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Interaction Between an Embedded Crack and an Interface Crack in Nonhomogeneous Coating System

E.E. Theotokoglou[1,a] and G.H. Paulino[2,b]

[1] Faculty of Applied Sciences, Dept. of Mechanics-Lab. of Strength of Materials
The National Technical University of Athens, Zographou Campus,
Theocaris Bld., GR-0157 73, Athens, Greece

[2] Department of Civil and Environmental Engineering
University of Illinois at Urbana-Champain, 2209 Newmark Laboratory
205 N. Mathews Ave., Urbana, IL 61801-2352, USA

[a]stathis@central.ntua.gr, [b]paulino@uiuc.edu

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Abstract. A general methodology is constructed for the fundamental solution of a crack in the homogeneous half-plane interacting with a crack at the interface between the homogeneous elastic half-plane and the nonhomogeneous elastic coating in which the shear modulus varies exponentially with one coordinate. The problem is solved under plane strain or generalized plane stress condition using the Fourier integral transform method. The stress field in the homogeneous half plane is evaluated by the superposition of two states of stresses, one is associated with a local coordinate system in the infinite fractured plate, while the other in the infinite half plane defined in a structural coordinate system.

Introduction

Turbine systems and aerospace applications require the use of structural ceramics to protect the hot sections. The thermomechanical mismatch between metal and ceramics induces high residual stresses responsible for cracking and spallation. Functionally graded materials (FGM) are composites with predetermined, continuously varying mechanical properties that reduce the residual stresses in composites [1-3]. FGM can be described as two-phase particulate composites where the volume fractions of its constituents differ continuously in the thickness direction. A number of authors have investigated cracks in nonhomogeneous materials via the singular integral equation method [4-12].

A very interesting problem in layered structures such as ceramic-coated metal substrates, is the interface crack problems [5-10]. In this work we consider the problem of a crack in the homogeneous half-plane interacting with a crack at the interface between the non-homogeneous coating and the homogeneous half-plane. A very important problem, that has not yet been addressed, is that thermal bariers always include some thickness of pure ceramic material [3].

In the plane elasticity problem shown in Fig. 1, it is assumed that the cracked half-plane is homogeneous with elastic constants μ_1, κ_1, the coating is nonhomogeneous with elastic parameters $\mu_2(y)$, $\kappa_2(y)$, and μ_2 is approximated by

$$\mu_2(y) = \mu_1 e^{\beta y}, \quad 0 < y < h_2 \tag{1}$$

where μ_i is the shear modulus, $\kappa_i = 3 - 4\nu_i$ for plane strain and $\kappa_i = (3-\nu_i)/(1+\nu_i)$ for generalized plane stress, ν_i being the Poisson's ratio $(i = 1,2)$ and β is a real constant that represents the coefficient of nonhomogeneity. In previous studies [6-8], it was shown that the influence of the variation in Poisson's ratio on stress intensity factors is rather insignificant and, therefore, κ may be assumed to be constant through-out the medium.

First, the stress and displacement fields are computed for the FGM. In the second step, the crack in the homogeneous half-plane is considered considering the superposition of the solution of the infinite half-plane with the crack with the solution of the infinite plane without the crack. Introducing the boundary conditions all the unknown are expressed in terms of the slopes of the crack displacement discontinuities along the cracks at the infinite half-plane and at the interface. Finally from the perturbation problem a system of four integral equations are derived. In this way we succeeded in expressing all the unknown coefficients in terms of the slopes of the crack displacement coefficients and in reducing the solution of the whole problem to the solution of a system of four singular integral equations.

