_j) - \mathscr{S}_{ij}^N f_j - t_i$$

$$\overset{(14)}{=} \mathscr{D}_{ij}^N \mathrm{e}_j^{t(1)}$$

so that

$$\mathrm{e}_i^{t(2)} = \mathscr{D}_{ij}^N \mathrm{e}_j^{t(1)}. \tag{25}$$

3.2. Problem 2: traction boundary conditions

Solve the Navier–Cauchy equations

$$(\lambda + \mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} = \mathbf{0} \text{ in } B$$

subject to the boundary conditions

$$\mathbf{t} = \mathbf{g} \text{ on } \partial B, \tag{26}$$

where the tractions satisfy the consistency conditions of static equilibrium

$$\int_{\partial B} \mathbf{t}\ \mathrm{d}s = \mathbf{0}, \tag{27}$$

$$\int_{\partial B} (\mathbf{r} \times \mathbf{t})\ \mathrm{d}s = 0. \tag{28}$$

It is known that the solution to the above problem exists, and is unique up to a rigid body motion (Fung [6]). The space of two-dimensional rigid body motions may be characterized as (Chen and Zhou [61])

$$\mathscr{R} := \mathbf{r}_0 + \boldsymbol{\omega} \times \mathbf{r}, \tag{29}$$

where $\mathbf{r}_0 \in \mathbb{R}^2$ is a translation, and $\boldsymbol{\omega} = \omega\ \mathbf{k}$ is an axial vector representing a rotation.

The first integral equation formulation for the problem follows from the displacement BIE (13). One has an integral equation of the second kind for the (unknown) displacement

$$u_i + \mathscr{T}_{ij} u_j = \mathscr{U}_{ij} g_j =: h_i^1 \tag{30}$$

and using the traction HBIE (14)

$$-\mathscr{S}_{ij}^N u_j = g_i - \mathscr{D}_{ij}^N g_j =: h_i^2 \tag{31}$$

one obtains an integral equation of the first kind for the displacement.

It is important to mention again that the solution of the traction prescribed BVP is arbitrary within a rigid body motion, and, to eliminate this arbitrariness, one must work in a restricted function space as has been done before [3] for Neumann problems in potential theory. An elegant practical way to solve traction prescribed problems in linear elasticity is outlined in a recent paper by Lutz et al. [62] where the singular matrix from the BIE is suitably regularized at the *discretized level* by eliminating rigid body modes.

3.2.1. Error estimate for the primary problem

The error estimation technique is analogous to the Neumann problem investigated previously by Menon et al. [3]. First, construct an approximation to the displacement field, $u_i^{(1)}$. Next find $t_i^{(2)}$, and iterated approximation to the traction, and use it to estimate the error in the primary solution.

Step 1: Solve the displacement BIE (30) for the displacement $u_i^{(1)}$

$$(\mathscr{I}_{ij} + \mathscr{T}_{ij})u_j^{(1)} = \mathscr{U}_{ij}g_j. \tag{32}$$

Step 2: Use the traction HBIE (14) to obtain $t_i^{(2)}$

$$t_i^{(2)} = \mathscr{D}_{ij}^N g_j - \mathscr{S}_{ij}^N u_j^{(1)}. \tag{33}$$

Define the hypersingular residual

$$r_i^t = t_i^{(1)} - t_i^{(2)} = g_i - t_i^{(2)}. \tag{34}$$

Also, the error in the displacement is redefined here as

$$\mathrm{e}_i^{u(1)} = u_i^{(1)} - u_i. \tag{35}$$

One can now show that

$$r_i^t \overset{(34)}{=} g_i - t_i^{(2)}$$

$$\overset{(33)}{=} g_i - (\mathscr{D}_{ij}^N g_j - \mathscr{S}_{ij}^N u_j^{(1)})$$

$$\overset{(35)}{=} g_i - \mathscr{D}_{ij}^N g_j + \mathscr{S}_{ij}^N (u_j + \mathrm{e}_j^{u(1)})$$

$$\overset{(14)}{=} \mathscr{S}_{ij}^N \mathrm{e}_j^{u(1)}$$

so that

$$r_i^t = \mathscr{S}_{ij}^N \mathrm{e}_j^{u(1)}. \tag{36}$$

Theorem 2. *The hypersingular traction residual bounds the error in the displacement globally*

$$C_1\|r_i^t\| \le \|\mathrm{e}_i^{u(1)}\| \le C_2\|r_i^t\|.$$

Proof. The proof is quite analogous to that of Theorem 1. It follows from using Eq. (36). □

