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# Integral equations with hypersingular kernels—theory and applications to fracture mechanics

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**Abstract**

Hypersingular integrals of the type

$$I_\alpha(T_n, m, r) = \oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{(s-r)^\alpha}\,\mathrm{d}s, \quad |r| < 1$$

and

$$I_\alpha(U_n, m, r) = \oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{(s-r)^\alpha}\,\mathrm{d}s, \quad |r| < 1$$

are investigated for general integers $\alpha$ (positive) and $m$ (nonnegative), where $T_n(s)$ and $U_n(s)$ are the Chebyshev polynomials of the first and second kinds, respectively. Exact formulas are derived for the cases $\alpha = 1, 2, 3, 4$ and $m = 0, 1, 2, 3$; most of them corresponding to new solutions derived in this paper. Moreover, a systematic approach for evaluating these integrals when $\alpha > 4$ and $m > 3$ is provided. The integrals are also evaluated as $|r| > 1$ in order to calculate the stress intensity factors. Examples involving crack problems are given and discussed with emphasis on the linkage between mathematics and mechanics of fracture. The examples include classical linear elastic fracture mechanics, functionally graded materials, and gradient elasticity theory. An appendix, with an alternative derivation of the formulae, supplements the paper.



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## 1. Introduction

Finite and boundary element methods are two of the most frequently used numerical approaches for solving crack problems in fracture mechanics. An alternative approach is the integral equation method, which is more efficient (e.g. it reduces a partial differential equation (PDE) in two dimensions to an one-dimensional integral equation) and, in general, is more accurate than the aforementioned methods. The accuracy of the integral equation method relies on the analytical evaluation of singular kernels to cancel the singularity. In general, the cancellation of singularity is not trivial, in particular, in the case of hypersingular integrals. This is the main concern of this paper.

Integral equations arising in static crack problems in fracture mechanics are typically Fredholm integral equations of the form

$$\int_c^d \mathrm{kernel}(x,t)D(t)\,\mathrm{d}t = p(x), \quad c < x < d, \tag{1}$$

where $\mathrm{kernel}(x,t)$ is, in general, a singular function of $(x,t)$; $D(t)$ is the unknown, called density function; $p(x)$ is some known (input) function corresponding to the loading on the crack faces; and the interval $(c,d)$ refers to the crack surfaces where $2a = d - c$ denotes the crack length. By means of the Fourier transform method, we write

$$\mathrm{kernel}(x,t) = \int_{-\infty}^{\infty} K(\xi)\mathrm{e}^{\mathrm{i}(t-x)\xi}\,\mathrm{d}\xi. \tag{2}$$

The singular part of the kernel can be separated from the regular part, by decomposing the Fourier transform as

$$K(\xi) = \underbrace{K_\infty(\xi)}_{\text{singular}} + \underbrace{[K(\xi) - K_\infty(\xi)]}_{\text{nonsingular}}, \tag{3}$$

which can be accomplished through asymptotic analysis (discussed later in this paper). Such analysis is not an easy task for complicated $K(\xi)$. This is another relevant issue to be addressed in this paper.

Once the decomposition (3) is accomplished, the integral Eq. (1) can be rewritten as

$$\neq\!\!\!\!\!\!\int_c^d \frac{c_\alpha D(t)}{(t-x)^\alpha}\,\mathrm{d}t + \int_c^d k(x,t)D(t)\,\mathrm{d}t + f(x) = p(x), \quad c < x < d, \tag{4}$$

where $\neq\!\!\!\!\!\!\int$ denotes an improper integral; $c_\alpha$ is a constant associated to the singular kernel $1/(t-x)^\alpha$; $k(x,t)$ is the nonsingular (regular) kernel; $f(x)$ is a function standing for the free term; and $\alpha$ is a positive integer which determines the degree of the singularity. If $\alpha = 1$, the integral Eq. (4) is called a Cauchy singular integral equation, and the singular term is evaluated as a Cauchy principal-value (CPV) integral. If $\alpha \geqslant 2$, it is called a hypersingular integral equation and the singular term is evaluated as a Hadamard finite-part (HFP) integral [1–7]. Here the notation $⨍$ and $⨎$ refer to CPV and HFP integrals, respectively. Most papers published in the literature involve either $\alpha = 1$ [8–10] or $\alpha = 2$ [11–13], and not much has been done with $\alpha \geqslant 3$. Thus another focus of this paper is to deal with hypersingular integral equations with $\alpha \geqslant 3$ which arise naturally in gradient elasticity theories (see Example 3 in Section 7).

After normalization (discussed in Section 3.1), the singular and/or hypersingular integrals which involve Chebyshev polynomials ($T_n$, first kind; $U_n$, second kind) and weight function $(1-s^2)^{m-1/2}$, $m \geqslant 0$, with singularity $\alpha \geqslant 1$ are defined by

$$I_\alpha(T_n, m, r) = ⨎_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{(s-r)^\alpha} \, \mathrm{d}s, \quad |r| < 1 \tag{5}$$

and

$$I_\alpha(U_n, m, r) = ⨎_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{(s-r)^\alpha} \, \mathrm{d}s, \quad |r| < 1. \tag{6}$$

The scope of this paper is as follows. First, Cauchy singular integrals, i.e. $\alpha = 1$, are evaluated and exact formulas are derived for general $m$. The new results here are closed form analytical solutions for $I_1(T_n, m, r)$, $m \geqslant 1$ and $I_1(U_n, m, r)$, $m \geqslant 2$. Once $I_1(T_n, m, r)$ and $I_1(U_n, m, r)$ are known, hypersingular integrals $I_\alpha(T_n, m, r)$ and $I_\alpha(U_n, m, r)$, $\alpha \geqslant 2$, can be found by successive differentiation (with respect to $r$) in the sense of finite-part integrals; formulas for $I_2(T_n, m, r)$, $m \geqslant 1$ and $I_2(U_n, m, r)$, $m \geqslant 2$ are derived in this manner. In the cases where $\alpha \geqslant 3$, evaluation of hypersingular integrals becomes tedious and the formulas are lengthy. Thus $I_\alpha(T_n, m, r)$ and $I_\alpha(U_n, m, r)$ are provided only for $\alpha = 3, 4$ and general $m$.

## 2. Related work

Singular integral equations have played an active role in the field of solid mechanics, particularly in the solution of fracture mechanics problems. According to the notation introduced in equation (4), these equations can be classified by the order of singularity $\alpha$. The case $\alpha = 1$ has been widely used and well developed [14]. A rich field of application of singular integral equations is in fracture mechanics of bimaterial [8,15,16] and nonhomogeneous materials. In the latter case, the investigation of crack behavior in nonhomogeneous materials has found many applications to functionally graded materials (FGMs) (e.g. [17–23]). Another use of singular integral equations involves FGMs for high temperature applications, so that thermal stress intensity factors (SIFs) can be numerically calculated [24–27]. Application of hypersingular integral equations for $\alpha \geqslant 2$ can be found in [3,11–13].

Quadrature formulas which involve hypersingular integrals with $\alpha \leqslant 2$ have been drawing a considerable amount of concentration [5,28–32] after Kutt [1] first introduced the HFP idea in his work. One of the key steps in the derivations involves the fact that higher order singular integrals can be obtained from lower order ones by a careful exchangeability of differentiation and integration [4,5], e.g.

$$\oint_{-1}^{1} \frac{g(s)}{(s-r)^{\alpha+1}}\,\mathrm{d}s = \frac{1}{\alpha}\oint_{-1}^{1} \frac{\partial}{\partial r}\left[\frac{g(s)}{(s-r)^{\alpha}}\right]\mathrm{d}s = \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}r}\oint_{-1}^{1} \frac{g(s)}{(s-r)^{\alpha}}\,\mathrm{d}s, \quad |r|<1. \tag{7}$$

For instance, in order to find

$$\oint_{-1}^{1} \frac{g(s)}{(s-r)^{2}}\,\mathrm{d}s, \quad |r|<1,$$

it suffices to know how to evaluate

$$\int_{-1}^{1} \frac{g(s)}{s-r}\,\mathrm{d}s, \quad |r|<1.$$

This important concept is applied later in this paper.

Another motivation for numerical evaluation of hypersingular integrals is due to the boundary element method, and the reader is directed to the literature in the field such as the recent book by Bonnet [33]; the review articles by Tanaka et al. [34] and Sladek and Sladek [35]; or the papers by Liu and Rizzo [36], Toh and Mukherjee [37], Paulino et al. [38], and Hui and Mukherjee [39]. Research work has been focused on singularity with $\alpha = 2$ for two-dimensional problems and $\alpha = 3$ for three-dimensional problems (see Eq. (4)).

## 3. Theoretical aspects

First, relevant concepts involving integration and approximation are given. These concepts position the contribution of the work with respect to the available literature. Next, a discussion on the influence of the density function on the corresponding singular integral equation formulation is presented. Afterwards, basic properties of the Chebyshev polynomials are provided. These properties are heavily used in the analytical derivations that follow.

### 3.1. Integration and approximation

As far as the integration and numerical procedures are concerned, the integral Eq. (4) may be normalized through the following change of variables

$$s = \frac{2}{d-c}\left(t - \frac{c+d}{2}\right) \quad \text{and} \quad r = \frac{2}{d-c}\left(x - \frac{c+d}{2}\right), \tag{8}$$

which leads to the normalized version of the integral equation (4) written as [1]

[1] The notations in this paper have been chosen as following: $x$ and $t$ refer to the physical quantities and have dimension of "Length"; $r$ and $s$ are normalized (dimensionless) variables, corresponding to $x$ and $t$, respectively.

$$\fint_{-1}^{1} \frac{D(s)}{(s-r)^{\alpha}}\,\mathrm{d}s + \int_{-1}^{1} \mathscr{K}(r,s)D(s)\,\mathrm{d}s + F(r) = P(r), \quad -1 < r < 1. \tag{9}$$

The density function $D(s)$ is further assumed to have the representation

$$D(s) = R(s)W(s). \tag{10}$$

The weight function $W(s)$ determines the singular behavior of the solution $D(s)$ and has the form

$$W(s) = (1-s)^{m_1}(1+s)^{m_2}. \tag{11}$$

In general, $m_1 \neq m_2$, and the corresponding integrals, which involve Jacobi polynomials $P_n^{(m_1,m_2)}(s)$, are of the type

$$\fint_{-1}^{1} \frac{(1-s)^{m_1}(1+s)^{m_2}P_n^{(m_1,m_2)}(s)}{s-r}\,\mathrm{d}s \tag{12}$$

and can be expressed in terms of the gamma and the hypergeometric functions [40,41]. In this paper, only the case $m_1 = m_2$ is considered and $m_1$, $m_2$ are set to be

$$m_1 = m_2 = m - \tfrac{1}{2}. \tag{13}$$

Thus $W(s)$ can be expressed as

$$W(s) = (1-s^2)^{m-1/2}, \quad m = 0, 1, 2, \ldots. \tag{14}$$

The value of $m$ is determined by the order of singularity $\alpha$. As $\alpha = 1$, one may apply Muskhelishvili's procedure [42,43] to the corresponding (Cauchy) singular integral equation and find $m = 0$. Thus the solution $D(s)$ to the Cauchy singular integral Eq. (9) takes the form

$$D(s) = \frac{R(s)}{\sqrt{1-s^2}}. \tag{15}$$

In this case, which consists of the majority of the work involving applications of integral equations to fracture mechanics [15,44,45], $R(s)$ is chosen to be

$$R(s) = \sum_{n}^{\infty} a_n T_n(s) \tag{16}$$

and because of that, the CPV integral $I_1(T_n, 0, r)$ can be evaluated exactly [3,40,46]:

$$I_1(T_n, 0, r) = \fint_{-1}^{1} \frac{T_n(s)}{(s-r)\sqrt{1-s^2}}\,\mathrm{d}s = \begin{cases} 0, & n = 0, \\ \pi U_{n-1}(r), & n \geqslant 1. \end{cases} \tag{17}$$

Another reason for choosing the expansion (16) is that with respect to the weight function $W(s) = 1/\sqrt{1-s^2}$, the class of the Chebyshev polynomials of first kind $T_n(s)$ is an orthogonal family [40]:

$$\int_{-1}^{1} \frac{T_m(s)T_n(s)}{\sqrt{1-s^2}}\,\mathrm{d}s = \begin{cases} \pi, & m=n=0 \\ \pi/2, & m=n; \quad m,n=1,2,3,\ldots \\ 0, & m\neq n; \quad m,n=0,1,2,\ldots \end{cases} \tag{18}$$

With this orthogonal property a Galerkin-type method (Krenk [28]) may be applied to find the coefficients $a_n$ in Eq. (16).

