

# Thermoelastic contact mechanics for a flat punch sliding over a graded coating/substrate system with frictional heat generation

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## Abstract

The problem of thermoelastic contact mechanics for the coating/substrate system with functionally graded properties is investigated, where the rigid flat punch is assumed to slide over the surface of the coating involving frictional heat generation. With the coefficient of friction being constant, the inertia effects are neglected and the solution is obtained within the framework of steady-state plane thermoelasticity. The graded material exists as a nonhomogeneous interlayer between dissimilar, homogeneous phases of the coating/substrate system or as a nonhomogeneous coating deposited on the substrate. The material nonhomogeneity is represented by spatially varying thermoelastic moduli expressed in terms of exponential functions. The Fourier integral transform method is employed and the formulation of the current thermoelastic contact problem is reduced to a Cauchy-type singular integral equation of the second kind for the unknown contact pressure. Numerical results include the distributions of the contact pressure and the in-plane component of the surface stress under the prescribed thermoelastic environment for various combinations of geometric, loading, and material parameters of the coated medium. Moreover, in order to quantify and characterize the singular behavior of contact pressure distributions at the edges of the flat punch, the stress intensity factors are defined and evaluated in terms of the solution to the governing integral equation.

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## 1. Introduction

The load transfer mechanism, through contacts between different solid bodies, has been a subject of vital importance in the context of its close relationship with the maintenance issues of components and parts of mechanical and structural assemblages. In order to achieve enhanced reliability and durability of such contacting bodies, especially near and at the surfaces, the utilization of the coated system has become a common practice in a broad range of modern technological applications. This is because the coating materials are essentially designed to possess superior properties to play the role of protecting the underlying substrate against the detrimental wear-, heat-, and corrosion-related damages (Schulz et al., 2003).

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When applying a conventional homogeneous coating to the substrate, however, the inevitable presence of sharp interface with the apparent mismatch of thermophysical properties may render the coating and the interface with the substrate susceptible to failure, mainly stemming from high stress concentrations, poor bonding strength, and brittleness of the coating materials. It is therefore, very likely that the enhancements gained by the coating are counteracted by the risk of such drawback. As an innovative way of coping with these limitations, functionally graded materials featuring smooth spatial variations of properties can be used in the form of an interlayer between the coating and the substrate or as a graded coating to replace the homogeneous coating. The mismatch in the properties of the constituents of the coated system can thus be minimized leading to the improved structural and tribological performances (Miyamoto et al., 1999).

Extensive overviews and thorough descriptions on the fundamental concepts and solutions to a variety of contact problems for the homogeneous materials may be found in the monographs by Gladwell (1980), Johnson (1985), and Hills et al. (1993). On the other hand, the studies on the contact mechanics involving the graded, nonhomogeneous properties appear to be relatively restricted. In an effort to keep up with the increasing usage of the graded media, Suresh and his coworkers, however, have made some mechanistic and phenomenological contributions in a series of papers dealing with the contact mechanics of the graded materials. Specifically, Giannakopoulos and Suresh (1997a, b) considered the axisymmetric problem of graded half-spaces subjected to a concentrated load and to frictionless flat, spherical, and conical indenters, respectively, in which the elastic modulus was assumed to vary according to a power or an exponential function. Analysis of parabolic frictionless indentation of the graded medium was also undertaken by Suresh et al. (1997) where comparison was made between the finite element solution and the experimental results. Subsequently, Jitcharoen et al. (1998) showed that the appropriate gradual variation of the elastic modulus significantly alters the stress field around the indenter, leading to the suppression of Hertzian cracking at the edges of the contact region. It was further illustrated that the controlled gradients in the mechanical properties of compositions and structures offer unique opportunities for the design of surfaces with improved resistance to sliding-contact deformation and damage that cannot be realized in the conventional homogeneous material (Suresh et al., 1999; Suresh, 2001). The closed-form analytical solutions to the plane elasticity problem of rigid punches on the graded substrate were given by Giannakopoulos and Pallot (2000) by considering the elastic modulus that increases monotonically with depth in terms of a power-law variation.

Besides, Stephens et al. (2000) investigated the initial yielding behavior in a hard coating/substrate system with functionally graded interface under frictional Hertzian contact based on the finite element modeling. It was indicated that the appropriate gradients in yield strength or elastic modulus could result in benefits to the reliability of the coated system compared to the case of an ungraded substrate. Guler and Erdogan (2004, 2007) examined the contact mechanics of graded coatings bonded to homogeneous substrates and loaded by frictional rigid punches with various profiles, and Guler and Erdogan (2006) also provided the solution to the problem of frictional contact between two deformable elastic solids with graded coatings, while El-Borgi et al. (2006) studied the frictionless receding contact behavior of a graded layer pressed against a homogeneous semi-infinite substrate. Moreover, Dag and Erdogan (2002) studied the surface cracking of a graded medium loaded by a sliding rigid stamp, where it was suggested that the contact problem for a graded half-plane has no solution when the medium exhibits exponentially decaying stiffness. In particular, for both frictionless and frictional indentation analyses of graded coatings with arbitrary spatial variations of shear modulus, Ke and Wang (2006, 2007) applied a multilayered model in conjunction with the transfer matrix approach (Bahar, 1972) and derived relevant singular integral equations for the unknown contact pressure distributions. It is worthwhile to mention that the results of aforementioned contact mechanics analyses could find applications where the surface wear and damage due to sliding contact are a serious concern such as in the design of load transfer components with the material property gradation near the surface (Suresh and Needleman, 1996). For the review on another class of boundary value problems pertaining to the graded, nonhomogeneous media containing crack-like flaws, the interested readers are referred to Paulino et al. (2003) and Walters et al. (2004).

In many sliding contact problems encountered in practice, a significant amount of heat may be generated due to friction entailing the thermoelastic distortion of the contacting interface, which in turn affects the contact pressure distribution and vice versa, giving rise to coupled thermomechanical response (Barber and Comninou, 1989). The thermoelastic contact problems of this type involve moving heat sources and combined normal and tangential loadings. To simplify the incumbent analysis with the frictional heating, an implicit

assumption of small sliding speed of the punch was employed by Barber (1976) discarding the effect of the convection term in the associated heat conduction analysis, and later by Yevtushenko and Kulchytsky-Zhyhailo (1995) and by Levytyskyi and Onyshkevych (1996) for the axisymmetric and plane contact configurations, respectively. A similar approach was also undertaken by Kulchytsky-Zhyhailo and Yevtushenko (1998) and Kulchytsky-Zhyhailo (2001) for thermoelastic contact analyses of layered and three-dimensional half-spaces, respectively; and more recently, by Lin and Ovaert (2006) for two-dimensional thermoelastic contact problem of a rigid indenter sliding against an anisotropic half-plane with frictional heating, yielding some interesting results.