3.2.2. The displacement residual

In the traction boundary condition problem, the unknown is the displacement but Eq. (36) relates the traction residual to the error in the displacement. It is proved below, however, that the traction residual is also equal to a suitably defined displacement residual for this problem.

The HBIE (14), with u_i added to both sides of it, and upon rearrangement, becomes

$$u_i = (\mathscr{I}_{ij} - \mathscr{S}_{ij}^N)u_j - (\mathscr{I}_{ij} - \mathscr{D}_{ij}^N)g_j. \tag{37}$$

Let $u_i^{(1)}$ be the solution of the BIE (13). Iterate (37) with this solution and define

$$u_i^{(2)} = (\mathscr{I}_{ij} - \mathscr{S}_{ij}^N)u_j^{(1)} - (\mathscr{I}_{ij} - \mathscr{D}_{ij}^N)g_j. \tag{38}$$

Define the displacement residual

$$r_t^u \equiv u_i^{(1)} - u_i^{(2)}. \tag{39}$$

One can now show that

$$r_i^u \equiv u_i^{(1)} - u_i^{(2)}$$

$$\overset{(38)}{=} u_i^{(1)} - (\mathscr{I}_{ij} - \mathscr{S}_{ij}^N)u_j^{(1)} + (\mathscr{I}_{ij} - \mathscr{D}_{ij}^N)g_j$$

$$= \mathscr{S}_{ij}^N u_j^{(1)} + g_i - \mathscr{D}_{ij}^N g_j$$

$$\overset{(14)}{=} \mathscr{S}_{ij}^N u_j^{(1)} - \mathscr{S}_{ij}^N u_j$$

$$\overset{(35)}{=} \mathscr{S}_{ij}^N \mathrm{e}_j^{u(1)}$$

$$\overset{(36)}{=} r_i^t \tag{40}$$

so that

$$r_i^u = r_i^t \tag{41}$$

and r_i^t can be replaced by r_i^u in Theorem 2!

Remark 2. An analogous result in potential theory appears in Menon et al. [3].

3.3. Problem 3: mixed boundary conditions

The general boundary value problem in linear elasticity is Solve the Navier–Cauchy equations

$$(\lambda + \mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} = \mathbf{0} \text{ in } B$$

subjected to the boundary conditions

$$\mathbf{Au} + \mathbf{Bt} = \mathbf{f} \text{ on } \partial B, \tag{42}$$

where the matrices $\mathbf{A}$ and $\mathbf{B}$ and the vector $\mathbf{f}$ are prescribed quantities.

This class of problems is the one most commonly encountered in linear elasticity. Indeed, all the numerical examples presented in Section 5 of this paper have mixed boundary conditions imposed upon them. In this case, however, a heuristic approach to error estimation is adopted here.

3.3.1. Traction residual

One computes the traction components $t_j^{(1)}$ on ∂B by solving the primary BIE (13) and then obtains the HBIE iterate $t_j^{(2)}$ from the HBIE (14). As before, the traction residual is defined as

$$r_i^t = t_i^{(1)} - t_i^{(2)}. \tag{43}$$

The corresponding pointwise error measure is as follows. At a fixed boundary point, if the traction is specified in one direction, and the displacement in the other, then the error in the boundary data is the error in displacement in the first direction, and the error in traction in the second direction. This issue is discussed further in Section 4 of this paper.