If $\alpha = 2$, then $m = 1$, and the solution $D(s)$ to the hypersingular integral Eq. (9) is characterized by

$$D(s) = R(s)\sqrt{1-s^2}. \tag{19}$$

Correspondingly, $R(s)$ is chosen to be

$$R(s) = \sum_{n}^{\infty} b_n U_n(s) \tag{20}$$

because of the same reasons for the case $\alpha = 1$, namely, analytical evaluation and orthogonal property. With respect to the first reason, the HFP integral $I_2(U_n, 1, r)$ can be evaluated analytically [3]:

$$I_2(U_n, 1, r) = \mathop{{\int\!\!\!\!\!\!=}}\nolimits_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{(s-r)^2}\,\mathrm{d}s = -(n+1)\pi U_n(r), \quad n \geqslant 0. \tag{21}$$

According to the second reason, by orthogonality,

$$\int_{-1}^{1} U_m(s)U_n(s)\sqrt{1-s^2}\,\mathrm{d}s = \begin{cases} \pi/2, & m=n; \quad m,n=0,1,2,\ldots, \\ 0, & m\neq n; \quad m,n=0,1,2,\ldots \end{cases} \tag{22}$$

and one may apply Galerkin-type methods [28] to find the coefficients $b_n$ in Eq. (20).

When $\alpha = 3$, then $m \geqslant 1$. For instance, with $m = 2$, the weight function $W(s) = (1-s^2)^{3/2}$, neither $T_n(s)$ nor $U_n(s)$ is an orthogonal family. *However, if collocation method is applied, one does not need the orthogonal property, as long as the expansion function $R(s)$ is chosen such that*

$$\mathop{{\int\!\!\!\!\!\!=}}\nolimits_{-1}^{1} \frac{R(s)(1-s^2)^{3/2}}{(s-r)^3}\,\mathrm{d}s$$

*can be evaluated analytically.* For example, if $R(s)$ is expanded as a Chebyshev polynomial of the first kind $T_n(s)$ or the second kind $U_n(s)$, i.e.

$$R(s) = \sum_{n}^{\infty} a_n T_n(s) \quad \text{or} \quad R(s) = \sum_{n}^{\infty} b_n U_n(s) \tag{23}$$

then the evaluation of

$$I_\alpha(T_n, m, r) = \oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{(s-r)^\alpha}\,\mathrm{d}s \quad \text{or} \quad I_\alpha(U_n, m, r) = \oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{(s-r)^\alpha}\,\mathrm{d}s$$

for general $m = 0, 1, 2, \dots$ and $\alpha = 1, 2, 3, \dots$ is a necessary step for the numerical approach to the integral equation (9). This is the one of main tasks in this paper and is addressed in Sections 4 and 5.

### 3.2. Selection of the density function

Usually the unknown function $D(t)$ in Eq. (1) can be chosen as the displacement profile (e.g. $u(t)$—a displacement function), the (first) derivative of the displacement function ($\mathrm{d}u(t)/\mathrm{d}t$, denoted by $\phi(t)$—the slope function), or a higher derivative of $u(t)$. The choice of the unknown function $D(t)$ will affect the degree of singularity in the formulation. For example, consider the standard mode III crack problem in a free space [47] and a linear elastic fracture mechanics (LEFM) setting. If $D(t)$ is chosen to be the slope function $\phi(t)$, then the governing integral equation is the Cauchy singular integral equation

$$D(t) \equiv \phi(t), \qquad \int_c^d \frac{\phi(t)}{t-x}\mathrm{d}t = p(x), \quad c < x < d. \tag{24}$$

However, if $D(t)$ is chosen to be the displacement function $w(t)$, then the hypersingular integral equation with $\alpha = 2$ is obtained,

$$D(t) \equiv w(t), \qquad \oint_c^d \frac{w(t)}{(t-x)^2}\,\mathrm{d}t = p(x), \quad c < x < d. \tag{25}$$

The differences between the above two formulations are discussed next.

Note that the higher singularity in (25) does *not* constitute more difficulty in solving the equation since hypersingular integrals can be evaluated exactly with suitable basis functions [48]. So the choice of the density functions should be dictated by considerations other than the order of singularity in the resulting equations. There are situations, however, in which the displacements appear to be more natural than the slopes, e.g., in periodic [13] or three-dimensional crack problems. Sometimes, a formulation with higher singular kernels results in simpler nonsingular kernels in the decomposition (3) (see Section 7 for discussion of examples).

### 3.3. Properties of Chebyshev polynomials

The evaluation of Cauchy singular and hypersingular integrals which involve the Chebyshev polynomials $T_n(s)$ and $U_n(s)$ highly depends on the special properties of these polynomials. They are listed here for the sake of completeness and because they will be of much use later in the development of this work. Most of them (but not all) can be found in [3,40].

- Definition of Chebyshev polynomials of the first kind:

$$T_n(s) = \cos[n\cos^{-1}(s)], \quad n = 0, 1, 2, \dots \tag{26}$$

- Definition of Chebyshev polynomials of the second kind:

$$U_n(s) = \frac{\sin[(n+1)\cos^{-1}(s)]}{\sin[\cos^{-1}(s)]}, \quad n = 0, 1, 2, \dots \tag{27}$$

- Iterative (recursive) properties:

$$sT_n(s) = \tfrac{1}{2}[T_{n+1}(s) + T_{n-1}(s)], \quad n \geqslant 1, \tag{28}$$

$$sU_n(s) = \tfrac{1}{2}[U_{n+1}(s) + U_{n-1}(s)], \quad n \geqslant 1, \tag{29}$$

$$T_n(s) = \tfrac{1}{2}[U_n(s) - U_{n-2}(s)], \quad n \geqslant 2, \tag{30}$$

$$U_n(s)(1-s^2) = sT_{n+1}(s) - T_{n+2}(s), \quad n \geqslant 0. \tag{31}$$

By means of Eq. (28), one may rewrite Eq. (31) above as

$$U_n(s) = \frac{1}{2(1-s^2)}[T_n(s) - T_{n+2}(s)], \quad n \geqslant 0. \tag{32}$$

Thus an additional equality, which is useful in handling cubic hypersingular integrals can be derived: [2]

$$\begin{aligned} U_n(s)(1-s^2)^{3/2} &\overset{(32)}{=} \tfrac{1}{2}[T_n(s) - T_{n+2}(s)]\sqrt{1-s^2} \\ &\overset{(30)}{=} -\tfrac{1}{2}\left[\tfrac{1}{2}U_{n+2}(s) - U_n(s) + \tfrac{1}{2}U_{n-2}(s)\right]\sqrt{1-s^2}, \quad n \geqslant 2 \\ &= -\tfrac{1}{4}[U_{n+2}(s) - 2U_n(s) + U_{n-2}(s)]\sqrt{1-s^2}, \quad n \geqslant 2. \end{aligned} \tag{33}$$

- Derivatives:

$$\frac{\mathrm{d}T_n(s)}{\mathrm{d}s} = nU_{n-1}(s), \quad n \geqslant 1, \tag{34}$$

$$\frac{\mathrm{d}U_n(s)}{\mathrm{d}s} = \frac{1}{1-s^2}\left[\frac{n+2}{2}U_{n-1}(s) - \frac{n}{2}U_{n+1}(s)\right], \quad n \geqslant 1. \tag{35}$$

[2] The equation number is stacked above the equal sign to show how the equations are being derived and connected.

## 4. Cauchy singular integral formulas ($\alpha = 1$)

This section mainly evaluates $I_1(T_n, m, r)$ and $I_1(U_n, m, r)$, which are defined in Eqs. (5) and (6). The new result here is that the singular integral formulas are found for general $m$. In order to obtain this new result, two well known Cauchy singular integral formulas are introduced [40]: one is already stated in Eq. (17), and the other one is

$$I_1(U_n, 1, r) = \fint_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{s-r}\,\mathrm{d}s = -\pi T_{n+1}(r), \quad n \geqslant 0, \tag{36}$$

which can be obtained as follows

$$\begin{aligned} I_1(U_n, 1, r) &= \fint_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{s-r}\,\mathrm{d}s \overset{(32)}{=} \frac{1}{2}\fint_{-1}^{1} \frac{T_n(s)-T_{n+2}(s)}{\sqrt{1-s^2}(s-r)}\,\mathrm{d}s \\ &\overset{(17)}{=} \frac{\pi}{2}[U_{n-1}(r) - U_{n+1}(r)] \\ &\overset{(30)}{=} -\pi T_{n+1}(r). \end{aligned}$$

The integral formulas for $m = 0, 1, 2, 3, \ldots$ are derived below. The general formulas have the restriction of minimum $n$. For instance, Eqs. (30) and (33) are only true for $n \geqslant 2$, and Eqs. (34) and (35) are valid for $n \geqslant 1$. The lower $n$ terms can not be derived by general formulas, and are given in [49].

### 4.1. $I_1(T_n, m, r), m = 0, 1, 2, 3$

- $I_1(T_n, 0, r)$:
  This is Eq. (17).
- $I_1(T_n, 1, r)$:

$$\begin{aligned} \fint_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{s-r}\mathrm{d}s &\overset{(30)}{=} \frac{1}{2}\fint_{-1}^{1} \frac{[U_n(s) - U_{n-2}(s)]\sqrt{1-s^2}}{s-r}\,\mathrm{d}s \\ &\overset{(36)}{=} \frac{\pi}{2}[T_{n-1}(r) - T_{n+1}(r)], \quad n \geqslant 2. \end{aligned} \tag{37}$$

- $I_1(T_n, 2, r)$:

$$\begin{aligned} \fint_{-1}^{1} \frac{T_n(s)(1-s^2)^{3/2}}{s-r}\,\mathrm{d}s &\overset{(30)}{=} \fint_{-1}^{1} \frac{\frac{1}{2}[U_n(s) - U_{n-2}(s)](1-s^2)^{3/2}}{s-r}\,\mathrm{d}s \\ &\overset{(40)}{=} \frac{\pi}{8}\{[T_{n-1}(r) - 2T_{n+1}(r) + T_{n+3}(r)] - [T_{n-3}(r) - 2T_{n-1}(r) + T_{n+1}(r)]\}. \\ &= -\frac{\pi}{8}[T_{n-3}(r) - 3T_{n-1}(r) + 3T_{n+1}(r) - T_{n+3}(r)], \quad n \geqslant 4. \end{aligned} \tag{38}$$