As can be inferred from the foregoing, the earlier attempts made for the contact analysis of the graded materials are limited to the isothermal loading conditions and it thus appears that little has been done to date for the nonisothermal counterpart. The present paper is, therefore, devoted to the problem of plane thermoelastic contact mechanics of the coating/substrate system with graded properties. It is assumed that the rigid flat punch slides slowly over the surface of the coating with frictional heat generated at the contacting interface being flowed into the coated system; thereby, the inertia effects are neglected and the problem is considered using the framework of steady-state thermoelasticity. The graded material is treated as a nonhomogeneous interlayer between the dissimilar, homogeneous phases of the coated system or as a nonhomogeneous coating directly deposited on the substrate, with the corresponding spatially continuous thermoelastic moduli expressed by the exponential variations. As the method of solution and analysis, the Fourier integral transform method and the transfer matrix approach are employed, leading to the derivation of a Cauchy-type singular integral equation of the second kind for the unknown contact pressure. Implicit in this particular formulation is the requirement that the punch remain in complete thermoelastic contact with the surface of the coating. Numerical results are obtained to address the effects of various geometric, loading, and material parameters of the coated medium on the distributions of the contact pressure and the in-plane surface stress component under the given thermoelastic environment. Furthermore, with a view to quantifying the degree of criticality or the magnitude of the local intensification of singular stresses that build up inherently near the edges of the sliding frictional flat punch, in parallel with the concept used for characterizing the singular behavior of crack-tip stresses in linear elastic materials, the stress intensity factors are defined and evaluated at the locations of contact edges.

It should be remarked that for a punch sliding over the surface of the thermoelastic medium with generation of frictional heating, the history of the temperature variation can be taken into account by adding a convective term to the heat equation. In the present investigation of steady-state thermoelastic contact, however, such a convection term in the heat equation as well as the inertia terms in the Navier–Cauchy equations is suppressed, which is justifiable via the presumption of slow sliding speed of the punch such that the effect of convection would be much smaller than that of conduction.

## 2. Problem statement and formulation

The configuration of the contact problem to be considered is depicted in Fig. 1, where a homogeneous coating layer is deposited on a substrate with a graded, nonhomogeneous interlayer in-between. A flat punch of width  $2c$  is pressed against the surface of the coating by a normal force  $P$  and slides to the right at a uniform speed  $V$ . A frictional tangential force  $Q = \mu_f P$  is developed at the contacting interface by the Coulomb-type friction, with  $\mu_f$  being the constant coefficient of friction, generating the frictional heat flux  $q_f$ . To make the problem tractable, the following assumptions are made:

- the punch is rigid and nonconductive so that the flow of heat is directed only into the coated medium;
- the free surface outside the contact area is thermally insulated;
- the motion of the punch is slow so that inertia effects are neglected;
- the contact area is stationary with respect to the coating with no separation between the punch and the contact surface.

The coating, the graded interlayer, and the substrate are distinguished in order from the top with the thickness  $h_j$ ,  $j = 1, 2$ , and semi-infinite, respectively. After denoting thermal conductivity coefficients, shear

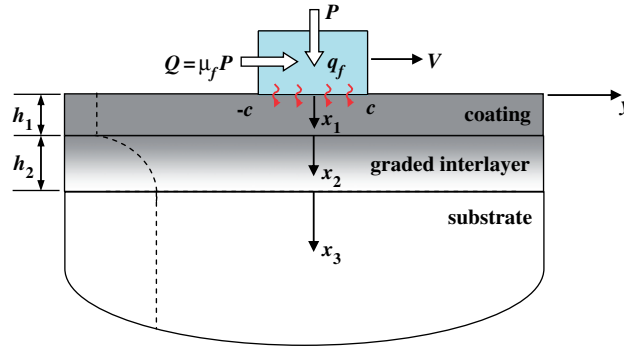


Fig. 1. Schematic of the thermoelastic contact problem involving graded properties and frictional heat generation.

moduli, and thermal expansion coefficients as  $k_j$ ,  $\mu_j$ , and  $\alpha_j$ ,  $j = 1,2,3$ , respectively, those of the interlayer are approximated as

$$k_2(x) = k_1 e^{\delta x}, \quad \mu_2(x) = \mu_1 e^{\beta x}, \quad \alpha_2(x) = \alpha_1 e^{\gamma x}, \tag{1}$$

where in the local coordinates  $(x,y) = (x_j,y)$ ,  $j = 1,2,3$ , the material gradation parameters  $\delta$ ,  $\beta$ , and  $\gamma$  (dimensionally, the reciprocal of characteristic length) are specified to render the continuous transition of the thermoelastic moduli from the coating to the substrate

$$\delta = \frac{1}{h_2} \ln \left( \frac{k_3}{k_1} \right), \quad \beta = \frac{1}{h_2} \ln \left( \frac{\mu_3}{\mu_1} \right), \quad \gamma = \frac{1}{h_2} \ln \left( \frac{\alpha_3}{\alpha_1} \right) \tag{2}$$

and the Poisson’s ratios are assumed to be constant as  $\nu_j = \nu$ ,  $j = 1,2,3$ . When  $h_1 \rightarrow 0$ , the current three-layer homogeneous coating/substrate system with a graded interlayer becomes that of a two-layer graded coating such that the thermoelastic properties of the coating vary continuously from  $(k_1, \mu_1, \alpha_1)$  at its surface to  $(k_3, \mu_3, \alpha_3)$  at the nominal interface with the substrate.

Let  $u_j(x,y)$  and  $v_j(x,y)$ ,  $j = 1,2,3$ , be the displacement components in the  $x$ - and  $y$ -directions, respectively, and  $\Theta_j(x,y)$ ,  $j = 1,2,3$ , be the temperature field measured from the reference stress-free temperature. The Duhamel–Neumann constitutive relations for the plane thermoelasticity are written as (Nowinski, 1978)

$$\sigma_{jxx} = \frac{\mu_j}{\kappa - 1} \left[ (1 + \kappa) \frac{\partial u_j}{\partial x} + (3 - \kappa) \frac{\partial v_j}{\partial y} - 4\alpha_j^* \Theta_j \right], \tag{3a}$$

$$\sigma_{jyy} = \frac{\mu_j}{\kappa - 1} \left[ (1 + \kappa) \frac{\partial v_j}{\partial y} + (3 - \kappa) \frac{\partial u_j}{\partial x} - 4\alpha_j^* \Theta_j \right], \tag{3b}$$

$$\tau_{jxy} = \mu_j \left( \frac{\partial u_j}{\partial y} + \frac{\partial v_j}{\partial x} \right), \quad j = 1, 2, 3, \tag{3c}$$

where  $\kappa = 3 - 4\nu$ ,  $\alpha_j^* = (1 + \nu)\alpha_j$  for the plane strain and  $\kappa = (3 - \nu)/(1 + \nu)$ ,  $\alpha_j^* = \alpha_j$  for the plane stress.