3.3.2. Stress residual

The stress residual is another important quantity in this work. The primary BIE (13) is solved first. This yields the boundary tractions and displacements $t_j^{(1)}$ and $u_j^{(1)}$. The boundary stresses $\sigma_{ij}^{(1)}$ are next obtained from the boundary values of the tractions and the tangential derivatives of the displacements, together with Hooke's law. This is a well-known procedure in the BIE literature (see, for example, Mukherjee [63] or Sladek and Sladek [64]).

Next, the iterated boundary stress is obtained from the HBIE (5) as follows.

$$\sigma_{ij}^{(2)} = \mathscr{D}_{ijk} t_k^{(1)} - \mathscr{S}_{ijk} u_k^{(1)} \tag{44}$$

where the required operators are defined in a manner analogous to the ones in Eq. (11) and (12), i.e.

$$(\mathscr{D}_{ijk} v_k)(p) := \int_{\partial B} D_{ijk}(p, Q) v_k(Q)\, \mathrm{d}s_Q, \tag{45}$$

$$(\mathscr{S}_{ijk} v_k)(p) := \int_{\partial B} S_{ijk}(p, Q) v_k(Q)\, \mathrm{d}s_Q \tag{46}$$

and the LTB of the above operators is used in Eq. (44). Also, Eq. (44) is collocated only at regular boundary points (where the boundary is locally smooth) inside boundary elements.

One now gets the error in stress, for the BIE and the HBIE iterate, respectively, as

$$\mathrm{e}_{ij}^{s(1)} = \sigma_{ij}^{(1)} - \sigma_{ij}, \tag{47}$$

$$\mathrm{e}_{ij}^{s(2)} = \sigma_{ij}^{(2)} - \sigma_{ij} \tag{48}$$

and the stress residual is defined as

$$r_{ij}^s = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}. \tag{49}$$

Remark 3. The stress residual, defined above in Eq. (49), can also be used for problems with displacement or traction boundary conditions, which are special cases of problems with mixed boundary conditions.

4. Element error indicators

The main objective of error estimation is the development of suitable element error indicators, which are denoted by η_i. These indicators should satisfy the following criteria

$$C_1 \eta_i \le \|\mathrm{e}\|_{A(\partial B_i)} \le C_2 \eta_i, \tag{50}$$

$$D_1 \sum_{i=1}^{N} \eta_i^2 \le \|\mathrm{e}\|_A^2 \le D_2 \sum_{i=1}^{N} \eta_i^2, \tag{51}$$

where A is a suitable norm, $A(\partial B_i)$ denotes the restriction of this norm to the ith element, and C_1, C_2, D_1 and D_2 are appropriate constants. It is often difficult to prove these properties analytically, and one usually takes recourse to numerical experiments. As in the potential theory case [3], this method leads to two natural error indicators. The first is based on the traction residual defined in Eq. (23), and the second is based on the stress residual defined in Eq. (49).

These 'pointwise' error measures may be used to define element error indicators. The following are proposed: the first based on the traction residual and the second on the stress residual

$$\eta_j^t := \|r_i^t\|_{L^2(\partial B_j)}, \tag{52}$$

$$\eta_j^s := \|r_{kl}^s\|_{L^2(\partial B_j)}. \tag{53}$$

Note that the subscript j refers to the jth element — the error indicator is a scalar, not a vector. The L^2 norm is used for convenience.

A few words about norms are in order. A detailed investigation of norms is beyond the scope of the present work and the reader is referred to the applied mathematics literature on the subject (see, for example, Sloan [33]). The choice of norm in this work is of an exploratory nature and is subject to the limitations associated with the L^2 norm. Other norms may also be used in conjunction with the present residual-based error estimators. This is planned for the future.

The error estimates, as defined above, do not depend directly on the boundary conditions on an element. The traction residual has been shown to be related to the pointwise error in the boundary unknowns for Dirichlet and Neumann problems in elasticity (Section 3, Theorems 1 and 2). Note that, even though the traction residual uses the difference in the primary and iterated tractions, but no explicit information about the displacement, it has been proved in Section 3.2.2 that the traction residual is equal to the displacement residual for a traction prescribed boundary value problem.