- $I_1(T_n, 3, r)$:

$$\begin{aligned}\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{5/2}}{s-r}\,\mathrm{d}s &\overset{(30)}{=} \oint_{-1}^{1} \frac{\frac{1}{2}[U_n(s)-U_{n-2}(s)](1-s^2)^{5/2}}{s-r}\,\mathrm{d}s \\ &\overset{(41)}{=} \frac{\pi}{32}\{[T_{n-5}(r)-4T_{n-3}(r)+6T_{n-1}(r)-4T_{n+1}(r)+T_{n+3}(r)] \\ &\quad -[T_{n-3}(r)-4T_{n-1}(r)+6T_{n+1}(r)-4T_{n+3}(r)+T_{n+5}(r)]\}, \quad n \geqslant 6 \\ &= \frac{\pi}{32}[T_{n-5}(r)-5T_{n-3}(r)+10T_{n-1}(r)-10T_{n+1}(r)+5T_{n+3}(r)-T_{n+5}(r)], \\ &\quad n \geqslant 6. \end{aligned} \tag{39}$$

### 4.2. $I_1(U_n, m, r), m = 1, 2, 3$

- $I_1(U_n, 1, r)$:
  This is Eq. (36).
- $I_1(U_n, 2, r)$:

$$\begin{aligned}\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{3/2}}{s-r}\,\mathrm{d}s &\overset{(32)}{=} \frac{1}{2}\oint_{-1}^{1} \frac{[T_n(s)-T_{n+2}(s)]\sqrt{1-s^2}}{s-r}\,\mathrm{d}s \\ &\overset{(37)}{=} \frac{\pi}{4}\{[T_{n-1}(r)-T_{n+1}(r)]-[T_{n+1}(r)-T_{n+3}(r)]\}, \quad n \geqslant 2 \\ &= \frac{\pi}{4}[T_{n-1}(r)-2T_{n+1}(r)+T_{n+3}(r)], \quad n \geqslant 2. \end{aligned} \tag{40}$$

- $I_1(U_n, 3, r)$:

$$\begin{aligned}\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{5/2}}{s-r}\,\mathrm{d}s &\overset{(32)}{=} \frac{1}{2}\oint_{-1}^{1} \frac{[T_n(s)-T_{n+2}(s)](1-s^2)^{3/2}}{s-r}\,\mathrm{d}s \\ &\overset{(38)}{=} \frac{\pi}{16}\{[T_{n-1}(r)-3T_{n+1}(r)+3T_{n+3}(r)-T_{n+5}(r)] \\ &\quad -[T_{n-3}(r)-3T_{n-1}(r)+3T_{n+1}(r)-3T_{n+3}(r)]\}, \quad n \geqslant 4 \\ &= -\frac{\pi}{16}[T_{n-3}(r)-4T_{n-1}(r)+6T_{n+1}(r)-4T_{n+3}(r)+T_{n+5}(r)], \quad n \geqslant 4. \end{aligned} \tag{41}$$

### 4.3. $I_1(T_n, m, r)$ and $I_1(U_n, m, r)$

At this point one may easily see the procedural steps above, which take advantage of recursive properties (30) and (32) between the Chebyshev polynomials $T_n(s)$ and $U_n(s)$. [3] For instance,

[3] Another way of evaluating $I_1(T_n, m, r)$ and $I_1(U_n, m, r)$ is by using the recursive property through $T_n(s)$ only, which is discussed in Appendix A.

$$\fint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m+\frac{1}{2}}}{s-r} ds$$

$$\Downarrow \ (30)$$

$$\fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m+\frac{1}{2}}}{s-r} ds$$

$$\Downarrow \ (32)$$

$$\fint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-\frac{1}{2}}}{s-r} ds$$

$$\Downarrow \ (30)$$

$$\fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-\frac{1}{2}}}{s-r} ds$$

$$\Downarrow \ (32)$$

$$\vdots$$

Equation (17) or (36)

Fig. 1. Evaluation of $I_1(T_n, m, r)$ and $I_1(U_n, m, r)$ for general $m$. The procedure reduces integrals with higher $m$ to those with lower $m$.

evaluation of $I_1(T_n, 4, r) = \fint_{-1}^{1} T_n(s)(1-s^2)^{7/2}/(s-r)\,\mathrm{d}s$ can be reduced to evaluation of $I_1(U_n, 4, r) = \fint_{-1}^{1} U_n(s)(1-s^2)^{7/2}/(s-r)\,\mathrm{d}s$. which, in turn, can be reduced to evaluation of $I_1(T_n, 3, r) = \fint_{-1}^{1} T_n(s)(1-s^2)^{5/2}/(s-r)\,\mathrm{d}s$. After a suitable number of steps, this reduction leads to either (17) or (36). This procedure is summarized in Fig. 1.

- $I_1(T_n, m, r)$, where $m \geqslant 1$, and $n \geqslant 2m$

$$\fint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{s-r}\,\mathrm{d}s = \pi(-1)^{m+1}\left(\frac{1}{2}\right)^{2m-1} \sum_{j=0}^{2m-1} (-1)^j \binom{2m-1}{j} T_{n+1-2m+2j}(r). \tag{42}$$

- $I_1(U_n, m, r)$, where $m \geqslant 2$, and $n \geqslant 2m-2$

$$\fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{s-r}\,\mathrm{d}s = \pi(-1)^{m}\left(\frac{1}{2}\right)^{2m-2} \sum_{j=0}^{2m-2} (-1)^j \binom{2m-2}{j} T_{n+3-2m+2j}(r). \tag{43}$$

The usual notation

$$\binom{m}{j} = \frac{(m)!}{j!(m-j)!}$$

denotes the binomial coefficients.

## 5. Hypersingular integral formulas ($\alpha \geqslant 2$)

Once a Cauchy singular integral formula has been reached, all other hypersingular integral formulas may be obtained successively by taking differentiation with respect to $r$, and making use of the finite-part integral formula (7).

### 5.1. $I_2(T_n, m, r)$

By means of

$$\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{(s-r)^2}\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}r}\int_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{s-r}\,\mathrm{d}s,$$

one readily obtains

- $I_2(T_n, 0, r)$ (see also [3]):

$$\oint_{-1}^{1} \frac{T_n(s)}{\sqrt{1-s^2}(s-r)^2}\,\mathrm{d}s = \pi\frac{\mathrm{d}}{\mathrm{d}r}U_{n-1}(r) \overset{(35)}{=} \frac{\pi}{1-r^2}\left[\frac{n+1}{2}U_{n-2}(r) - \frac{n-1}{2}U_n(r)\right], \quad n \geqslant 2. \tag{44}$$

- $I_2(T_n, 1, r)$:

$$\oint_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{(s-r)^2}\,\mathrm{d}s = \frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}r}[T_{n-1}(r) - T_{n+1}(r)] \overset{(34)}{=} \frac{\pi}{2}[(n-1)U_{n-2}(r) - (n+1)U_n(r)], \quad n \geqslant 2. \tag{45}$$

- $I_2(T_n, 2, r)$:

$$\begin{aligned}\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{3/2}}{(s-r)^2}\,\mathrm{d}s &= -\frac{\pi}{8}\frac{\mathrm{d}}{\mathrm{d}r}[T_{n-3}(r) - 3T_{n-1}(r) + 3T_{n+1}(r) - T_{n+3}(r)] \\ &\overset{(34)}{=} -\frac{\pi}{8}[(n-3)U_{n-4}(r) - 3(n-1)U_{n-2}(r) + 3(n+1)U_n(r) \\ &\quad - (n+3)U_{n+2}(r)], \quad n \geqslant 4.\end{aligned} \tag{46}$$

- $I_2(T_n, 3, r)$:

$$\begin{aligned}
&\fint_{-1}^{1} \frac{T_n(s)(1-s^2)^{5/2}}{(s-r)^2}\,\mathrm{d}s \\
&\quad = \frac{\pi}{32}\frac{\mathrm{d}}{\mathrm{d}r}[T_{n-5}(r) - 5T_{n-3}(r) + 10T_{n-1}(r) - 10T_{n+1}(r) + 5T_{n+3}(r) - T_{n+5}(r)] \\
&\quad \overset{(34)}{=} \frac{\pi}{32}[(n-5)U_{n-6}(r) - 5(n-3)U_{n-4}(r) + 10(n-1)U_{n-2}(r) - 10(n+1)U_n(r) \\
&\qquad + 5(n+3)U_{n+2}(r) - (n+5)U_{n+4}(r)], \quad n \geqslant 6.
\end{aligned} \tag{47}$$

- $I_2(T_n, m, r)$, where $m \geqslant 1$, and $n \geqslant 2m+1$:

$$\begin{aligned}
\fint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{(s-r)^2}\,\mathrm{d}s &= \pi(-1)^{m+1}\left(\frac{1}{2}\right)^{2m-1} \\
&\quad \times \sum_{j=0}^{2m-1}(-1)^j\binom{2m-1}{j}(n+1-2m+2j)U_{n-2m+2j}(r).
\end{aligned} \tag{48}$$

*5.2. $I_2(U_n, m, r)$*

The following equality

$$\fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{(s-r)^2}\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}r}\fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{s-r}\,\mathrm{d}s,$$

leads to:

- $I_2(U_n, 1, r)$ (see also [3]):

$$\fint_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{(s-r)^2}\,\mathrm{d}s = -\pi\frac{\mathrm{d}}{\mathrm{d}r}T_{n+1}(r) \overset{(34)}{=} -\pi(n+1)U_n(r), \quad n \geqslant 0, \tag{49}$$

which is the same as (21).

- $I_2(U_n, 2, r)$:

$$\begin{aligned}
\fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{3/2}}{(s-r)^2}\,\mathrm{d}s &= \frac{\pi}{4}\frac{\mathrm{d}}{\mathrm{d}r}[T_{n-1}(r) - 2T_{n+1}(r) + T_{n+3}(r)] \\
&\overset{(34)}{=} \frac{\pi}{4}[(n-1)U_{n-2}(r) - 2(n+1)U_n(r) + (n+3)U_{n+2}(r)], \quad n \geqslant 2.
\end{aligned} \tag{50}$$

- $I_2(U_n, 3, r)$:

$$\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{5/2}}{(s-r)^2}\,\mathrm{d}s = -\frac{\pi}{16}\frac{\mathrm{d}}{\mathrm{d}r}[T_{n-3}(r) - 4T_{n-1}(r) + 6T_{n+1}(r) - 4T_{n+3}(r) + T_{n+5}(r)]$$
$$\overset{(34)}{=} -\frac{\pi}{16}[(n-3)U_{n-4}(r) - 4(n-1)U_{n-2}(r) + 6(n+1)U_n(r) - 4(n+3)U_{n+2}(r) + (n+5)U_{n+4}(r)], \quad n \geqslant 4. \tag{51}$$

- $I_2(U_n, m, r)$, where $m \geqslant 2$, and $n \geqslant 2m-1$:

$$\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{(s-r)^2}\,\mathrm{d}s = \pi(-1)^m \left(\frac{1}{2}\right)^{2m-2} \times \sum_{j=0}^{2m-2} (-1)^j \binom{2m-2}{j} (n+3-2m+2j)U_{n+2-2m+2j}(r). \tag{52}$$

### 5.3. $I_3(T_n, m, r)$

By means of

$$\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{(s-r)^3}\,\mathrm{d}s = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{(s-r)^2}\,\mathrm{d}s,$$

one obtains:

- $I_3(T_n, 0, r)$:

$$\oint_{-1}^{1} \frac{T_n(s)}{\sqrt{1-s^2}(s-r)^3}\,\mathrm{d}s = \frac{\pi}{8(1-r^2)^2}[(n+1)(n+2)U_{n-3}(r) - 2(n^2-3)U_{n-1}(r) + (n-1)^2 U_{n+1}(r)], \quad n \geqslant 3. \tag{53}$$

- $I_3(T_n, 1, r)$:

$$\oint_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{(s-r)^3}\,\mathrm{d}s = \frac{\pi}{8(1-r^2)}[(n^2-n)U_{n-3}(r) - 2(n^2+2)U_{n-1}(r) + (n^2+n)U_{n+1}(r)], \quad n \geqslant 3. \tag{54}$$