The steady-state heat conduction equations for the coated system subjected to the frictional heating induced by the slowly moving punch are given by

$$\nabla^2 \Theta_j + \delta \frac{\partial \Theta_j}{\partial x} = 0, \quad j = 1, 2, 3 \tag{4}$$

and the Navier–Cauchy equations of equilibrium governing the thermoelastic behavior in the absence of body forces are expressed as

$$\nabla^2 u_j + \frac{2}{\kappa - 1} \left( \frac{\partial^2 u_j}{\partial x^2} + \frac{\partial^2 v_j}{\partial x \partial y} \right) + \frac{\beta}{\kappa - 1} \left[ (1 + \kappa) \frac{\partial u_j}{\partial x} + (3 - \kappa) \frac{\partial v_j}{\partial y} \right] = \frac{4\alpha_j^* e^{\beta x}}{\kappa - 1} \left[ (\beta + \gamma) \Theta_j + \frac{\partial \Theta_j}{\partial x} \right], \tag{5a}$$

$$\nabla^2 v_j + \frac{2}{\kappa - 1} \left( \frac{\partial^2 v_j}{\partial y^2} + \frac{\partial^2 u_j}{\partial x \partial y} \right) + \beta \left( \frac{\partial v_j}{\partial x} + \frac{\partial u_j}{\partial y} \right) = \frac{4\alpha_j^* e^{\gamma x}}{\kappa - 1} \frac{\partial \Theta_j}{\partial y}, \quad j = 1, 2, 3, \tag{5b}$$

where  $\delta \neq 0$ ,  $\beta \neq 0$ ,  $\gamma \neq 0$  for the graded interlayer ( $j = 2$ ) and  $\delta = 0$ ,  $\beta = 0$ ,  $\gamma = 0$  for the homogeneous constituents ( $j = 1, 3$ ). Note that  $\alpha_2^* = \alpha_1^*$  when  $j = 2$ .

In view of the fact that the entire frictional heat flows into the coated medium through the contact area without any loss to the surroundings (Hills and Barber, 1985), and that the problem is treated within the linear thermoelasticity framework, the heat conduction analysis is to be first considered subjected to the thermal boundary and interface conditions written in the local coordinates  $(x, y) = (x_j, y)$ ,  $j = 1, 2, 3$ , as

$$k_1 \frac{\partial \Theta_1}{\partial x}(0, y) = \begin{cases} -q_f(y), & |y| < c, \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

$$\Theta_j(h_j, y) = \Theta_{j+1}(0, y), \quad \frac{\partial \Theta_j}{\partial x}(h_j, y) = \frac{\partial \Theta_{j+1}}{\partial x}(0, y), \quad j = 1, 2, \quad |y| < \infty, \tag{7}$$

$$\Theta_3(\infty, y) = 0, \quad |y| < \infty, \tag{8}$$

so that the result can be made available for incorporation into the ensuing thermal stress analysis. In this case, the displacements are known a priori within the contact area via the prescribed punch profile and the tractions beneath the punch are unknown, with the following set of mixed contact boundary and interface conditions (see Fig. 1):

$$\tau_{1xy}(0, y) = \mu_f \sigma_{1xx}(0, y), \quad u_1(0, y) = u_0, \quad |y| < c, \tag{9}$$

$$\sigma_{1xx}(0, y) = 0, \quad \tau_{1xy}(0, y) = 0, \quad |y| > c, \tag{10}$$

$$u_j(h_j, y) = u_{j+1}(0, y), \quad v_j(h_j, y) = v_{j+1}(0, y), \quad j = 1, 2, \quad |y| < \infty, \tag{11}$$

$$\sigma_{jxx}(h_j, y) = \sigma_{(j+1)xx}(0, y), \quad \tau_{jxy}(h_j, y) = \tau_{(j+1)xy}(0, y), \quad j = 1, 2, \quad |y| < \infty, \tag{12}$$

$$u_3(\infty, y) = 0, \quad v_3(\infty, y) = 0, \quad |y| < \infty, \tag{13}$$

where  $u_0$  is the indentation depth at the surface of the coating and the unique solution of the problem requires that the overall equilibrium condition be met such that

$$\int_{-c}^c \sigma_{1xx}(0, y) dy = -P. \tag{14}$$

In addition, based on the initial assumption that the frictional tangential traction inside the contact area is totally responsible for the heating effect under consideration, the magnitude of heat flux  $q_f$  into the thermally conducting coated medium is equal to the rate of frictional heat generated according to the following relation (Joachim-Ajao and Barber, 1998):

$$q_f(y) = -V \tau_{1xy}(0, y), \quad |y| < c. \tag{15}$$

To solve the governing field equations of thermoelasticity in Eqs. (4) and (5), the Fourier integral transform method is employed. The general solutions for the temperature in the local coordinates,  $(x, y) = (x_j, y)$ ,  $j = 1, 2, 3$ , that fulfill the regularity condition in Eq. (8) are readily obtained as

$$\Theta_1(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (A_{11} e^{sx} + A_{12} e^{-sx}) e^{-isy} ds, \tag{16}$$

$$\Theta_2(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 A_{2j} e^{\lambda_j x - isy} ds, \tag{17}$$

$$\Theta_3(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_{31} e^{-|s|x-isy} ds, \tag{18}$$

where  $s$  is the transform variable,  $i = (-1)^{1/2}$ , with  $\lambda_j(s), j = 1, 2$ , given by

$$\lambda_1 = -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + s^2}, \quad \lambda_2 = -\frac{\delta}{2} - \sqrt{\frac{\delta^2}{4} + s^2} \tag{19}$$

and  $A_{kj}(s), k = 1, 2, j = 1, 2$ , and  $A_{31}(s)$  are arbitrary unknowns that can be determined by applying the thermal boundary and interface conditions in Eqs. (6) and (7).