In general, at a local level, on a particular element, the error will depend on the boundary conditions. In mixed boundary value problems, for instance, the traction may be prescribed in the x_1-direction and the displacement in the x_2-direction at a boundary point. The errors are, therefore, in the displacement in the x_1-direction, and in the traction in the x_2-direction. Ideally, the traction residual based element error indicator will capture the L^2 norm of these errors on an element, even for mixed boundary value problems.

The stress residual is also used as a measure of the error in stress on the boundary. At any boundary point in a 2D problem, at most two components of the stress are known from the prescribed boundary conditions. Thus, there is always some error in a computed stress tensor at a boundary point. Numerical experiments presented below (Section 5) suggest that this error is effectively tracked by the stress residual bases error indicator.

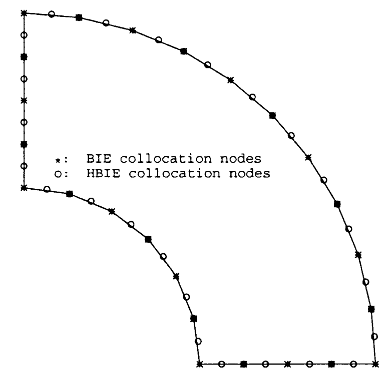

Fig. 1. Discretized domain for the Lamé problem with 12 elements.

5. Numerical examples

Two basic problems from the theory of planar elasticity are considered in this section. The numerical implementation consists of two modules: a standard code for two-dimensional elastostatics, and a set of routines that calculate the hypersingular residual for error-estimation. For the first part (i.e. the BIE), a code due to Becker [65] is employed. This code uses collocation with quadratic isoparametric elements. Numerical integration is done using Gaussian quadrature, except on elements that contain the collocation point. Singular integration is avoided using the rigid-body mode, i.e., diagonal terms are evaluated by summing the offdiagonal terms. For the second part of the code (i.e the HBIE), collocation is carried out at points on the boundary where it is locally smooth, and which are inside boundary elements, in order to determine the components of stress using the traction BIE. The numerical method used for evaluation of the necessary hypersingular integrals here is due to Guiggiani [12]. In the following examples, the stress tensor at a boundary point is evaluated by using the HBIE (5) at three boundary points inside a boundary element, and then a quadratic polynomial is employed to approximate the stress components over each element.

5.1. Example 1: Lamé's problem of a thick-walled cylinder under internal pressure

Consider an infinitely long hollow cylinder subjected to an internal pressure $p = 1$. The inner radius $r_i = 3$, and the outer radius $r_o = 6$. Material properties are also chosen of $O(1)$, namely Young's modulus, $E = 1.0$, and Poisson's ratio $\nu = 0.3$. Consistent units are assumed throughout this paper.

Symmetry is employed and the problem is formulated as a mixed boundary value problem on a quarter of the cylinder. The mesh used to solve this problem is shown in Fig. 1. Notice that the mesh is not biased a priori in the sense that the element density is not increased on parts of the boundary where the error is expected to be high.

Fig. 2 presents a comparison of the computed and analytical solutions for the stress components σ_{11} at the collocation points used for the HBIE. Note that the continuous line for the exact solution is just used as a matter of convenience and does not have any meaning except at discrete points, since the x-axis is the collocation point number. The results for σ_{12} and σ_{22} display similar accuracy and are not shown here.

More importantly, a pointwise comparison between the absolute value of the error and pointwise measurements of the hypersingular residual is considered next. Unlike the numerical examples in potential theory [3], the hypersingular residual and error are often of opposite signs in this elasticity example. Also, it is seen that the residual is not an upper bound as it underestimates the error at some points. Since the stress residual is a symmetric tensor with three independent components, a comparison between the pointwise error and stress residual in each direction is carried out here. The error in stress, for the BIE and the HBIE iterate, and the stress residual, are defined in Eqs. (47)–(49).

Now consider a comparison of absolute values of errors, and the residual in σ_{11} in Fig. 3. Similar plots for σ_{12} are shown in Fig. 4. It is seen that the stress residual provides good pointwise tracking of the error on a relatively coarse mesh.