The problem of the Nonhomogeneous Coating

By using standard Fourier transforms for the nonhomogeneous layer 2 (Fig. 1), we have

$$u_2(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{n=1}^{4}D_n(\xi)A_n(\xi)e^{\lambda_n y}e^{-ix\xi}ds\,,\quad \upsilon_2(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{n=1}^{4}A_n(\xi)e^{\lambda_n y}e^{-ix\xi}d\xi \tag{2}$$

where $A_1(\xi),\ldots,A_4(\xi)$ are unkbown functions, $\lambda_1,\ldots,\lambda_4$ are the roots of the characteristic equation, given by [8,12]

$$\left[\lambda_n^2+\beta\lambda_n-\xi^2\right]^2+\frac{3-\kappa}{\kappa+1}\beta^2\xi^2=0 \tag{3}$$

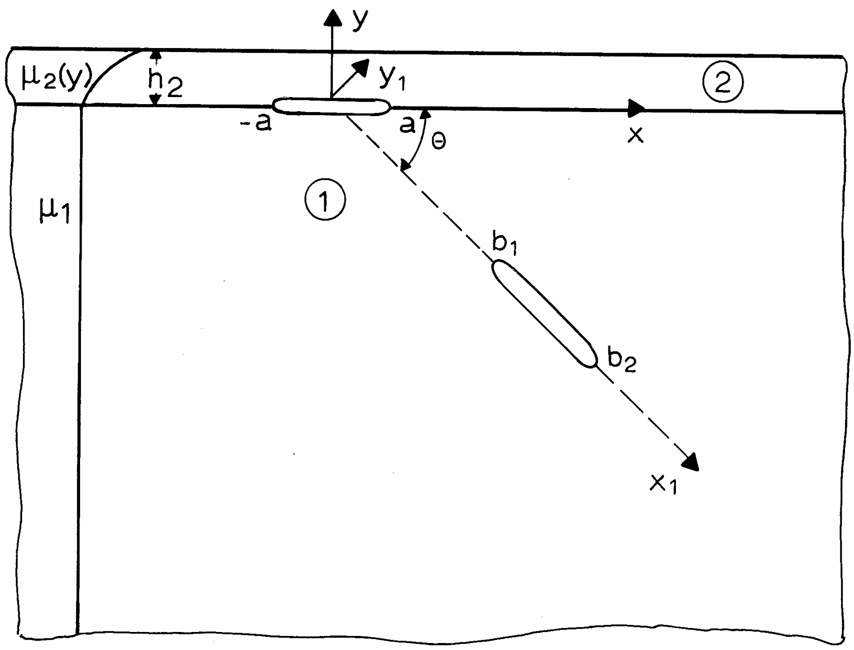

Fig. 1: A crack in the homogeneous half-plane interacting with a crack at the interface.

and

$$D_n(\xi)=\frac{2i\xi\lambda_n+i\beta\xi(\kappa-1)}{-\xi^2(\kappa+1)+\lambda_n^2(\kappa-1)+\beta\lambda_n(\kappa-1)};\quad n=1,2,3,4 \tag{4}$$

Using the strain-displacement relations and the constitutive equations [8,13], the stress field is given finally by

$$\sigma_{2yy}(x,y)=\frac{\mu_1}{(\kappa-1)\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{n=1}^{4}G_n(\xi)A_n(\xi)e^{\lambda_n y}e^{-ix\xi}d\xi\,;\quad y>0$$
$$\sigma_{2xy}(x,y)=\frac{\mu_1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{n=1}^{4}H_n(\xi)A_n(\xi)e^{\lambda_n y}e^{-ix\xi}d\xi\,;\quad y>0 \qquad (5)$$

where

$$G_n(\xi)=\lambda_n(\kappa+1)-i\xi D_n(3-\kappa),\quad H_n(\xi)=\lambda_n D_n-i\xi\,;\quad n=1,2,3,4 \qquad (6)$$

From our analysis, we have 4 *unknowns* $A_1(\xi)$, $A_2(\xi)$, $A_3(\xi)$ and $A_4(\xi)$.