- $I_3(T_n, 2, r)$:

$$\int_{-1}^{1} \frac{T_n(s)(1-s^2)^{3/2}}{(s-r)^3}\,\mathrm{d}s = \frac{\pi}{32(1-r^2)}\{-(n+3)(n+2)U_{n+3}(r) + [(n+3)(n+4)+3n(n+1)]U_{n+1}(r) - [3(n+1)(n+2)+3(n-1)(n-2)]U_{n-1}(r) + [3n(n-1)+(n-3)(n-4)]U_{n-3}(r) - (n-3)(n-2)U_{n-5}(r)\}, \quad n \geqslant 5. \tag{55}$$

- $I_3(T_n, 3, r)$:

$$\int_{-1}^{1} \frac{T_n(s)(1-s^2)^{5/2}}{(s-r)^3}\,\mathrm{d}s = \frac{\pi}{128(1-r^2)}[(n^2+9n+20)U_{n+5}(r) - 6(n^2+6n+10)U_{n+3}(r) + 15(n^2+3n+4)U_{n+1}(r) - 20(n^2+2)U_{n-1}(r) + 15(n^2-3n+4)U_{n-3}(r) - 6(n^2-6n+10)U_{n-5}(r) + (n^2-9n+20)U_{n-7}(r)], \quad n \geqslant 7. \tag{56}$$

- $I_3(T_n, m, r)$, where $m \geqslant 1$, and $n \geqslant 2m+2$:

$$\int_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{(s-r)^3}\,\mathrm{d}s = (-1)^{m+1}\left(\frac{1}{2}\right)^{2m+1}\frac{\pi}{1-r^2}\sum_{j=0}^{2m-1}(-1)^j\binom{2m-1}{j}(n+1-2m+2j) \times [(n+2-2m+2j)U_{n-1-2m+2j}(r) - (n-2m+2j)U_{n+1-2m+2j}(r)]. \tag{57}$$

### 5.4. $I_3(U_n, m, r)$

By means of

$$\int_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{(s-r)^3}\,\mathrm{d}s = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\int_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{(s-r)^2}\,\mathrm{d}s,$$

one gets:

- $I_3(U_n, 1, r)$:

$$\int_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{(s-r)^3}\,\mathrm{d}s = \frac{\pi}{4(1-r^2)}\{-(n^2+3n+2)U_{n-1}(r) + (n^2+n)U_{n+1}(r)\}, \quad n \geqslant 1. \tag{58}$$

- $I_3(U_n, 2, r)$:

$$\fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{3/2}}{(s-r)^3}\,\mathrm{d}s = \frac{\pi}{16(1-r^2)}[-(n^2+5n+6)U_{n+3}(r) + (3n^2+9n+12)U_{n+1}(r) - (3n^2+3n+6)U_{n-1}(r) + (n^2-n)U_{n-3}(r)], \quad n \geqslant 3. \tag{59}$$

- $I_3(U_n, 3, r)$:

$$\fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{5/2}}{(s-r)^3}\,\mathrm{d}s = \frac{\pi}{64(1-r^2)}[(n^2+9n+20)U_{n+5}(r) - (5n^2+31n+54)U_{n+3}(r) + (10n^2+34n+48)U_{n+1}(r) - (10n^2+6n+20)U_{n-1}(r) + (5n^2-11n+12)U_{n-3}(r) - (n^2-5n+6)U_{n-5}(r)], \quad n \geqslant 5. \tag{60}$$

- $I_3(U_n, m, r)$, where $m \geqslant 2$, and $n \geqslant 2m$:

$$\fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{(s-r)^3}\,\mathrm{d}s = (-1)^m \left(\frac{1}{2}\right)^{2m} \frac{\pi}{1-r^2} \sum_{j=0}^{2m-2} (-1)^j \binom{2m-2}{j} (n+3-2m+2j) \times [(n+4-2m+2j)U_{n+1-2m+2j}(r) - (n+2-2m+2j)U_{n+3-2m+2j}(r)]. \tag{61}$$

### 5.5. $I_4(T_n, m, r)$

By means of

$$\fint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{(s-r)^4}\,\mathrm{d}s = \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}r}\fint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{(s-r)^3}\,\mathrm{d}s,$$

one reaches the following results:

- $I_4(T_n, 0, r)$:

$$\fint_{-1}^{1} \frac{T_n(s)}{(s-r)^4\sqrt{1-s^2}}\,\mathrm{d}s = \frac{\pi}{48(1-r^2)^3}[(n^3+6n^2+11n+6)U_{n-4}(r) - (3n^3+6n^2-25n-44)U_{n-2}(r) + (3n^3-5n^2-19n+37)U_n(r) - (n^3-5n^2+7n-3)U_{n+2}(r)], \quad n \geqslant 4. \tag{62}$$

- $I_4(T_n, 1, r)$:

$$\oint_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{(s-r)^4}\,\mathrm{d}s = \frac{\pi}{48(1-r^2)^2}[(n^3-n)U_{n-4}(r) - (3n^3+9n+12)U_{n-2}(r) + (3n^3+9n-12)U_n(r) - (n^3-n)U_{n+2}(r)], \quad n \geqslant 4. \tag{63}$$

- $I_4(T_n, 2, r)$:

$$\begin{aligned}\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{3/2}}{(s-r)^4}\,\mathrm{d}s = \frac{\pi}{192(1-r^2)^2}[&(n^3+6n^2+11n+6)U_{n+4}(r) \\ &- (5n^3+18n^2+43n+30)U_{n+2}(r) \\ &+ (10n^3+12n^2+134n-36)U_n(r) \\ &- (10n^3-12n^2+134n+36)U_{n-2}(r) \\ &+ (5n^3-18n^2+43n-30)U_{n-4}(r) \\ &- (n^3-6n^2+11n-6)U_{n-6}(r)], \quad n \geqslant 6.\end{aligned} \tag{64}$$

- $I_4(T_n, 3, r)$:

$$\begin{aligned}\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{5/2}}{(s-r)^4}\,\mathrm{d}s = \frac{\pi}{384(1-r^2)^2}\Bigg[&-\left(\frac{1}{2}n^3+6n^2+\frac{47}{2}n+30\right)U_{n+6}(r) \\ &+ \left(\frac{7}{2}n^3+30n^2+\frac{197}{2}n+120\right)U_{n+4}(r) \\ &- \left(\frac{21}{2}n^3+54n^2+\frac{327}{2}n+180\right)U_{n+2}(r) \\ &+ \left(\frac{35}{2}n^3+30n^2+\frac{325}{2}n+90\right)U_n(r) \\ &- \left(\frac{35}{2}n^3-30n^2+\frac{325}{2}n-90\right)U_{n-2}(r) \\ &+ \left(\frac{21}{2}n^3-54n^2+\frac{327}{2}n-180\right)U_{n-4}(r) \\ &- \left(\frac{7}{2}n^3-30n^2+\frac{197}{2}n-120\right)U_{n-6}(r) \\ &+ \left(\frac{1}{2}n^3-6n^2+\frac{47}{2}n-30\right)U_{n-8}(r)\Bigg], \quad n \geqslant 8.\end{aligned} \tag{65}$$

- $I_4(T_n, m, r)$, where $m \geqslant 1$, and $n \geqslant 2m+3$:

$$\begin{aligned}\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{(s-r)^4}\,\mathrm{d}s = (-1)^{m+1}\left(\frac{1}{2}\right)^{2m+2}\frac{1}{3}\frac{\pi}{(1-r^2)^2}\sum_{j=0}^{2m-1}(-1)^j\binom{2m-1}{j}(n+1-2m+2j) \\ \times \{[(n+2-2m+2j)(n+3-2m+2j)]U_{n-2-2m+2j}(r) \\ -[2(n-2m+2j)^2+4(n-2m+2j)-6]U_{n-2m+2j}(r) \\ +[(n-2m+2j)(n-1-2m+2j)]U_{n+2-2m+2j}(r)\}. \end{aligned} \tag{66}$$

### 5.6. $I_4(U_n, m, r)$:

By means of

$$\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{(s-r)^4}\,\mathrm{d}s = \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}r}\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{(s-r)^3}\,\mathrm{d}s,$$

one obtains

- $I_4(U_n, 1, r)$:

$$\begin{aligned}\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{1/2}}{(s-r)^4}\,\mathrm{d}s = \frac{\pi}{24(1-r^2)^2}[-(2n^3+9n^2+11n+6)U_{n-2}(r) \\ +(2n^3+6n^2-2n-6)U_n(r)-(n^3-n)U_{n+2}(r)], \quad n \geqslant 2. \end{aligned} \tag{67}$$

- $I_4(U_n, 2, r)$:

$$\begin{aligned}\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{3/2}}{(s-r)^4}\,\mathrm{d}s = \frac{\pi}{96(1-r^2)^2}[(n^3+6n^2+11n+6)U_{n+4}(r) \\ -(4n^3+18n^2+44n+30)U_{n+2}(r) \\ +(6n^3+18n^2+54n+42)U_n(r) \\ -(4n^3+6n^2+20n+18)U_{n-2}(r) \\ +(n^3-n)U_{n-4}(r)], \quad n \geqslant 4. \end{aligned} \tag{68}$$

- $I_4(U_n, 3, r)$:

$$\int_{-1}^{1} \frac{U_n(s)(1-s^2)^{5/2}}{(s-r)^4}\,ds = \frac{\pi}{192(1-r^2)^2}\Bigg[-\left(\frac{1}{2}n^3+6n^2+\frac{47}{2}n+30\right)U_{n+6}(r) + (3n^3+27n^2+93n+117)U_{n+4}(r) - \left(\frac{15}{2}n^3+45n^2+\frac{285}{2}n+165\right)U_{n+2}(r) + (10n^3+30n^2+110n+90)U_n(r) - \left(\frac{15}{2}n^3+\frac{105}{2}n\right)U_{n-2}(r) + (3n^3-9n^2+21n-15)U_{n-4}(r) - \left(\frac{1}{2}n^3+3n^2+\frac{11}{2}n-3\right)U_{n-6}(r)\Bigg], \quad n \geqslant 6. \tag{69}$$

- $I_4(U_n, m, r)$, where $m \geqslant 2$, and $n \geqslant 2m+1$:

$$\int_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{(s-r)^4}\,ds = (-1)^m\left(\frac{1}{2}\right)^{2m+1}\frac{1}{3}\frac{\pi}{(1-r^2)^2}\sum_{j=0}^{2m-2}(-1)^j\binom{2m-2}{j}(n+3-2m+2j) \times \{[(n+4-2m+2j)(n+5-2m+2j)]U_{n-2m+2j}(r) - [2(n-2m+2j)^2+12(n-2m+2j)+10]U_{n+2-2m+2j}(r) + [(n+2-2m+2j)(n+1-2m+2j)]U_{n+4-2m+2j}(r)\}. \tag{70}$$

## 6. Evaluation of stress intensity factors

An important task is to evaluate the SIFs at both crack tips, since the propagation of a crack starts around its tips. In mode III crack problems, standard SIFs can be calculated from

$$K_{\rm III}(d) = \lim_{x\to d^+}\sqrt{2\pi(x-d)}\,\sigma_{yz}(x,0), \quad (x > d) \tag{71}$$

and

$$K_{\rm III}(c) = \lim_{x\to c^-}\sqrt{2\pi(c-x)}\,\sigma_{yz}(x,0), \quad (x < c). \tag{72}$$

Note that the limit is taken from outside of the crack surfaces and towards both tips. Usually the left-hand-side of integral Eq. (4) is the expression for $\sigma_{yz}(x,0)$ which is valid for $x$ is inside the crack surfaces $(c,d)$ as well as outside of $(c,d)$. Thus to calculate SIFs, the key is to evaluate the following integrals (after proper normalization and a change of variables described in Eq. (8)):

$$S_\alpha(T_n, m, r) = \int_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-(1/2)}}{(s-r)^\alpha}\,\mathrm{d}s, \quad r \notin (-1,1) \tag{73}$$

and

$$S_\alpha(U_n, m, r) = \int_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-(1/2)}}{(s-r)^\alpha}\,\mathrm{d}s, \quad r \notin (-1,1). \tag{74}$$

Note that the above integrals are not singular as $x \neq t$ for $t \in (c,d)$ and $x \notin (c,d)$.