Of most practical importance (e.g. in adaptivity) is the performance of element error indicators. In particular, the

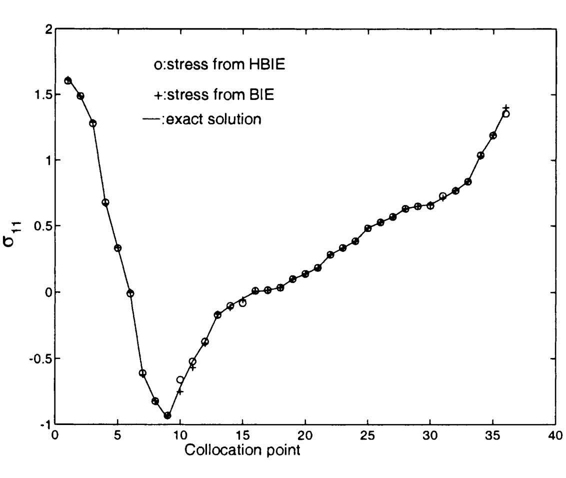

Fig. 2. Comparison of analytical and numerical solutions for σ_{11}.

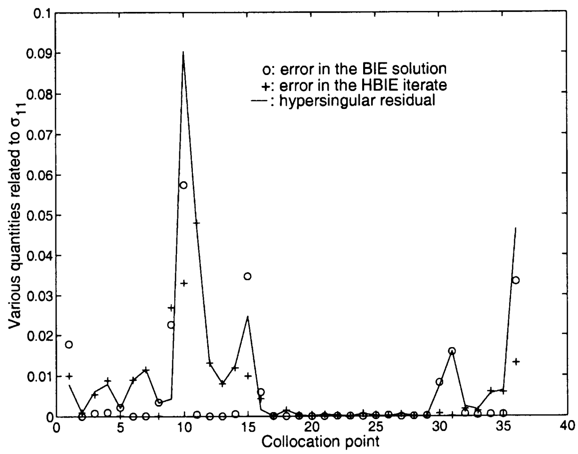

Fig. 3. Absolute values of error in the BIE solution, error in the HBIE iterate, and the hypersingular residual, for σ_{11}. All variables are unscaled.

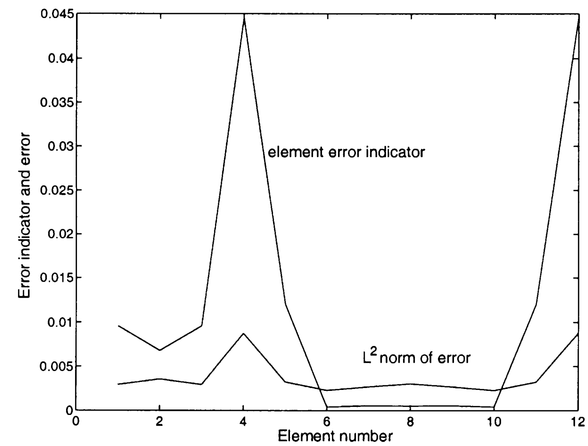

Fig. 5. Traction residual based element error indicator compared with error in the unspecified boundary data.

performance of the two indicators η_j^t and η_j^s defined in Eqs. (52) and (53), respectively, is studied here. The first is a traction residual based error indicator, and the second uses the stress residual.

The element error indicator based on the traction residual (η_j^t from Eq. (52)) is compared with the element based L^2 norm of the error in the unspecified boundary data in Fig. 5. On the other hand, the element error indicator based on the stress residual (η_j^s from Eq. (53)) is compared to the element based L^2 norm of the error in stress, on all the boundary elements, in Fig. 6. In both cases, the residuals are seen to capture the error trends quite effectively.

Remark 4. The comparison between the traction residual and error in the displacement is difficult unless one uses normalized values. A simple way to do this is to use non-dimensional quantities to begin with. This is the approach followed in this work.

5.2. Example 2: Kirsch's problem of an infinite plate with a circular hole

Consider an infinite plate with a hole of radius 1, subject to a traction $t_x = 1$ at infinity. Material properties are the same as in the previous example.