A Crack in the Homogeneous Half-Plane

The stress and the displacement fields in the cracked homogeneous half-plane, considering the superposition principle, are given by

$$u_1(x,y)=u_1^{(1)}(x,y)+u_1^{(2)}(x,y)\ ,\qquad \upsilon_1(x,y)=\upsilon_1^{(1)}(x,y)+\upsilon_1^{(2)}(x,y)$$
$$\sigma_{1ij}(x,y)=\sigma_{1ij}^{(1)}(x,y)+\sigma_{1ij}^{(2)}(x,y)\,;\quad i,j=x,y \qquad (7)$$

where the superscript (1) refers to the field components *in an infinite plane with a crack* and the superscript (2) to those in the *half-plane without the crack.*

Taking into consideration the Fourier transform for the displacements $u_1^{(2)}(x,y)$ and $\upsilon_1^{(2)}(x,y)$ along the x-coordinate, and the regularity condition, $\lim_{y\to-\infty} u_1^{(2)}=\lim_{y\to-\infty}\upsilon_1^{(2)}=0$, it is finally obtained

$$u_1^{(2)}(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(R_1(\xi)+yR_2(\xi))e^{|\xi|y}e^{-ix\xi}d\xi\,,\quad -\infty<y<0$$
$$\upsilon_1^{(2)}(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(Q_1(\xi)+yQ_2(\xi))e^{|\xi|y}e^{-ix\xi}d\xi\,,\quad -\infty<y<0 \qquad (8)$$

where

$$R_1(\xi)=-\frac{i\xi}{|\xi|}Q_1(\xi)-\frac{i\kappa}{\xi}Q_2(\xi),\quad R_2(\xi)=-i\frac{\xi}{|\xi|}Q_2(\xi)\,;\quad -\infty<y\le 0 \qquad (9)$$

Using the strain-displacement relations, and the constitutive equations, [8,12], we obtain the stresses for the region, $y<0$

$$\sigma_{1xx}^{(2)}=\frac{\mu_1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|\xi|\left(-2Q_1(\xi)+\left(-2y-(\kappa+3)\frac{1}{|\xi|}\right)Q_2(\xi)\right)e^{|\xi|y-ix\xi}d\xi\,,\quad -\infty<y<0$$

$$\sigma_{1yy}^{(2)}=\frac{\mu_1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|\xi|\left(2Q_1(\xi)+\left(2y+(\kappa-1)\frac{1}{|\xi|}\right)Q_2(\xi)\right)e^{|\xi|y-ix\xi}d\xi\,,\quad -\infty<y<0 \qquad (10)$$

$$\sigma_{1xy}^{(2)} = \mu_1 \gamma_{1xy} = \frac{-\mu_1 i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \xi \left(2Q_1(\xi) + \left(2y + \frac{\kappa+1}{|\xi|} \right) Q_2(\xi) \right) e^{|\xi| y - ix\xi} d\xi \,, \quad -\infty < y < 0$$

From our analysis 2 unknowns, $Q_1(\xi)$ and $Q_2(\xi)$, are yielded

The displacement field in the cracked infinite plane, according to Fourier transform, is given by

$$u_1^{(1)}(x_1, y_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U_{32}(y_1, \xi) e^{-ix_1\xi} d\xi \,, \quad \upsilon_1^{(1)}(x_1, y_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V_{32}(y_1, \xi) e^{-ix_1\xi} d\xi \,, \quad y_1 < 0 \qquad (11)$$

taking into account that $\lim_{y_1 \to +\infty} u_1^{(1)} = \lim_{y_1 \to +\infty} \upsilon_1^{(1)} = 0$, we get

$$U_{32}(y_1, \xi) = (R_{41}(\xi) + y_1 R_{42}(\xi)) e^{|\xi| y_1} \,, \quad V_{32}(y_1, \xi) = (Q_{41}(\xi) + y_1 Q_{42}(\xi)) e^{|\xi| y_1} \,, \quad y_1 < 0 \qquad (12)$$

where

$$R_{41}(\xi) = -\frac{i\xi}{|\xi|} Q_{41}(\xi) - \frac{i\kappa}{\xi} Q_{42}(\xi) \,, \quad R_{42}(\xi) = -\frac{i\xi}{|\xi|} Q_{42}(\xi) ; \quad y_1 < 0 \qquad (13)$$