Supplemented by the complementary solutions to the homogeneous part of the governing equations in Eqs. (5a) and (5b), the general solutions for the displacements in the homogeneous coating layer ( $\delta = 0, \beta = 0, \gamma = 0$  and  $(x, y) = (x_1, y)$ ) are also obtainable in terms of the Fourier integrals (Choi, 2003)

$$u_1(x, y) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \left\{ [F_{11} + F_{12} \left(x - \frac{\kappa}{s}\right)] e^{sx} - [F_{13} + F_{14} \left(x + \frac{\kappa}{s}\right)] e^{-sx} \right\} e^{-isy} ds + \frac{2\alpha_1^*}{1 + \kappa} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ A_{11} \left(x + \frac{1}{s}\right) e^{sx} + A_{12} \left(x - \frac{1}{s}\right) e^{-sx} \right] e^{-isy} ds, \tag{20}$$

$$v_1(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [(F_{11} + F_{12}x) e^{sx} + (F_{13} + F_{14}x) e^{-sx}] e^{-isy} ds - \frac{2\alpha_1^*}{1 + \kappa} \frac{i}{2\pi} \int_{-\infty}^{\infty} x(A_{11} e^{sx} - A_{12} e^{-sx}) e^{-isy} ds, \tag{21}$$

where  $F_{1j}(s), j = 1, \dots, 4$ , are arbitrary unknowns.

For the graded, nonhomogeneous interlayer ( $\delta \neq 0, \beta \neq 0, \gamma \neq 0$  and  $(x, y) = (x_2, y)$ ), the general expressions of the displacement components can be obtained as

$$u_2(x, y) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^4 F_{2j} m_j e^{n_j x - isy} ds + \frac{4\alpha_1^* e^{\gamma x}}{\kappa - 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 A_{2j} \frac{\Phi_j}{\Delta_j} e^{\lambda_j x - isy} ds, \tag{22}$$

$$v_2(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^4 F_{2j} e^{n_j x - isy} ds + \frac{4\alpha_1^* e^{\gamma x}}{\kappa - 1} \frac{i}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 A_{2j} \frac{\Omega_j}{\Delta_j} e^{\lambda_j x - isy} ds, \tag{23}$$

where  $F_{2j}(s), j = 1, \dots, 4$ , are arbitrary unknowns,  $n_j(s), j = 1, \dots, 4$ , are the roots of the characteristic equation

$$(n^2 + \beta n - s^2)^2 + \left(\frac{3 - \kappa}{1 + \kappa}\right) \beta^2 s^2 = 0 \tag{24}$$

from which it can be shown that

$$n_j = -\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + s^2 - i(-1)^j \beta s \left(\frac{3 - \kappa}{1 + \kappa}\right)^{1/2}}, \quad \text{Re}(n_j) > 0, \quad j = 1, 2, \tag{25a}$$

$$n_j = -\frac{\beta}{2} - \sqrt{\frac{\beta^2}{4} + s^2 + i(-1)^j \beta s \left(\frac{3 - \kappa}{1 + \kappa}\right)^{1/2}}, \quad \text{Re}(n_j) < 0, \quad j = 3, 4 \tag{25b}$$

and  $m_j(s), j = 1, \dots, 4$ , are written for each root  $n_j(s), j = 1, \dots, 4$ , as

$$m_j = \frac{(\kappa - 1)(n_j^2 + \beta n_j) - (1 + \kappa)s^2}{[2n_j + (\kappa - 1)\beta]s}. \tag{26}$$

Besides, the thermoelastic constants,  $\Phi_j(s)$ ,  $\Omega_j(s)$ , and  $\Delta_j(s)$ ,  $j = 1, 2$ , in the particular solutions in Eqs. (22) and (23) are defined as

$$\Phi_j = (\beta + \gamma + \lambda_j) \left[ \left( \frac{\kappa - 1}{\kappa + 1} \right) P_j - \frac{4\kappa s^2}{\kappa^2 - 1} \right] + s^2 Q_j, \tag{27a}$$

$$\Omega_j = s(\beta + \gamma + \lambda_j) \left[ Q_j + 2\beta \left( \frac{\kappa - 2}{\kappa - 1} \right) \right] - s P_j, \tag{27b}$$

$$\Delta_j = \left[ \left( \frac{\kappa - 1}{\kappa + 1} \right) P_j - \frac{4\kappa s^2}{\kappa^2 - 1} \right] P_j + s^2 \left[ Q_j + 2\beta \left( \frac{\kappa - 2}{\kappa - 1} \right) \right] Q_j \tag{27c}$$

in which  $P_j(s)$  and  $Q_j(s)$ ,  $j = 1, 2$ , are given by

$$P_j = \left( \frac{\kappa + 1}{\kappa - 1} \right) (\gamma + \lambda_j)(\beta + \gamma + \lambda_j) - s^2, \tag{28a}$$

$$Q_j = \frac{2(\gamma + \lambda_j) + \beta(3 - \kappa)}{\kappa - 1}. \tag{28b}$$

The general solutions for the displacements in the semi-infinite homogeneous substrate ( $\delta = 0$ ,  $\beta = 0$ ,  $\gamma = 0$  and  $(x, y) = (x_3, y)$ ), satisfying the boundedness conditions in Eq. (13), are written as

$$u_3(x, y) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{s}{|s|} \left[ F_{31} + F_{32} \left( x + \frac{\kappa}{|s|} \right) \right] e^{-|s|x - isy} ds + \frac{2\alpha_3^*}{1 + \kappa} \frac{1}{2\pi} \int_{-\infty}^{\infty} A_{31} \left( x - \frac{1}{|s|} \right) e^{-|s|x - isy} ds, \tag{29}$$

$$v_3(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (F_{31} + F_{32}x) e^{-|s|x - isy} ds + \frac{2\alpha_3^*}{1 + \kappa} \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{s}{|s|} A_{31} x e^{-|s|x - isy} ds, \tag{30}$$

where  $F_{3j}(s)$ ,  $j = 1, 2$ , are arbitrary unknowns.

As can be seen in the above, there are a total of 15 unknowns;  $A_{kj}(s)$ ,  $k = 1, 2$ ,  $j = 1, 2$ , and  $A_{31}(s)$  in the general solutions for the heat conduction problem;  $F_{kj}(s)$ ,  $k = 1, 2$ ,  $j = 1, \dots, 4$ , and  $F_{3j}(s)$ ,  $j = 1, 2$ , for the thermal stress problem. Among these, the expressions for  $A_{kj}(s)$ ,  $k = 1, 2$ ,  $j = 1, 2$ , and  $A_{31}(s)$  determined from the thermal boundary and interface conditions in Eqs. (6) and (7) are listed in the Appendix. Subsequently, the interface conditions for the displacements and stresses, Eqs. (11) and (12), can be applied to eliminate the eight out of the 10 unknowns for the thermoelastic field and the mixed conditions in Eqs. (9) and (10) would yield, in principle, a pair of integral equations for the remaining two unknowns.

### 3. Application of boundary and interface conditions

In the thermoelastic contact problem at hand, the contact pressure distribution is to be determined from the requirement that the sum of the isothermal displacements due to the normal and tangential tractions and the nonisothermal displacement due to the heat input should be constant beneath the punch for the contact to be maintained. The next step in the solution procedure would be to obtain the expressions for the displacements in the coating layer subjected to arbitrary tractions and heat flux acting in a certain area on the boundary of the coated medium and the interface conditions as well. To accomplish such routine algebraic manipulations in a judicious manner, the transfer matrix approach (Bahar, 1972) is employed, with the corresponding result to be made use of in the following section in deriving an integral equation for the unknown contact pressure.