The displacement and stress fields for this problem may be found in Timoshenko and Goodier [66]. In order to simulate this problem with a finite geometry, the boundaries of a finite square domain are subjected to tractions computed from the exact solution of an infinite plate subjected to traction at infinity. Using symmetry, only a quarter of the plate is used in the computer model. The mesh is shown in Fig. 7.

The important comparison is between the computed element error indicators and the L^2 norms of the error on each element (Figs. 8 and 9). Fig. 8 uses errors in the unspecified boundary data while Fig. 9 uses errors in stress components. It is seen that the traction based error indicator underestimates the error on some elements. The stress based

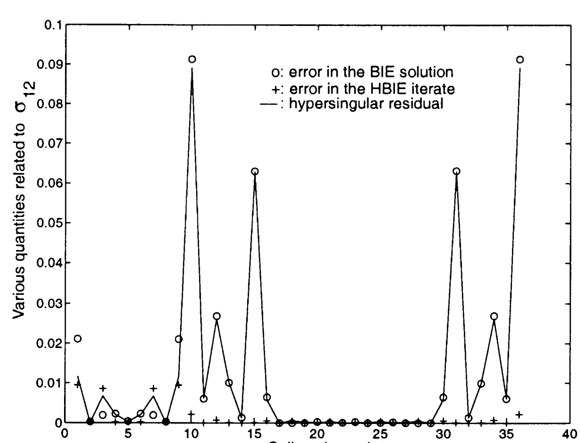

Fig. 4. Absolute values of error in the BIE solution, error in the HBIE iterate, and the hypersingular residual, for σ_{12}. All variables are unscaled.

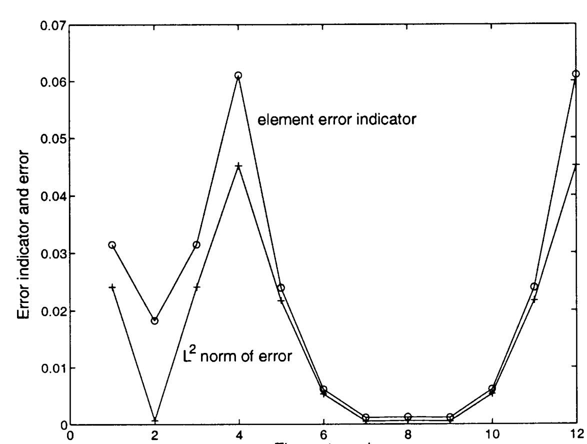

Fig. 6. Stress residual based element error indicator compared with error in boundary stresses.

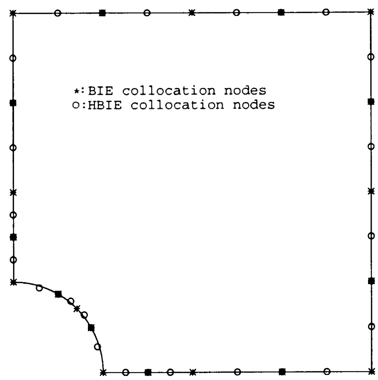

Fig. 7. Discretized domain for the problem of a plate with a hole with 10 elements. Plate side = 4 units, holeradius = 1 unit of length.

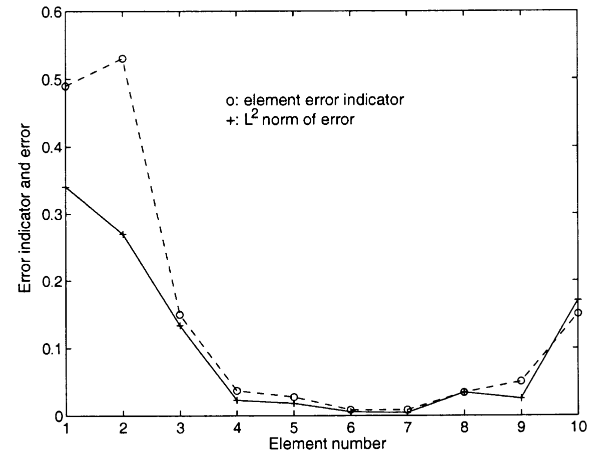

Fig. 9. Stress residual based element error indicator compared with error in boundary stresses.

error indicator performs better and accurately captures the error in stress on most of the elements.