Using the strain-displacement relations [8,12], it is obtained for the region $y_1 < 0$

$$\sigma_{1y_1y_1}^{(1)} = \frac{\mu_1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\xi| \left(2Q_{41}(\xi) + \left(2y_1 + \frac{(\kappa-1)}{|\xi|} \right) Q_{42}(\xi) \right) e^{|\xi| y_1 - ix_1\xi} d\xi \,, \quad y_1 < 0$$

$$\sigma_{1x_1y_1}^{(1)} = -\frac{\mu_1 i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \xi \left(2Q_{41}(\xi) + \left(2y_1 + \frac{(\kappa+1)}{|\xi|} \right) Q_{42}(\xi) \right) e^{|\xi| y_1 - ix_1\xi} d\xi \,, \quad y_1 < 0 \qquad (14)$$

$$\sigma_{1x_1x_1}^{(1)} = -\frac{\mu_1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\xi| \left(2Q_{41}(\xi) + \left(2y_1 + \frac{(\kappa+3)}{|\xi|} \right) Q_{42}(\xi) \right) e^{|\xi| y_1 - ix_1\xi} d\xi \,, y_1 < 0$$

We introduce the slopes of the crack displacement discontinuity along the crack (b_1, b_2),

$$f_3(x_1) = \frac{\partial}{\partial x_1} \left(\upsilon_1^{(1)}(x_1, 0^+) - \upsilon_1^{(1)}(x_1, 0^-) \right), \quad f_4(x_1) = \frac{\partial}{\partial x_1} \left(u_1^{(1)}(x_1, 0^+) - u_1^{(1)}(x_1, 0^-) \right); \quad b_1 < x_1 < b_1 \qquad (15)$$

with the following properties

$$f_n(x_1) = 0 \,, \quad 0 < x_1 < b_1 \,, \quad x_1 > b_2 \,, \quad \int_{-b_1}^{b_2} f_n(x_1) dx_1 = 0 ; \quad n = 3,4 \qquad (16)$$

From the application of the inverse Fourier transform, it is finally obtained

$$\begin{Bmatrix} Q_{41}(\xi) \\ Q_{42}(\xi) \end{Bmatrix} = \begin{bmatrix} \frac{1}{2i\xi} & -\frac{(\kappa-1)}{(\kappa+1)} \frac{1}{2|\xi|} \\ -\frac{|\xi|}{i\xi(\kappa+1)} & \frac{1}{\kappa+1} \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} \qquad (17)$$

with

$$A(\xi)=\frac{1}{\sqrt{2\pi}}\int_{b_1}^{b_2} f_3(t)e^{i\xi t}dt\,,\quad B(\xi)=\frac{1}{\sqrt{2\pi}}\int_{b_1}^{b_2} f_4(t)e^{i\xi t}dt \tag{18}$$

Boundary Conditions along the Coating and along the Interface between the Coating and the Homogeneous Half-Plane

From the boundary condition along $y=h_2$ (Fig. 1),

$$\sigma_{2yy}(x,h_2)=0\,,\quad \sigma_{2xy}(x,h_2)=0\,;\quad -\infty<x<\infty \tag{19}$$

and the relations (5), we have

$$\begin{aligned} A_3(\xi)&=e^{(\lambda_1-\lambda_3)h_2}\frac{(H_1G_4-H_4G_1)}{H_4G_3-H_3G_4}A_1(\xi)+e^{(\lambda_2-\lambda_3)h_2}\frac{(H_2G_4-H_4G_2)}{H_4G_3-H_3G_4}A_2(\xi)=G_{31}(\xi)A_1(\xi)+G_{32}(\xi)A_2(\xi)\\ A_4(\xi)&=e^{(\lambda_1-\lambda_4)h_2}\frac{(G_3H_1-G_1H_3)}{H_3G_4-G_3A_4}A_1(\xi)+e^{(\lambda_2-\lambda_4)h_1}\frac{(G_3H_2-G_2H_3)}{H_3G_4-G_3A_4}A_2(\xi)=G_{41}(\xi)A_1(\xi)+G_{42}(\xi)A_2(\xi)\end{aligned} \tag{20}$$

and from the boundary conditions along $y=0$,

$$\sigma_{2yy}(x,0^+)=\sigma_{1yy}(x,0^-),\quad \sigma_{2xy}(x,0^+)=\sigma_{1xy}(x,0^-);\quad -\infty<x<\infty \tag{21}$$