From the general solutions in Eqs. (20)–(23), (29), (30) and the constitutive relations in Eqs. (3a)–(3c), the displacements and tractions in the coated system can be written in the Fourier-transformed domain as

$$\mathbf{f}_j(x, s) = \mathbf{T}_j(x, s)\mathbf{a}_j(s) + \mathbf{f}_{jT}(x, s), \quad j = 1, 2, 3, \tag{31}$$

where  $\mathbf{f}_j(x, s)$ ,  $j = 1, 2, 3$ , are state vectors containing the physical variables that need to be determined for the given constituents,  $\mathbf{a}_j(s)$ ,  $j = 1, 2, 3$ , are vectors for the arbitrary unknowns in the general solutions of thermoelasticity equations such that

$$\mathbf{f}_j(x, s) = \{ \bar{u}_j(x, s)/i, \bar{v}_j(x, s), \bar{\sigma}_{jxx}(x, s)/i, \bar{\tau}_{jxy}(x, s) \}^T, \quad j = 1, 2, 3, \tag{32a}$$

$$\mathbf{a}_j(s) = \{ F_{j1}(s), F_{j2}(s), F_{j3}(s), F_{j4}(s) \}^T, \quad j = 1, 2, \tag{32b}$$

$$\mathbf{a}_3(s) = \{ F_{31}(s), F_{32}(s) \}^T \tag{32c}$$

and  $\mathbf{T}_j(x, s)$ ,  $j = 1, 2, 3$ , are matrices which are a function of not only the variables  $x$  and  $s$ , but also the elastic parameters of the constituents, and  $4 \times 4$  for the coating and the graded interlayer ( $j = 1, 2$ ) and  $4 \times 2$  for the substrate ( $j = 3$ ), while  $\mathbf{f}_{jT}(x, s)$ ,  $j = 1, 2, 3$ , are vectors indicating the nonisothermal effect originating from the nonhomogeneous part of the governing equations in Eqs. (5a) and (5b).

In terms of the state vectors, the appropriate boundary and interface conditions can be expressed as

$$\mathbf{f}_1^-(s) = \mathbf{f}_0(s) = \{ \bar{u}_1^-(s)/i, \bar{v}_1^-(s), \bar{\sigma}(s)/i, \bar{\tau}(s) \}^T, \tag{33}$$

$$\mathbf{f}_1^+(s) = \mathbf{f}_2^-(s), \quad \mathbf{f}_2^+(s) = \mathbf{f}_3^-(s), \tag{34}$$

where the superscript  $-/+$  denotes the upper/lower surfaces of the constituents and  $\bar{\sigma}(s)$  and  $\bar{\tau}(s)$  refer to the transformed normal and shear tractions acting on the upper surface of the coating layer

$$\bar{\sigma}(s) = \int_{-c}^c \sigma(r) e^{isr} dr, \quad \bar{\tau}(s) = \int_{-c}^c \tau(r) e^{isr} dr \tag{35}$$

and the applications of the boundary and interface conditions, Eqs. (33) and (34), to the state vector equations in Eq. (31) can remove the unknown vectors  $\mathbf{a}_j(s)$ ,  $j = 1, 2$ , in the coating and the interlayer so that the surface values of the field quantities in Eq. (33) are written in terms of the unknown vector in the substrate  $\mathbf{a}_3(s)$  such that

$$\mathbf{f}_0(s) = \mathbf{G}(s)\mathbf{a}_3(s) + \mathbf{r}_0(s), \tag{36}$$

where  $\mathbf{G}(s)$  is a  $4 \times 2$  transfer matrix between the substrate and the upper surface of the coating and  $\mathbf{r}_0(s)$  is a vector of length four containing the thermal loading:

$$\mathbf{G}(s) = \prod_{j=1}^3 \mathbf{H}_j(s), \tag{37a}$$

$$\mathbf{r}_0(s) = -i\mathbf{r}(s)\bar{q}_f(s) = \left[ \prod_{j=1}^2 \mathbf{H}_j(s) \right] [\mathbf{f}_{3T}^-(s) - \mathbf{f}_{2T}^+(s)] + \mathbf{H}_1(s) [\mathbf{f}_{2T}^-(s) - \mathbf{f}_{1T}^+(s)] + \mathbf{f}_{1T}^-(s) \tag{37b}$$

in which the matrix functions  $\mathbf{H}_j(s)$ ,  $j = 1, 2, 3$ , and the transformed heat flux  $\bar{q}_f(s)$  are given by

$$\mathbf{H}_j(s) = \mathbf{T}_j^-(s) [\mathbf{T}_j^+(s)]^{-1}, \quad j = 1, 2, \quad \mathbf{H}_3(s) = \mathbf{T}_3^-(s), \tag{38a}$$

$$\bar{q}_f(s) = \int_{-c}^c q_f(r) e^{isr} dr. \tag{38b}$$

The transfer matrix equation in Eq. (36) is then solved, eliminating the unknown vector  $\mathbf{a}_3(s)$ , for the transformed surface displacements,  $\bar{u}_1^-(s) = \bar{u}_1(0, s)$  and  $\bar{v}_1^-(s) = \bar{v}_1(0, s)$ , directly in terms of the transformed surface tractions,  $\bar{\sigma}(s)$  and  $\bar{\tau}(s)$ , and heat flux,  $\bar{q}_f(s)$ . Upon taking the inverse Fourier transform, one can show that

$$u_1(0, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [N_{11}(s)\bar{\sigma}(s) + iN_{12}(s)\bar{\tau}(s) + L_1(s)\bar{q}_f(s)] e^{-isy} ds, \quad |y| < \infty, \tag{39}$$



$$v_1(0, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [-iN_{21}(s)\bar{\sigma}(s) + N_{22}(s)\bar{\tau}(s) - iL_2(s)\bar{q}_f(s)] e^{-isy} ds, \quad |y| < \infty, \quad (40)$$

where  $N_{jk}(s)$ ,  $j, k = 1, 2$ , are elements of the  $2 \times 2$  matrix and  $L_j(s)$ ,  $j = 1, 2$ , are those of the vector of two units in length

$$\mathbf{N}(s) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} G_{31} & G_{32} \\ G_{41} & G_{42} \end{bmatrix}^{-1}, \quad (41a)$$

$$\mathbf{L}(s) = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} - \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} G_{31} & G_{32} \\ G_{41} & G_{42} \end{bmatrix}^{-1} \begin{Bmatrix} r_3 \\ r_4 \end{Bmatrix} \quad (41b)$$

in which the matrix  $\mathbf{N}(s)$  depends only on the elastic parameters of the constituents of the coated system and the vector  $\mathbf{L}(s)$  has the dependency on the thermoelastic moduli of such constituents.