The strategy to evaluate $S_\alpha(T_n, m, r)$ and $S_\alpha(U_n, m, r)$ for general integers $\alpha$ (positive) and $m$ (nonnegative) is similar to the process for evaluating $I_\alpha(T_n, m, r)$ and $I_\alpha(U_n, m, r)$. It consists of evaluating the integrals $S_1(T_n, m, r)$ and $S_1(U_n, m, r)$ by means of the reduction procedure described in Section 4.3, and taking differentiation (with respect to $r$) to obtain $S_\alpha(T_n, m, r)$ and $S_\alpha(U_n, m, r)$ for $\alpha \geqslant 2$. The relevant derivations are provided below. The range of $r$ is restricted to $|r| > 1$ for each formula provided in this Section 6.

### 6.1. $S_1(T_n, m, r)$ and $S_1(U_n, m, r)$

- $S_1(T_n, 0, r)$:
  This is a well known integral [46]:

$$S_1(T_n, 0, r) = \int_{-1}^{1} \frac{T_n(s)}{(s-r)\sqrt{1-s^2}}\,\mathrm{d}s = -\pi \frac{\left(r - \sqrt{r^2-1}|r|/r\right)^n}{\sqrt{r^2-1}|r|/r}, \quad n \geqslant 0. \tag{75}$$

- $S_1(U_n, 1, r)$:

$$\int_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{s-r}\,\mathrm{d}s \overset{(32)}{=} \frac{1}{2}\int_{-1}^{1} \frac{T_n(s)}{(s-r)\sqrt{1-s^2}}\,\mathrm{d}s - \frac{1}{2}\int_{-1}^{1} \frac{T_{n+2}(s)}{(s-r)\sqrt{1-s^2}}\,\mathrm{d}s$$
$$\overset{(75)}{=} -\pi\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^{n+1}, \quad n \geqslant 0. \tag{76}$$

- $S_1(T_n, 1, r)$:

$$\int_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{s-r}\,\mathrm{d}s \overset{(30)}{=} \frac{1}{2}\int_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{s-r}\,\mathrm{d}s - \frac{1}{2}\int_{-1}^{1} \frac{U_{n-2}(s)\sqrt{1-s^2}}{s-r}\,\mathrm{d}s$$
$$\overset{(76)}{=} \pi\frac{|r|}{r}\sqrt{r^2-1}\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^n, \quad n \geqslant 2. \tag{77}$$

- $S_1(U_n, 2, r)$:

$$\int_{-1}^{1} \frac{U_n(s)(1-s^2)^{3/2}}{s-r}\,\mathrm{d}s \overset{(32)}{=} \frac{1}{2}\int_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{s-r}\,\mathrm{d}s - \frac{1}{2}\int_{-1}^{1} \frac{T_{n+2}(s)\sqrt{1-s^2}}{s-r}\,\mathrm{d}s \overset{(77)}{=} \pi(r^2-1)\left(r-\frac{|r|}{r}\sqrt{r^2-1}\right)^{n+1}, \quad n \geqslant 2. \tag{78}$$

- $S_1(T_0, 2, r)$:

$$\int_{-1}^{1} \frac{T_0(s)(1-s^2)^{3/2}}{s-r}\,\mathrm{d}s = \pi(r^2-1)\left(r-\frac{|r|}{r}\sqrt{r^2-1}\right). \tag{79}$$

- $S_1(T_1, 2, r)$:

$$\int_{-1}^{1} \frac{T_1(s)(1-s^2)^{3/2}}{s-r}\,\mathrm{d}s = \frac{\pi}{2}(r^2-1)\left(r-\frac{|r|}{r}\sqrt{r^2-1}\right)^{2}. \tag{80}$$

- $S_1(T_n, 2, r)$:

$$\int_{-1}^{1} \frac{T_n(s)(1-s^2)^{3/2}}{s-r}\,\mathrm{d}s \overset{(30),(78)}{=} -\frac{\pi|r|}{r}(r^2-1)^{3/2}\left(r-\frac{|r|}{r}\sqrt{r^2-1}\right)^{n}, \quad n \geqslant 2. \tag{81}$$

Thus we obtain the following formulas for $S_1(T_n, m, r)$ and $S_1(U_n, m, r)$:

$$S_1(T_n, m, r) = \int_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{s-r}\,\mathrm{d}s = \pi(-1)^{m+1}\frac{|r|}{r}(r^2-1)^{m-1/2}\left(r-\frac{|r|}{r}\sqrt{r^2-1}\right)^{n}, \quad m \geqslant 0 \quad \text{and} \quad n \geqslant 2m. \tag{82}$$

$$S_1(U_n, m, r) = \int_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{s-r}\,\mathrm{d}s = \pi(-1)^{m}(r^2-1)^{m-1}\left(r-\frac{|r|}{r}\sqrt{r^2-1}\right)^{n}, \quad m \geqslant 1 \quad \text{and} \quad n \geqslant 2m-2. \tag{83}$$

### 6.2. $S_2(T_n, m, r)$ and $S_2(U_n, m, r)$

Differentiating (with respect to $r$) the formulas for $S_1(T_n, m, r)$ and $S_1(U_n, m, r)$, we obtain the formulas for $S_2(T_n, m, r)$ and $S_2(U_n, m, r)$.

$$\int_{-1}^{1} \frac{U_n(s)(1-s^2)^{3/2}}{(s-r)^2}\,\mathrm{d}s = -\pi(n+1)\frac{|r|}{r}\sqrt{r^2-1}\left(r-\frac{|r|}{r}\sqrt{r^2-1}\right)^{n+1} + 2\pi r\left(r-\frac{|r|}{r}\sqrt{r^2-1}\right)^{n+1}, \quad n \geqslant 0 \tag{84}$$

and

$$\int_{-1}^{1} \frac{T_n(s)(1-s^2)^{3/2}}{(s-r)^2}\,\mathrm{d}s = -\frac{\pi|r|}{r}(r^2-1)^{3/2}\left(r-\frac{|r|}{r}\sqrt{r^2-1}\right)^{n}, \quad n \geqslant 2. \tag{85}$$

Formulas (84) and (85) are used in calculating SIFs for the examples discussed in Section 7.

### 6.3. $S_3(T_n, m, r)$ and $S_3(U_n, m, r)$

The following formulas are obtained by differentiating twice (with respect to $r$) the corresponding formulas obtained in Section 6.1.

$$\int_{-1}^{1} \frac{U_n(s)(1-s^2)^{3/2}}{(s-r)^3}\,\mathrm{d}s = \frac{\pi}{2}\left[(n^2+2n+3) - 3(n+1)\frac{|r|}{\sqrt{r^2-1}}\right] \times \left(r-\frac{|r|}{r}\sqrt{r^2-1}\right)^{n+1}, \quad n \geqslant 0, \tag{86}$$

$$\int_{-1}^{1} \frac{T_1(s)(1-s^2)^{3/2}}{(s-r)^3}\,\mathrm{d}s = \frac{3\pi}{2}\left(r-\frac{|r|}{r}\sqrt{r^2-1}\right)^{2}\left(1-\frac{|r|}{\sqrt{r^2-1}}\right), \tag{87}$$

$$\int_{-1}^{1} \frac{T_0(s)(1-s^2)^{3/2}}{(s-r)^3}\,\mathrm{d}s = \frac{3\pi}{2}\left(r-\frac{|r|}{r}\sqrt{r^2-1}\right)\left(1-\frac{|r|}{\sqrt{r^2-1}}\right), \tag{88}$$

$$\int_{-1}^{1} \frac{T_n(s)(1-s^2)^{3/2}}{(s-r)^3}\,\mathrm{d}s = \frac{\pi}{4}\left\{\left(r-\frac{|r|}{r}\sqrt{r^2-1}\right)^{n+1}\left[(n^2+2n+3) - 3(n+1)\frac{|r|}{\sqrt{r^2-1}}\right] \times \left(r-\frac{|r|}{r}\sqrt{r^2-1}\right)^{n-1}\left[(n^2-2n+3) - 3(n-1)\frac{|r|}{\sqrt{r^2-1}}\right]\right\}, \quad n \geqslant 2. \tag{89}$$

The above formulas are used in calculating the SIFs for the examples discussed in the Section 7.

## 7. Examples

Three examples are presented here which emphasize various aspects of singular integral equation formulations and their linkage to fracture mechanics. These examples are:

1. Internal mode I crack in an infinite strip.
2. Mode III crack problem in nonhomogeneous materials.
3. Gradient elasticity theory applied to a mode III crack.

The first and last examples consider homogeneous materials, and the second example considers nonhomogeneous materials, which has relevant applications to the field of FGM [48,50]. The first two examples are from classical elasticity, and the last one is from gradient elasticity theory. The first example involves mode I cracks and the last two examples involve mode III cracks. All the examples are formulated by using hypersingular integral equations. For the first two examples the order of singularity $\alpha$ is 2, and for the last example $\alpha$ is 3. A detailed comparison between $U_n$ and $T_n$ representations is given in the first example. A discussion on the influence of the density function on the order of singularity of the integral equation is presented in the second and third examples. The description of the examples is summarized in Table 1. It is worth mentioning that the crack displacement profiles are plotted with respect to the normalization parameters adopted here.

### 7.1. Internal mode I crack in an infinite strip

Consider a crack in an infinite strip of homogeneous material, as illustrated by Fig. 2. The governing partial differential equations (PDEs) and boundary conditions are:

$$
\begin{aligned}
&\nabla^2 u(x,y) + \frac{2}{\kappa-1}\left(\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial x \partial y}\right) = 0, \quad -\infty < x,y < \infty, \\
&\nabla^2 v(x,y) + \frac{2}{\kappa-1}\left(\frac{\partial^2 v(x,y)}{\partial y^2} + \frac{\partial^2 v(x,y)}{\partial x \partial y}\right) = 0, \quad -\infty < x,y < \infty, \\
&\sigma_{xx}(0,y) = \sigma_{xy}(0,y) = \sigma_{xx}(h,y) = \sigma_{xy}(h,y) = 0, \quad -\infty < y < \infty, \\
&\sigma_{xy}(x,0) = 0, \quad 0 < x < h, \\
&\sigma_{yy}(x,0) = -p(x), \quad x \in (c,d), \\
&v(x,0) = 0, \quad x \notin [c,d],
\end{aligned}
\tag{90}
$$

where $u$ and $v$ are the $x$ and $y$ components of the displacement vector; $\sigma_{ij}$ is the stress tensor; $\kappa$ is an elastic constant ($\kappa = 3 - 4v$ for plane strain, $\kappa = (3 - v)/(1 + v)$ for plane stress, and $v$ is the Poisson's ratio). This problem has been studied by Kaya and Erdogan [3] by means of a $U_n$ representation, and it has also been used as a benchmark problem by Kabir et al. [51]. Here both $U_n$ and $T_n$ are employed and compared.