A_1 and A_2 are expressed in terms of Q_1, Q_2, f_3 and f_2 . We introduce

$$f_1(x)=\frac{\partial}{\partial x}\left(\upsilon_2(x,0^+)-\upsilon_1(x,0^-)\right),\quad f_2(x)=\frac{\partial}{\partial x}\left(u_2(x,0^+)-u_1(x,0^-)\right);\quad -a<x<a \tag{22}$$

which are the slopes of the crack displacement discontinuities along the interface crack $(-a,a)$, with the following properties

$$f_n(x)=0\,,\quad |x|>a\,,\quad \int_{-a}^{a} f_n(x)dx=0\,;\quad n=1{,}2 \tag{23}$$

Application of the inverse Fourier transform to equations (22), and taking into considerations relations (2), (7), (8), (11) and (12), we get

$$\begin{aligned}\int_{-a}^{a} f_1(t)e^{i\xi t}dt&=(-i\xi)\sum_{i=1}^{4}A_n(\xi)+i\xi Q_1(\xi)+\frac{i\xi\tan\theta}{\sqrt{2\pi}}U_{32}\left(t\sin\theta,\frac{\xi}{\cos\theta}\right)+\\ &\quad+\frac{i\xi}{\sqrt{2\pi}}V_{32}\left(t\sin\theta,\frac{\xi}{\cos\theta}\right)\\ \int_{-a}^{a} f_2(t)e^{i\xi t}dt&=(-i\xi)\sum_{i=1}^{4}D_n(\xi)A_n(\xi)+|\xi|Q_1(\xi)+\kappa Q_2(\xi)+\frac{i\xi}{\sqrt{2\pi}}U_{32}\left(t\sin\theta,\frac{\xi}{\cos\theta}\right)+\\ &\quad+\frac{i\xi\tan\theta}{\sqrt{2\pi}}V_{32}\left(t\sin\theta,\frac{\xi}{\cos\theta}\right)\end{aligned} \tag{24}$$

where $D_n(\xi)$ are given by (4), $A_n(\xi)$ by (20) and (21) and U_{32}, V_{32} by (12).

From the solution of (24), we have $Q_1(\xi)$ and $Q_2(\xi)$ in terms of unknowns distribution of dislocations f_1, f_2, f_3 and f_4.

Conclusions

From our analysis, all the unknown coefficients are expressed in terms of the slopes of the crack displacement discontinuities, along the crack at the homogeneous half-plane and along the crack at the interface between the coating and the homogeneous half-plane.

The proposed procedure needs neither the inverse of a coefficients matrix [8,9] which may be create big numerical problems nor the stiffness matrix procedure [10].

Taking into consideration the four boundary conditions come from the perturbation problem, namely

$$\sigma_{1y_1y_1}(x_1,0^-)=p_1(x_1),\quad \sigma_{1x_1y_1}(x_1,0^-)=p_2(x_1);\quad b_1<x_1<b_2$$
$$\sigma_{2yy}(x,0^-)=p_3(x),\quad \sigma_{2xy}(x,0^-)=p_4(x);\quad -a<x<a$$

where p_1, p_2, p_3 and p_4 are the traction forces on the crack surfaces, σ_{2ij} $(i,j=x,y)$ are given by (5) and σ_{1ij} $(i,j=x_1,y_1)$ by (7) after an appropriate coordinate transformation, a system of four integral equation that solve the general problem is derived.

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Functionally Graded Materials VIII
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