#### 4. Integral equation for the thermoelastic contact mechanics

In order to dictate the correct nature of singularities the current contact problem may have, and to exclude the possibility of rigid body displacements, with the substitution of Eqs. (35) and (38b), the displacements in Eqs. (39) and (40) are differentiated with respect to the variable  $y$  to yield

$$\frac{\partial u_1}{\partial y}(0, y) = -\frac{1}{2\pi} \int_{-c}^c [iK_{11}(y, r)\sigma(r) - K_{12}(y, r)\tau(r) + iK_{13}(y, r)q_f(r)] dr, \quad |y| < \infty, \quad (42)$$

$$\frac{\partial v_1}{\partial y}(0, y) = -\frac{1}{2\pi} \int_{-c}^c [K_{21}(y, r)\sigma(r) + iK_{22}(y, r)\tau(r) + K_{23}(y, r)q_f(r)] dr, \quad |y| < \infty, \quad (43)$$

where the kernel functions,  $K_{jk}(y, r)$ ,  $j = 1, 2$ ,  $k = 1, 2, 3$ , are written as

$$K_{jk}(y, r) = \int_{-\infty}^{\infty} sN_{jk}(s)e^{is(r-y)} ds, \quad j = 1, 2, \quad k = 1, 2, \quad (44)$$

$$K_{j3}(y, r) = \int_{-\infty}^{\infty} sL_j(s)e^{is(r-y)} ds, \quad j = 1, 2 \quad (45)$$

together with the asymptotic behavior of the integrands for the variable  $s$  as follows:

$$\lim_{|s| \rightarrow \infty} sN_{11}(s) = \lim_{|s| \rightarrow \infty} sN_{22}(s) = N_1^\infty \frac{s}{|s|} = -\frac{\kappa + 1}{4\mu_1} \frac{s}{|s|}, \quad (46a)$$

$$\lim_{|s| \rightarrow \infty} sN_{12}(s) = \lim_{|s| \rightarrow \infty} sN_{21}(s) = N_2^\infty = -\frac{\kappa - 1}{4\mu_1}, \quad (46b)$$

$$\lim_{|s| \rightarrow 0} sL_1(s) = L_1^0(s) = -\frac{2\alpha_3^*}{k_3} \frac{1}{s}, \quad (46c)$$

$$\lim_{|s| \rightarrow 0} sL_2(s) = L_2^0(s) = -\frac{2\alpha_3^*}{k_3} \frac{\kappa}{\kappa + 1} \frac{1}{|s|}. \quad (46d)$$

After separating the leading terms from the kernels in Eqs. (44) and (45) and employing the Fourier representation of generalized functions (Friedman, 1991)

$$\int_0^\infty \sin s(r - y) ds = \frac{1}{r - y}, \quad (47a)$$

$$\int_0^\infty \cos s(r - y) ds = \pi\delta(r - y), \quad (47b)$$

where  $\delta(r-y)$  is the Dirac delta function, a pair of integral equations is obtained for the unknown contact tractions  $\sigma(y)$  and  $\tau(y)$  as

$$\begin{aligned} \frac{\kappa-1}{\kappa+1}\tau(y) + \frac{1}{\pi} \int_{-c}^c \frac{\sigma(r)}{r-y} dr - \frac{1}{\pi} \frac{4\mu_1}{\kappa+1} \int_{-c}^c \left[ k_{11}(y,r)\sigma(r) + k_{12}(y,r)\tau(r) + \frac{\alpha_1^*}{k_1} k_{13}(y,r)q_f(r) \right] dr \\ + \frac{4\mu_1\alpha_3^*}{(\kappa+1)k_3} \int_{-c}^c \operatorname{sgn}(r-y)q_f(r) dr = \frac{4\mu_1}{\kappa+1} \frac{\partial u_1}{\partial y}(0,y), \quad |y| < c, \end{aligned} \tag{48}$$

$$\begin{aligned} \frac{\kappa-1}{\kappa+1}\sigma(y) - \frac{1}{\pi} \int_{-c}^c \frac{\tau(r)}{r-y} dr - \frac{1}{\pi} \frac{4\mu_1}{\kappa+1} \int_{-c}^c \left[ k_{21}(y,r)\sigma(r) - k_{22}(y,r)\tau(r) + \frac{\alpha_1^*}{k_1} k_{23}(y,r)q_f(r) \right] dr \\ - \frac{1}{\pi} \frac{8\mu_1\alpha_3^*\kappa}{(\kappa+1)^2 k_3} \int_{-c}^c \ln|r-y|q_f(r) dr = \frac{4\mu_1}{\kappa+1} \frac{\partial v_1}{\partial y}(0,y), \quad |y| < c \end{aligned} \tag{49}$$

provided the surface slope of the punch profile is prescribed and the complete thermoelastic contact condition is sustained beneath the punch, where the kernels  $k_{jk}(y,r)$ ,  $j = 1,2$ ,  $k = 1,2,3$  are bounded and given by

$$k_{ij}(y,r) = \int_0^\infty [sN_{ij}(s) - N_1^\infty] \sin s(r-y) ds, \quad (i,j) = (1,1), (2,2), \tag{50a}$$

$$k_{ij}(y,r) = \int_0^\infty [sN_{ij}(s) - N_2^\infty] \cos s(r-y) ds, \quad (i,j) = (1,2), (2,1), \tag{50b}$$

$$k_{13}(y,r) = \frac{k_1}{\alpha_1^*} \int_0^\infty [sL_1(s) - L_1^0(s)] \sin s(r-y) ds, \tag{50c}$$

$$k_{23}(y,r) = \frac{k_1}{\alpha_1^*} \int_0^\infty [sL_2(s) - L_2^0(s)] \cos s(r-y) ds. \tag{50d}$$