Table 1
Description of the examples

| Description | Example 1 | Example 2 | Example 3 |
|---|---|---|---|
| Homogeneous material | √ | | √ |
| Nonomogeneous material | | √ | |
| Classical elasticity | √ | √ | |
| Gradient elasticity | | | √ |
| Crack mode | I | III | III |
| Density function | displacement ($v$) | displacement ($w$) | slope ($\phi$) |
| Degree of singularity $\alpha$ | 2 | 2 | 3 |
| Weight function exponent, $m-(1/2)$ | 1/2 | 1/2 | 3/2 |
| Representation | $U_n$, $T_n$ | $U_n$, $T_n$ | $T_n$ |

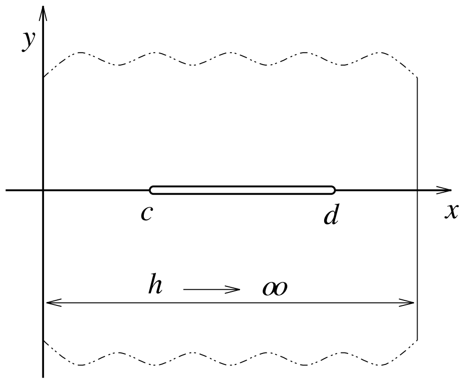

Fig. 2. A mode I crack in an infinite strip.

The governing integral equation can be written in the form given by Eq. (4) as in [3]

$$\neq\!\!\!\!\int_c^d \frac{\Delta v(t)}{(t-x)^2}\mathrm{d}t + \int_c^d k(x,t)\Delta v(t)\mathrm{d}t = -\pi\left(\frac{1+\kappa}{2G}\right)p(x), \quad c<x<d, \tag{91}$$

where $G \equiv \mu$ is the shear modulus, the primary variable is the crack opening displacement $\Delta v$ given by

$$\Delta v(x) = v(x,0^+) - v(x,0^-), \quad c<x<d$$

and the kernel $k(x,t)$ is given by Kaya and Erdogan [3, Eqs. (51)–(54c), p. 112]. It is worth noting that as $h\to\infty$ (see Fig. 2), the integral equation for the half plane is recovered (symmetry argument) and the kernel $k(x,t)$ is reduced to a much simpler form [4]

[4] To be consistent with the notation adopted in this paper, we have used symbols different from those used by Kaya and Erdogan [3]. For instance, upper case $K(t,x)$ is used by Kaya and Erdogan [3], instead of $k(t,x)$.

$$k(x,t) = \frac{-1}{(t+x)^2} + \frac{12x}{(t+x)^3} - \frac{12x^2}{(t+x)^4}.$$

After normalization, the corresponding integral equation can be written in a fashion similar to Eq. (9), i.e. [5]

$$\fint_{-1}^{1} \frac{D(s)}{(s-r)^2}\,\mathrm{d}s + \int_{-1}^{1} \mathscr{K}(r,s)D(s)\,\mathrm{d}s = P(r), \quad -1 < r < 1, \tag{92}$$

where $D(s)$ is the unknown function, the regular kernel is

$$\mathscr{K}(r,s) = \frac{-1}{\left[(r+s)+2\left(\frac{d+c}{d-c}\right)\right]^2} + \frac{12\left[s+\left(\frac{d+c}{d-c}\right)\right]}{\left[(r+s)+2\left(\frac{d+c}{d-c}\right)\right]^3} - \frac{12\left[s+\left(\frac{d+c}{d-c}\right)\right]^2}{\left[(r+s)+2\left(\frac{d+c}{d-c}\right)\right]^4}$$

and the loading function is

$$P(s) = -\pi\left(\frac{1+\kappa}{2G}\right)p\left(\left(\frac{d-c}{2}\right)s + \frac{d+c}{2}\right).$$

The case $c > 0$ represents an internal crack, which is the case of interest in this work. Based on the dominant behavior of the singular kernels of the integral equation (92), the solution takes the form

$$D(s) = R(s)\sqrt{1-s^2}.$$

Here the representation function $R(s)$ is approximated in terms of Chebyshev polynomials of first and second kinds, i.e.

$$R(s) = \sum_{n=0}^{N} a_n U_n(s) \quad \text{and} \quad R(s) = \sum_{n=0}^{N} b_n T_n(s).$$

The unknown coefficients $a_n$ and $b_n$ are determined by selecting an appropriate set of collocation points

$$r_j = \cos\left(\frac{(2n-1)\pi}{2(N+1)}\right), \ j = 1,2,\ldots,N+1; \quad \text{for } U_n \text{ representation},$$

$$r_j = \cos\left(\frac{n\pi}{N+2}\right), \ j = 1,2,\ldots,N+1; \quad \text{for } T_n \text{ representation}.$$

Once the solution is obtained, the SIFs can be calculated from [6]

[5] Again, the notation is different from the one adopted by Kaya and Erdogan [3].
[6] Kaya and Erdogan [3] do not consider the factor $\sqrt{\pi}$ in the definition of SIFs, equations (93) and (94). Note that this does not affect the normalized SIFs (see Table 2).

Table 2
Normalized SIFs for an internal crack in a half-plane

| $\frac{d+c}{d-c}$ | $N+1$ | $U_n$ representation | | $T_n$ representation | | Ref. [3] | |
|---|---|---|---|---|---|---|---|
| | | $\frac{K_I(c)}{p_0\sqrt{\pi(d-c)/2}}$ | $\frac{K_I(d)}{p_0\sqrt{\pi(d-c)/2}}$ | $\frac{K_I(c)}{p_0\sqrt{\pi(d-c)/2}}$ | $\frac{K_I(d)}{p_0\sqrt{\pi(d-c)/2}}$ | $\frac{K_I(c)}{p_0\sqrt{\pi(d-c)/2}}$ | $\frac{K_I(d)}{p_0\sqrt{\pi(d-c)/2}}$ |
| 1.01 | 15 | 3.6437 | 1.3292 | 3.8037 | 1.3313 | 3.6387 | 1.3298 |
| 1.05 | 10 | 2.1541 | 1.2535 | 2.1920 | 1.2543 | 2.1547 | 1.2536 |
| 1.1 | 10 | 1.7583 | 1.2108 | 1.7655 | 1.2111 | 1.7587 | 1.2108 |
| 1.2 | 6 | 1.4637 | 1.1625 | 1.4728 | 1.1632 | 1.4637 | 1.1626 |
| 1.3 | 6 | 1.3316 | 1.1331 | 1.3346 | 1.1335 | 1.3316 | 1.1331 |
| 1.4 | 6 | 1.2544 | 1.1123 | 1.2556 | 1.1125 | 1.2544 | 1.1123 |
| 1.5 | 4 | 1.2036 | 1.0966 | 1.2066 | 1.0969 | 1.2035 | 1.0967 |
| 2.0 | 4 | 1.0913 | 1.0539 | 1.0916 | 1.0540 | 1.0913 | 1.0539 |
| 3.0 | 4 | 1.0345 | 1.0246 | 1.0346 | 1.0246 | 1.0345 | 1.0246 |
| 4.0 | 4 | 1.0182 | 1.0141 | 1.0182 | 1.0141 | 1.0182 | 1.0141 |
| 5.0 | 4 | 1.0112 | 1.0092 | 1.0112 | 1.0092 | 1.0112 | 1.0092 |
| 10.0 | 4 | 1.0026 | 1.0024 | 1.0026 | 1.0024 | 1.0026 | 1.0024 |
| 20.0 | 4 | 1.0006 | 1.0006 | 1.0006 | 1.0006 | 1.0006 | 1.0006 |

$N+1$ terms are used in approximating the primary variable.

$$\begin{aligned} K_I(c) &= \lim_{x\to c^-}\sqrt{2\pi(c-x)}\sigma_{yy}(x,0) \quad (x<c) \\ &= \left(\frac{2G}{1+\kappa}\right)\lim_{x\to c^+}\frac{D(x)}{\sqrt{2\pi(x-c)}} \quad (x>c) \\ &= \left(\frac{2G}{1+\kappa}\right)\sqrt{\frac{d-c}{2\pi}}R(-1), \end{aligned} \tag{93}$$

$$\begin{aligned} K_I(d) &= \lim_{x\to d^+}\sqrt{2\pi(x-d)}\sigma_{yy}(x,0) \quad (x>d) \\ &= \left(\frac{2G}{1+\kappa}\right)\lim_{x\to d^-}\frac{D(x)}{\sqrt{2\pi(d-x)}} \quad (x<d) \\ &= \left(\frac{2G}{1+\kappa}\right)\sqrt{\frac{d-c}{2\pi}}R(+1), \end{aligned} \tag{94}$$

which are obtained from Eq. (91) by observing that its left-hand-side gives the stress component $\sigma_{yy}(x,0)$ outside the crack interval $(c,d)$.

Table 2 presents the SIFs at both tips of an internal crack in a half-plane ($h\to\infty$) under uniform load ($p(x)=p_0$) obtained with both $U_n$ and $T_n$ representations. First, it is worth noting that the present SIF results for the $U_n$ representation compare well with those reported in Table 1 (page 114) of the paper by Kaya and Erdogan [3] for the entire range of values describing the relative position of the crack, i.e. $1.01<(d+c)/(d-c)<20$. Next, comparing the SIFs obtained with the $U_n$ and $T_n$ representations in Table 2, we note that the results compare quite well, except when $(d+c)/(d-c)\approx 1.0$, and the discrepancy is bigger at the left-hand-side (LHS) than at the right-hand-side (RHS) crack tip. This occurs because of the "edge effect" [52]. If 42 terms (i.e. $N+1=42$) and $T_n$ representation are considered for the case $(d+c)/(d-c)=1.01$, then the

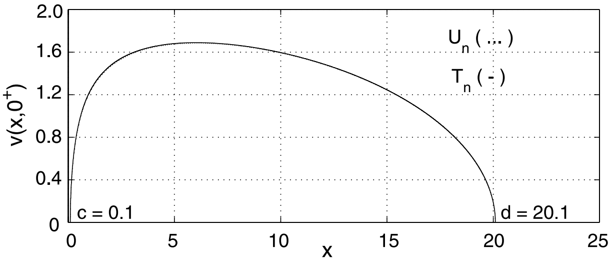


Fig. 3. Displacement profiles for a mode I crack in an infinite strip obtained by means of $U_n$ and $T_n$ representations ($N+1=15$). Here $c=0.1$, $d=20.1$, $2a=20$, and $(c+d)/(d-c)=1.01$. The crack is tilted to the left because of the "edge effect".

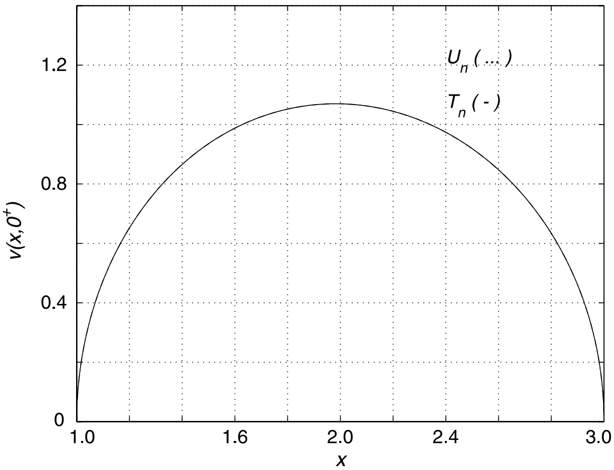


Fig. 4. Displacement profiles for a mode I crack in an infinite strip obtained by means of $U_n$ and $T_n$ representations ($N+1=8$). Here $c=1$, $d=3$, $2a=2$, and $(c+d)/(d-c)=2$.

normalized SIFs at the LHS and RHS crack tips are 3.6437 and 1.3302, respectively. Thus, when there is an "edge effect", the results are sensitive to the discretization adopted. Moreover, for the same number of collocation points, the level of accuracy attained with the $U_n$ representation is slightly different from that with the $T_n$ representation.

Figs. 3 and 4 compare the crack profiles for $U_n$ and $T_n$ representations. One may observe that the displacement profiles obtained from both representations practically agree within plotting accuracy, especially in Fig. 4. Note that the displacement profile in Fig. 3 is tilted to the left because of the "edge effect". Such effect is negligible in Fig. 4.