In conclusion, the error estimation method presented here has the advantage of capturing errors in the stress field. On the other hand, the usual residual techniques may only be used to compute the error in displacements. On physical grounds, a measure of the error in stress is preferable to a measure of displacement error. It may also be viewed as a stronger measure of convergence — i.e. the approximate displacement field, *and its gradient* have converged to the actual solution.

6. Concluding remarks

This paper has presented an analysis of error estimators in linear elasticity based on hypersingular residuals, defined in the context of BIEs. Proofs of relationships between residuals and actual errors are presented for displacement prescribed and for traction prescribed BVPs, while heuristic error estimators are proposed for mixed BVPs. The proofs are only valid under restricted conditions such as for bodies with smooth boundaries. Numerical results, however, are presented for mixed BVPs for bodies with corners; and are most encouraging.

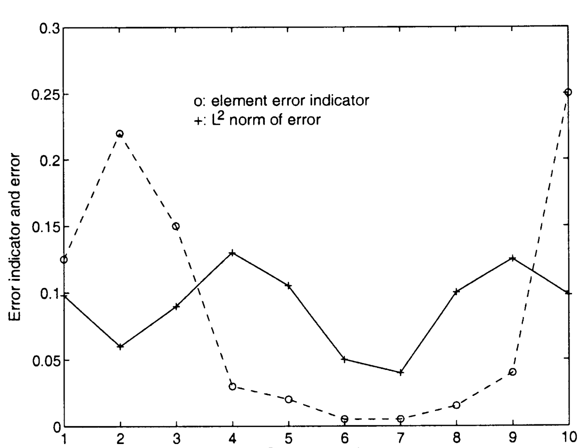

Fig. 8. Traction residual based element error indicator compared with error in the unspecified boundary data.

This promising idea of error estimation and adaptivity, using a ‘two level’ iteration scheme, has been extended recently to various types of boundary element methods, such as the symmetric Galerkin BEM (SGBEM) [5], the boundary node method (BNM) [6] and the boundary contour method (BCM) [7]. Further work along these lines is currently in progress.

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Appendix A. Kelvin kernels in planar elasticity

The Kelvin kernels in the BIE (1) are

$$U_{ij} = -\frac{1}{8\pi\mu(1-\nu)}[(3-4\nu)\delta_{ij}\log r - r_{,i}r_{,j}],$$

$$T_{ij} = -\frac{1}{4\pi(1-\nu)r} \times\left\{\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}] - (1-2\nu)(r_{,i}n_j - r_{,j}n_i)\right\}.$$

The corresponding kernals in the HBIE (4) are

$$D_{ijk} = \frac{1}{4\pi(1-\nu)r}[(1-2\nu)(\delta_{ik}r_{,j} + \delta_{jk}r_{,i} - \delta_{ij}r_{,k}) + 2r_{,i}r_{,j}r_{,k}],$$

$$S_{ijk} = \frac{\mu}{2\pi(1-\nu)r^2}\left\{2\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ij}r_{,k} + \nu(\delta_{jk}r_{,i} + \delta_{ik}r_{,j}) - 4r_{,i}r_{,j}r_{,k}] + (1-2\nu)(2n_k r_{,i}r_{,j} + n_j\delta_{ik} + n_i\delta_{jk}) + 2\nu(n_i r_{,j}r_{,k} + n_j r_{,i}r_{,k}) - (1-4\nu)n_k\delta_{ij}.\right.$$

In the above, r is the distance between a source point P and a field point Q, μ and ν are the shear modulus and Poisson's ratio, respectively, δ_{ik} are the components of the Kronecker delta and $,k \equiv \partial/\partial x_k(Q)$. Also, the components of the normal $\mathbf{n}$ as well as the normal derivative $\partial r/\partial n$ are evaluated at a field point Q.

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