Now that the punch slides relative to the coated medium against friction, generating the frictional heat, the following relations hold within the contact area:

$$\sigma_{1xx}(0,y) = \sigma(y) = -p(y), \quad |y| < c, \tag{51}$$

$$\tau_{1xy}(0,y) = \tau(y) = -\mu_f p(y), \quad |y| < c, \tag{52}$$

$$q_f(y) = \mu_f V p(y), \quad |y| < c, \tag{53}$$

hence the formulation of the thermoelastic contact problem is reduced to solving a Cauchy-type singular integral equation of the second kind for the unknown contact pressure  $p(y)$  as

$$\mu_f \frac{\kappa-1}{\kappa+1} p(y) + \frac{1}{\pi} \int_{-c}^c \frac{p(r)}{r-y} dr + \int_{-c}^c k_1(y,r)p(r) dr = f(y), \quad |y| < c, \tag{54}$$

where the function  $f(y)$  and the bounded kernel  $k_1(y,r)$  are written as

$$f(y) = \frac{4\mu_1}{\kappa+1} \frac{\partial u_1}{\partial y}(0,y), \tag{55}$$

$$k_1(y,r) = -\frac{1}{\pi} \frac{4\mu_1}{\kappa+1} \left\{ k_{11}(y,r) + \mu_f k_{12}(y,r) - \frac{\mu_f V \alpha_1^*}{k_1} \left[ k_{13}(y,r) - \pi \frac{\alpha_3^* k_1}{\alpha_1^* k_3} \operatorname{sgn}(r-y) \right] \right\} \tag{56}$$

subjected to the satisfaction of equilibrium with the resultant contact force  $P$

$$\int_{-c}^c p(y) dy = P. \tag{57}$$

Because the dominant singular kernel in the integral equation is the Cauchy-type, the contact pressure  $p(y)$  can be expressed as (Muskhelishvili, 1953)

$$p(y) = (c-y)^\lambda (c+y)^\omega F(y), \quad |y| < c, \tag{58}$$

where  $F(y)$  signifies a continuous and bounded function, and in the normalized interval as  $r = c\eta$ ,  $y = c\xi$ , the fundamental function that characterizes the nature of the contact pressure is found to be the weight function of Jacobi polynomials (Gradshteyn and Ryzhik, 1980). As a result, the solution to the singular integral equation can be expressed in terms of the series expansion such that

$$p(y) = \phi(\xi) = w(\xi) \sum_{n=0}^{\infty} c_n P_n^{(\chi, \omega)}(\xi), \quad w(\xi) = (1 - \xi)^\chi (1 + \xi)^\omega, \quad |\xi| < 1, \tag{59}$$

where  $c_n$ ,  $n \geq 0$ , are unknown coefficients to be evaluated via the procedure developed by Erdogan (1978),  $P_n^{(\chi, \omega)}(\xi)$  are the Jacobi polynomials associated with the weight function  $w(\xi)$ , and the physics of the problem for the rigid flat punch requires that both the constants  $\chi$  and  $\omega$  be negative and determined as

$$\chi = \frac{\theta}{\pi}, \quad \omega = -\frac{\theta}{\pi} - 1, \quad \tan \theta = -\frac{1}{\mu_f} \frac{\kappa + 1}{\kappa - 1}, \quad -1 < (\chi, \omega) < 0 \tag{60}$$

from which it is evident that the values of  $\chi$  and  $\omega$  as the powers of stress singularity at the leading ( $y = c$ ) and the trailing ( $y = -c$ ) edges of the punch, respectively, are dependent only on the friction coefficient  $\mu_f$  and the Poisson’s ratio through  $\kappa$ , as for the case of the isothermal counterpart (Guler and Erdogan, 2004).

With the surface slope of the flat punch being zero within the contact area by Eq. (9), i.e.,  $f(y) = 0$  in Eq. (55), after substituting from Eq. (59) into Eqs. (54) and (57), truncating the series at  $n = N$ , and regularizing the singular part based on the properties of the Jacobi polynomials (Gradshteyn and Ryzhik, 1980), one can show that the singular integral equation and the equilibrium condition become

$$\sum_{n=0}^N c_n \left[ -\frac{1}{2 \sin \pi \chi} P_{n-1}^{(-\chi, -\omega)}(\xi) + h_n(\xi) \right] = 0, \tag{61}$$

$$\sum_{n=0}^N c_n \int_{-1}^1 w(\xi) P_n^{(\chi, \omega)}(\xi) d\xi = \frac{P}{c}, \tag{62}$$

where the function  $h_n(\xi)$  is given by

$$h_n(\xi) = c \int_{-1}^1 k_1(\xi, \eta) P_n^{(\chi, \omega)}(\eta) (1 - \eta)^\chi (1 + \eta)^\omega d\eta \tag{63}$$

and the functional equations in Eqs. (61) and (62) can be recast into solvable form by means of the orthogonality of  $P_n^{(\chi, \omega)}(\xi)$  to construct a system of linear algebraic equations for  $c_n$ ,  $0 \leq n \leq (N + 1)$  as

$$c_0^* \theta_0^{(\chi, \omega)} = 1, \quad -\frac{\theta_k^{(-\chi, -\omega)}}{2 \sin \pi \chi} c_{k+1}^* + \sum_{n=0}^N d_{kn} c_n^* = 0, \quad k = 0, 1, 2, \dots, N, \tag{64}$$

along with the following definitions:

$$c_n^* = \frac{c}{P} c_n, \quad n = 0, 1, 2, \dots, (N + 1), \tag{65a}$$

$$d_{kn} = \int_{-1}^1 h_n(\xi) P_k^{(-\chi, -\omega)}(\xi) (1 - \xi)^{-\chi} (1 + \xi)^{-\omega} d\xi, \tag{65b}$$

$$\theta_k^{(\chi, \omega)} = \begin{cases} \frac{2^{\chi+\omega+1}}{2k + \chi + \omega + 1} \frac{\Gamma(k + \chi + 1) \Gamma(k + \omega + 1)}{k! \Gamma(k + \chi + \omega + 1)}, & k \geq 1, \\ \frac{2^{\chi+\omega+1} \Gamma(\chi + 1) \Gamma(\omega + 1)}{\Gamma(\chi + \omega + 2)}, & k = 0. \end{cases} \tag{65c}$$

By solving the system of equations in Eq. (64), the normal contact traction beneath the flat punch can be evaluated as

$$\sigma_{1xx}(0, y) = -p(y) = -P(c - y)^\lambda (c + y)^\omega \sum_{n=0}^{N+1} c_n^* P_n^{(\lambda, \omega)}\left(\frac{y}{c}\right), \quad |y| < c, \quad (66)$$

where the expression for the in-plane stress component  $\sigma_{1yy}(0, y)$  acting on the surface of the coating can be obtained from the constitutive relations in Eqs. (3a)–(3c) and Eq. (49) as (Guler and Erdogan, 2004)

$$\sigma_{1yy}(0, y) = -p(y) + \frac{2\mu_f}{\pi} \int_{-c}^c \frac{p(r)}{r - y} dr + 2 \int_{-c}^c k_2(y, r) p(r) dr - \frac{8\mu_1 \alpha_1^*}{\kappa + 1} \Theta_1(0, y), \quad |y| < \infty, \quad (67)$$

where the function  $k_2(y, r)$  is the bounded kernel such that

$$k_2(y, r) = \frac{1}{\pi} \frac{4\mu_1}{\kappa + 1} \left\{ k_{21}(y, r) - \mu_f k_{22}(y, r) - \frac{\mu_f V \alpha_1^*}{k_1} \left[ k_{23}(y, r) + \frac{2\alpha_3^* k_1}{\alpha_1^* k_3} \frac{\kappa}{\kappa + 1} \ln |r - y| \right] \right\}. \quad (68)$$