### *7.2. Mode III crack problem in nonhomogeneous materials [18]*

Consider the antiplane shear problem for the nonhomogeneous material shown in Fig. 5 with shear modulus variation given by

$$G(x) = G_0 e^{\beta x}, \tag{95}$$

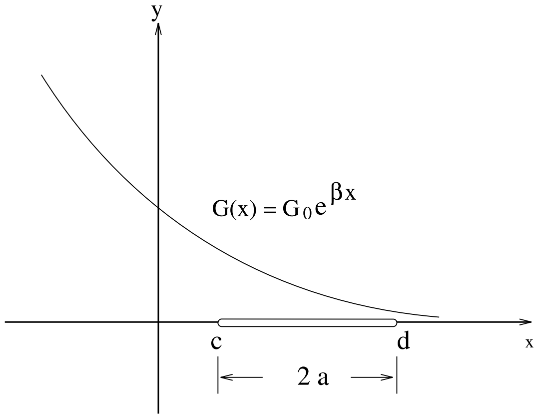


Fig. 5. The half plane of the antiplane shear problem for nonhomogeneous material with shear modulus $G(x) = G_0 e^{\beta x}$.

where $G_0$ and $\beta$ are material constants. Erdogan [18] has studied this problem in order to investigate the singular nature of the crack-tip stress field in bonded nonhomogeneous materials under antiplane shear loading. He uses a slope formulation, [7] while here we use a displacement formulation. To understand what can be gained through the displacement formulation, we first state the governing PDE and the boundary conditions for the crack problem:

$$\begin{aligned} &\nabla^2 w(x,y) + \beta \frac{\partial w(x,y)}{\partial x} = 0, && -\infty < x < \infty,\ y \geqslant 0, \\ &w(x,0) = 0, && x \notin [c,d], \\ &\sigma_{yz}(x,0^+) = p(x), && x \in (c,d), \end{aligned} \tag{96}$$

where $p(x)$ is the traction function along the crack surfaces $(c,d)$; and because of symmetry, only the upper half plane $y > 0$ is considered. By the Fourier transform we write $w(x,y)$ as

$$w(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [A(\xi) e^{\lambda(\xi) y}] e^{-ix\xi} \, d\xi, \tag{97}$$

where $A(\xi)$ is to be determined by the boundary conditions, and

$$[\lambda(\xi)]^2 = \xi^2 + i\beta\xi. \tag{98}$$

Because of the far field boundary condition, $\lim_{y\to\infty} w(x,y) = 0$, $\lambda(\xi)$ is found to have a non-positive real part which can be expressed as

$$\lambda(\xi) = \frac{-1}{\sqrt{2}} \sqrt{\sqrt{\xi^4 + \beta^2\xi^2} + \xi^2} - \frac{i}{\sqrt{2}} \operatorname{sgn}(\beta\xi) \sqrt{\sqrt{\xi^4 + \beta^2\xi^2} - \xi^2}, \tag{99}$$

---

[7] The governing integral equations are described by relationships (20)–(22) in [18].

where the sign function sgn(·) is defined as

$$\operatorname{sgn}(\eta) = \begin{cases} 1, & \eta > 0, \\ 0, & \eta = 0, \\ -1, & \eta < 0. \end{cases} \tag{100}$$

By applying the inverse Fourier transform to Eq. (97), one finds

$$A(\xi) = \frac{1}{\sqrt{2\pi}} \int_c^d w(t,0) \mathrm{e}^{\mathrm{i}t\xi} \, \mathrm{d}t, \tag{101}$$

which leads to the following integral equation: [8]

$$\sigma_{yz}(x,0) = p(x) = \frac{G(x)}{2\pi} \int_c^d k(x,t) w(t,0) \, \mathrm{d}t, \tag{102}$$

where

$$k(x,t) = \lim_{y \to 0^+} \int_{-\infty}^{\infty} [\lambda(\xi) \mathrm{e}^{\lambda(\xi)y}] \mathrm{e}^{\mathrm{i}(t-x)\xi} \, \mathrm{d}\xi. \tag{103}$$

The trade-off between a displacement-based and a slope-based formulation can be seen here if one recalls the issue regarding decomposition of the $K(\xi)$ described in (3). In summary,

$$K(\xi) = \lim_{y \to 0^+} \frac{\lambda(\xi)}{-\mathrm{i}\xi} \mathrm{e}^{\lambda(\xi)y} = \frac{\lambda(\xi)}{-\mathrm{i}\xi}, \quad \text{if slope formulation is used} \tag{104}$$

and

$$K(\xi) = \lim_{y \to 0^+} \lambda(\xi) \mathrm{e}^{\lambda(\xi)y} = \lambda(\xi), \quad \text{if displacement formulation is adopted.} \tag{105}$$

The decomposition of $K(\xi)$ in (104) is not straightforward because of the term $(-\mathrm{i}\xi)$ in the denominator, which would cause the logarithmic singularity. On the other hand, the decomposition of $K(\xi)$ in (105) can be achieved through a simple and straightforward asymptotic analysis:

$$\text{Real part of } \lambda(\xi) = \frac{-1}{\sqrt{2}} \sqrt{\sqrt{\xi^4 + \beta^2 \xi^2} + \xi^2} \overset{|\xi| \to \infty}{\sim} -|\xi|, \tag{106}$$

[8] Note that Eqs. (102) and (103) correspond to equations (20) and (21) in [18], respectively. However, the present notation is different from the one used by Erdogan [18], in which the dummy variable used for the Fourier transform is $\alpha$, $(a, b)$ stands for the crack surfaces, and $m(\alpha)$ corresponds to our $\lambda(\xi)$.

$$\mathrm{i} \times \text{Imaginary part of } \lambda(\xi) = \frac{-\mathrm{i}}{\sqrt{2}}\operatorname{sgn}(\beta\xi)\sqrt{\sqrt{\xi^4+\beta^2\xi^2}-\xi^2} \overset{|\xi|\to\infty}{\sim} -\frac{\mathrm{i}\beta}{2}\frac{|\xi|}{\xi}. \tag{107}$$

Thus the governing hypersingular integral equation is found to be

$$\frac{G(x)}{2\pi} \,\mathop{=\!\!\!\!\!\!\int}_c^d \left[\frac{2}{(t-x)^2} + \frac{\beta}{t-x} + N(x,t)\right] D(t)\,\mathrm{d}t = p(x), \quad c < x < d, \tag{108}$$

where we have let

$$D(t) = w(t,0) \tag{109}$$

and the nonsingular kernel is

$$N(x,t) = \int_0^\infty \left\{ \frac{-\beta^2\sqrt{\xi}\cos[(t-x)\xi]}{\left(\sqrt{\xi} + \frac{1}{\sqrt{2}}\sqrt{\sqrt{\xi^2+\beta^2}+\xi}\right)\left(\xi + \sqrt{\xi^2+\beta^2}\right)} + \frac{-\beta^4\sin[(t-x)\xi]}{\left(\beta + \sqrt{2}\operatorname{sgn}(\beta)\sqrt{\sqrt{\xi^4+\beta^2\xi^2}-\xi^2}\right)\left(2\xi^2+\beta^2+2\sqrt{\xi^4+\beta^2\xi^2}\right)} \right\} \mathrm{d}\xi. \tag{110}$$

Recall that the function $\operatorname{sgn}(\cdot)$ is defined by Eq. (100). As a consistency check, note that if $\beta = 0$, then both the Cauchy singular kernel $\beta/(x-t)$ and the nonsingular kernel $N(x,t)$ will be dropped from Eq. (108) so that equation (25) is recovered.

Fig. 6 shows numerical results for displacement profiles considering a crack with uniformly applied shear tractions $\sigma_{yz}(x,0) = -p_0(|x| < a)$, and various values of the material parameter $\beta$. Note that the cracks are tilted to the right because of material nonhomogeneity. Further numerical results, including SIFs at both tips of the crack and corresponding displacement profiles, are given by Chan et al. [48]. From a numerical point of view, they have shown that essentially the same results are obtained either by $U_n$ or $T_n$ representations for this specific problem [48].

### 7.3. Gradient elasticity applied to mode III cracks [53]

One of the most relevant aspects of the formulas derived in Sections 4 and 5 is the evaluation of hypersingular integrals $I_\alpha(T_n, m, r)$ with weight function $W(s) = (1-s^2)^{m-1/2}$, in which $m \geqslant 2$. This example illustrates this point for the case $m = 2$.

Fannjiang et al. [53] have presented a hypersingular integral equation formulation for a mode III crack in a material described by constitutive equations of gradient elasticity with both volumetric and surface energy gradient dependent terms. A similar study, using a different approach, has been conducted by Vardoulakis et al. [54] and Exadaktylos et al. [55]. Zhang et al. [56] and Shi et al. [57] have used the Wiener–Hopf technique of analytic continuation to get full-field solution

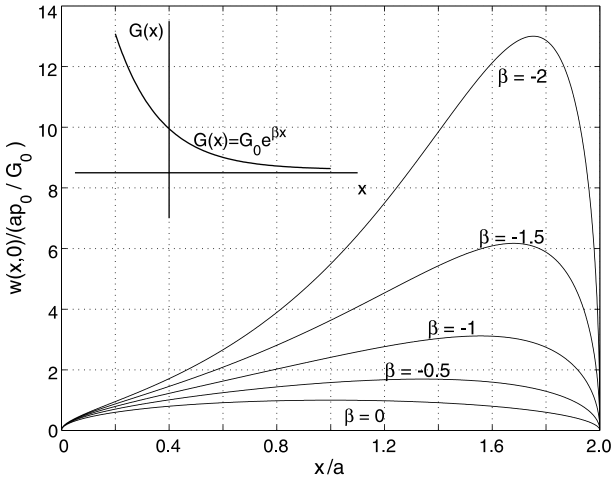


Fig. 6. The half plane of the antiplane shear problem for nonhomogeneous material with shear modulus $G(x) = G_0 e^{\beta x}$. The cracks are tilted to the right because of material nonhomogeneity.

for a semi-infinite crack in an infinite solid characterized by the higher-order elastic continuum theory proposed by Fleck and Hutchinson [58]. For this problem, the governing PDE is

$$-\ell^2 \nabla^4 w + \nabla^2 w = 0, \tag{111}$$

where $\ell$ is the characteristic length of the material associated to volumetric strain-gradient terms, and $w$ is the antiplane shear displacement. The boundary conditions are

$$\begin{aligned} &\sigma_{yz}(x,0) = p(x), && |x| < a, \\ &\mu_{yyz}(x,0) = 0, && -\infty < x < \infty, \\ &w(x,0) = 0, && |x| > a, \end{aligned} \tag{112}$$

where

$$\begin{aligned} \mu_{yyz} &= 2G(\ell^2 \partial\epsilon_{yz}/\partial y - \ell' \epsilon_{yz}), \\ \sigma_{yz} &= 2G(\epsilon_{yz} - \ell^2 \nabla^2 \epsilon_{yz}). \end{aligned}$$

$\ell'$ is the characteristic length of the material associated to surface strain-gradient terms, $\sigma_{ij}$ is the stress tensor and $\mu_{ijk}$ is the couple stress tensor. The notation of Exadaktylos et al. [55] and Paulino et al. [50] is adopted here. A brief discussion about the physical meaning of characteristic lengths in the context of Fleck and Hutchinson's [58] strain-gradient theory, which is a generalization of the higher-order linear elastic continuum theory [59,60], can be found in [57]. It is worth mentioning that experimental techniques in the field include micro-torsion [61], micro-bending [62], and micro-indentation [63]. Moreover, the characterization of actual materials, with respect to strain-gradient length-scale(s), is an active area of research and much remains to be done.

Enforcing the governing Eq. (111), imposing the boundary conditions (112), taking account of symmetry along the $x$-axis, and using Fourier transform method, Fannjiang et al. [53] have obtained the following governing hypersingular integral equation

$$\frac{G}{\pi}\oint_{-a}^{a}\left\{\frac{-2\ell^2}{(t-x)^3}+\frac{1-\left(\frac{\ell'}{2\ell}\right)^2}{t-x}+k_1(x,t)\right\}\phi(t)\,\mathrm{d}t-\frac{G\ell'}{2}\phi'(x)=p(x),\quad |x|<a, \tag{113}$$

which is a particular case of the integral equation presented by Paulino et al. [50] for nonhomogeneous materials with microstructure. In the above equation, the singularity order is $\alpha=3$, and the slope function $\phi(x)$ is defined as

$$\phi(x)=\frac{\partial w(x,0)}{\partial x}, \tag{114}$$

which satisfies the single-valuedness condition

$$\int_{-a}^{a}\phi(t)\,\mathrm{d}t=0 \tag{115}$$

and the "smooth closure condition"

$$\phi(a)=\phi(-a)=0 \tag{116}$$

for the solution of the fracture mechanics problem. Here $k_1(x,t)$ is the nonsingular kernel given by

$$k_1(x,t)=\int_0^{\infty}K_1(\xi;\ell,\ell')\sin[\xi(t-x)]\,\mathrm{d}\xi, \tag{117}$$

where

$$K_1(\xi;\ell,\ell')=\frac{(\ell'\xi/2)\left(\sqrt{\ell^2\xi^2+1}-\ell\xi\right)-[\ell'/(2\ell)]^2\left(\sqrt{\ell^2\xi^2+1}-\ell\xi\right)+(\ell'/\ell)^3/4}{\ell'/\ell-\left(\sqrt{\ell^2\xi^2+1}+\ell\xi\right)} \tag{118}$$

and $(-a,a)$ stands for the crack surfaces; $\ell'$ is the characteristic length responsible for surface strain-gradient terms; $G$ is the shear modulus of the material; $p(x)$ is the known loading function; $t$ is the integration variable, and $x$ is the collocation variable.

At $\ell'=0$, the behavior of the solution in terms of the density function $\phi(t)$, can be expressed as

$$\phi(t)=R(t)\sqrt{a^2-t^2}, \tag{119}$$

where $R(t)$ is an expansion of Chebyshev polynomials, either $T_n(t)$ or $U_n(t)$ (see [48]). Thus one can either use $I_3(T_n,1,r)$ or $I_3(U_n,1,r)$ to evaluate the corresponding hypersingular integrals that arise in solving the governing integral Eq. (113).

Note that the unknown density function is taken to be the first derivative of the displacement function, as described by Eq. (114). For this particular example, the decomposition of the original kernel into singular and nonsingular parts, stated by Eq. (3), can be accomplished by means of partial fractions [48]. In general, this step of decomposition is not an easy task, as discussed in Example 2. An alternative approach consists of considering the displacement function $w(x,0)$ as the unknown density function, and thus the decomposition of the original kernel becomes a routine asymptotic analysis. In this case, the following hypersingular integral equation with $\alpha = 4$ is obtained:

$$\frac{G}{\pi}\int_{-a}^{a}\left\{\frac{-6\ell^2}{(t-x)^4}+\frac{1-[\ell'/(2\ell)]^2}{(t-x)^2}+k_2(x,t)\right\}w(t)\,\mathrm{d}t-\frac{G\ell'}{2}w''(x)+\frac{G\ell'}{8\ell^2}[1+(\ell'/\ell)^2]w(x)$$
$$= p(x), \quad |x| < a, \tag{120}$$

in which the regular kernel $k_2(x,t)$ is given by

$$k_2(x,t) = \int_0^\infty K_2(\xi;\ell,\ell')\cos[\xi(t-x)]\,\mathrm{d}\xi, \tag{121}$$

where

$$K_2(\xi;\ell,\ell') = \frac{\ell'\left[2\ell\ell'^3\xi\left(\sqrt{\ell^2\xi^2+1}+\ell\xi\right)-\ell^2(3\ell^2+\ell'^2)\xi-\ell(\ell^2+\ell'^2)\sqrt{\ell^2\xi^2+1}+\ell'(\ell^2+\ell'^2)\right]}{-8\ell^4\left(\ell^2\xi+\ell\sqrt{\ell^2\xi^2+1}-\ell'\right)\left(\ell\xi+\sqrt{\ell^2\xi^2+1}\right)^2}. \tag{122}$$

With order of singularity $\alpha = 4$ and at the value $\ell' = 0$, the behavior of the density function $w(t)$ in Eq. (120) can be expressed by

$$w(t) = R(t)(a^2-t^2)^{3/2}. \tag{123}$$

This example motivates the whole work of this paper because the analytical evaluation of the hypersingular integral $I_4(T_n,2,r)$ or $I_4(U_n,2,r)$ is needed for successfully solving the governing integral equation (120). The expression for $I_4(T_n,2,r)$ and $I_4(U_n,2,r)$ are given by Eqs. (64) and (68), respectively.

The numerical results are presented in terms of generalized SIFs (Table 3) and displacement profiles (Fig. 7) by considering the slope function $\phi(x)$ as unknown in Eq. (113) and the $U_n$ expansion to $R(t)$ in Eq. (119). Table 3 shows the convergence of the SIFs by choosing different values of volumetric gradient dependent term, $\ell$, and letting surface energy gradient dependent term $\ell' = 0$. The displacement profile in Fig. 7 shows the cusping of the crack tips which is more physical than the parabolic shape of the crack tips in the classical linear elasticity without the cohesive-zone correction. Further numerical results and discussions are provided in Paulino et al. [50].

Table 3
Normalized generalized SIFs $K_{\rm III}(a)/(\sigma_0\sqrt{\pi a}) = (\ell/a)\sum_{n=0}^{N}(n+1)a_n$, where $a_n$ are the coefficients of the $U_n$ expantion to $R(t)$ in Eq. (119)

| $N$ | $\ell/a = 0.8$ | $\ell/a = 0.5$ | $\ell/a = 0.2$ | $\ell/a = 0.1$ | $\ell/a = 0.05$ | $\ell/a = 0.01$ | $\ell/a = 0.005$ |
|---|---|---|---|---|---|---|---|
| 11 | 0.52799576 | 0.69777466 | 0.89338314 | 0.94859611 | 0.97292423 | 0.61441745 | 0.34481071 |
| 21 | 0.52799576 | 0.69777466 | 0.89338314 | 0.94859983 | 0.97467051 | 0.97914609 | 0.83455315 |
| 31 | 0.52799576 | 0.69777466 | 0.89338314 | 0.94859983 | 0.97467051 | 0.99491034 | 0.98554527 |
| 41 | 0.52799576 | 0.69777466 | 0.89338314 | 0.94859983 | 0.97467051 | 0.99498732 | 0.99719326 |
| 51 | 0.52799576 | 0.69777466 | 0.89338314 | 0.94859983 | 0.97467051 | 0.99498737 | 0.99749398 |
| 61 | 0.52799576 | 0.69777466 | 0.89338314 | 0.94859983 | 0.97467051 | 0.99498737 | 0.99749685 |
| 71 | 0.52799576 | 0.69777466 | 0.89338314 | 0.94859983 | 0.97467051 | 0.99498737 | 0.99749685 |

Here $\ell' = 0$, and $\sigma_{yz}(x,0) = \sigma_0$, a uniform crack surface antiplane loading.

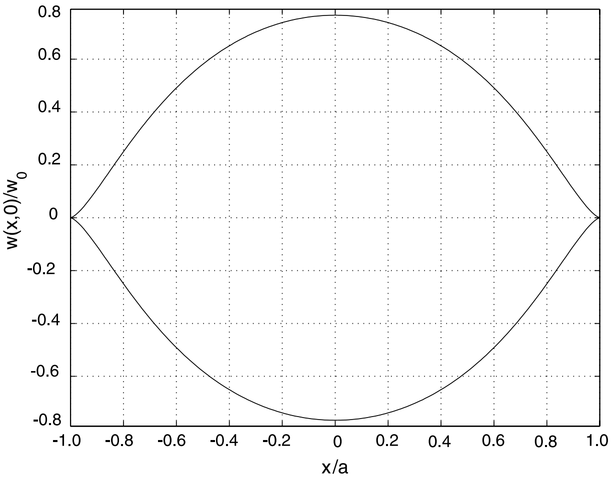


Fig. 7. Full crack displacement profile at $\ell/a = 0.2$ and $\ell'/a = 0.0$ under uniform crack surface antiplane loading $\sigma_{yz}(x,0) = -\sigma_0$; $w_0 = a\sigma_0/G_0$.

## 8. Concluding remarks

Closed form analytical solutions are provided here for a broad class of improper integrals with hypersingular kernels and density functions approximated by means of Chebyshev polynomials. Whenever possible, the symbolical and numerical tools of the computer algebra software MAPLE [9] [64–66] have been used to verify the proposed solutions. A systematic approach for evaluating integrals when higher order singularities is also given in the present paper.

The examples involve crack problems and aspects such as LEFM, nonhomogeneous materials, and gradient elasticity theory (see Table 1). All these problems are solved by means of hypersingular integral equation formulations. When classical elasticity is used, both $T_n$ and $U_n$ represen-

[9] MAPLE can evaluate some relatively simple CPV integrals. However, in general, computer algebra systems are very limited with respect to hypersingular integrals.

tations lead to essentially the same numerical results. For a crack problem in nonhomogeneous material, the difficulty that arises in splitting the singular and nonsingular parts from the original kernels can be circumvented by means of displacement based, rather than slope based, formulation.

As a closing remark, we note that as material property variation in space and higher order gradient continuum theories are considered, the formulation of the crack problem and the associated kernels can become quite involved. Thus, better analytical and numerical techniques are needed to successfully solve the governing singular integral equations. This paper is a contribution in this sense.

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## Appendix A. Alternative approach

Instead of using the recursive property between the Chebyshev polynomials $T_n(s)$ and $U_n(s)$, i.e. Eqs. (30) and (32), another way of evaluating $I_1(T_n, m, r)$ and $I_1(U_n, m, r)$ is by using the recursive property through $T_n(s)$ only. By Eqs. (30) and (32) one may obtain

$$T_n(s)(1 - s^2) = -\tfrac{1}{4}T_{n-2}(s) + \tfrac{1}{2}T_n(s) - \tfrac{1}{4}T_{n+2}(s), \quad n \geqslant 2. \tag{A.1}$$

Thus both the integrals $I_1(T_n, m+1, r)$ and $I_1(U_n, m+1, r)$ can be deduced from knowing $I_1(T_n, m, r)$. For example, an alternative way of deriving Eq. (37) is

$$\begin{aligned} \int_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{s-r}\,\mathrm{d}s &\overset{(A.1)}{=} -\frac{1}{4}\int_{-1}^{1} \frac{T_{n-2}(s)}{(s-r)\sqrt{1-s^2}} + \frac{1}{2}\int_{-1}^{1} \frac{T_n(s)}{(s-r)\sqrt{1-s^2}} - \frac{1}{4}\int_{-1}^{1} \frac{T_{n+2}(s)}{(s-r)\sqrt{1-s^2}} \\ &= \begin{cases} \frac{\pi}{2}[T_1(r) - T_3(r)], & n = 2 \\ \frac{\pi}{4}[U_{n-3}(r) + 2U_{n-1}(r) - U_{n+1}(r)], & n \geqslant 3 \end{cases} \\ &= \frac{\pi}{2}[T_{n-1}(r) - T_{n+1}(r)], \quad n \geqslant 2. \end{aligned}$$

This alternative derivation applies to all the other formulas derived in this paper, such as Eqs. (42), (43), (48), (52), (57), (61), (66) and (70).

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