Moreover, in order to characterize in a quantitative manner the severity of singular behavior of the contact pressure at the edges of the flat punch, in parallel with the concept used for analyzing the crack-like flaws in linear elastic materials, the stress intensity factors are defined and evaluated in terms of the solution to the integral equation as

$$K_T = \lim_{y \rightarrow -c} p(y)(c + y)^{-\omega} = K_{0T} \sum_{n=0}^{N+1} c_n^* P_n^{(\lambda, \omega)}(-1), \quad (69)$$

$$K_L = \lim_{y \rightarrow c} p(y)(c - y)^{-\lambda} = K_{0L} \sum_{n=0}^{N+1} c_n^* P_n^{(\lambda, \omega)}(+1), \quad (70)$$

where  $K_{0T} = P(2c)^\lambda$  and  $K_{0L} = P(2c)^\omega$  are the normalizing factors, and the suffix T refers to the trailing edge of the punch ( $y = -c$ ) and the suffix L is for the leading edge of the punch ( $y = c$ ).

## 5. Results and discussion

Numerical results are obtained for various combinations of geometric ( $h_1/h_2$ ), loading ( $V$ ,  $c/h_2$ ,  $\mu_f$ ), and thermoelastic parameters ( $k_1/k_3$ ,  $\mu_1/\mu_3$ ,  $\alpha_1/\alpha_3$ ) of the coated medium. The state of plane strain is assumed with the constant Poisson's ratio,  $\nu = 0.3$ . The Jacobi-type integrals in Eqs. (63) and (65b) are evaluated based on the Gauss–Jacobi quadrature formula with 60 collocation points and the improper integrals in Eqs. (50a)–(50d) are evaluated employing the Gauss–Legendre quadrature formula (Davis and Rabinowitz, 1984), with a 12-term expansion of the Jacobi polynomials in Eq. (59). The outlined numerical scheme is found to be sufficient for the solutions to converge to the desired degree of accuracy for the geometric, loading, and material configurations considered in this study, such that neither increasing the number of integration or collocation points nor increasing the number of terms in the expansion of the Jacobi polynomials has an appreciable effect on the results. It is to be remarked that for verification purposes, the work by Guler and Erdogan (2004) for isothermal contact of graded coatings can be recovered when  $h_1/h_2 = 0.0$  and  $V = 0.0$  as a particular case of the present formulation and implementation for thermoelastic contact.

### 5.1. Effects of geometric and loading parameters

Specific results are first presented for the material pair that is representative of a zirconia ceramic coating ( $ZrO_2$ ) deposited on a titanium-based metallic substrate (Ti–6Al–4V), with  $k_1/k_3 = 0.1125$ ,  $\mu_1/\mu_3 = 1.7674$ ,  $\alpha_1/\alpha_3 = 0.6903$  (Fujimoto and Noda, 2001) and  $h_1/h_2 = 1.0$ ,  $c/h_2 = 0.2$ ,  $\mu_f = 0.5$ , unless otherwise stated (see Table 1 for the actual material properties). To be additionally mentioned at this point is that in the resolution of thermoelastic contact problems, the excessive amount of heat flux between the contacting bodies may give rise to the separation of contacting surfaces, enforcing the contact stress distribution to turn into positive and tensile around the edges of the punch (Barber and Comninou, 1989). In order to circumvent such erratic

Table 1  
Thermoelastic properties of ceramic and metal (Fujimoto and Noda, 2001)

Properties	ZrO <sub>2</sub>	Ti-6Al-4V
Thermal conductivity: $k_1, k_3$	2.036 W (m K) <sup>-1</sup>	18.1 W (m K) <sup>-1</sup>
Elastic modulus: $E_1, E_3$	117.0 GPa	66.2 GPa
Thermal expansion coefficient: $\alpha_1, \alpha_3$	$7.11 \times 10^{-6} \text{ K}^{-1}$	$10.3 \times 10^{-6} \text{ K}^{-1}$

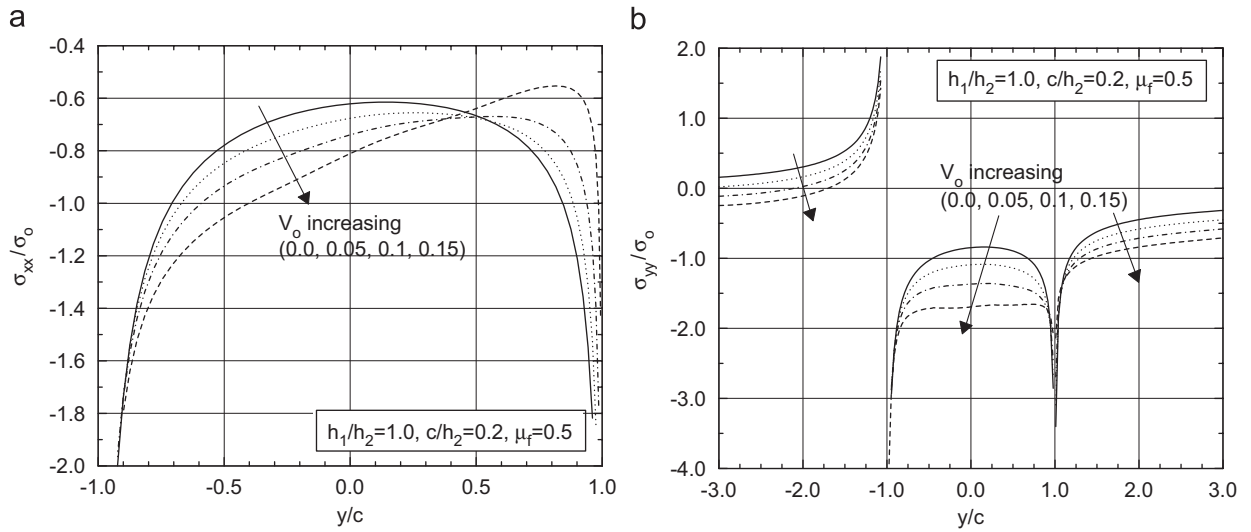


Fig. 2. (a) Contact stress  $\sigma_{1xx}(0,y)/\sigma_o$  and (b) in-plane stress  $\sigma_{1yy}(0,y)/\sigma_o$  distributions on the surface of the coated medium for different values of the nondimensional sliding speed  $V_o = \mu_1 \alpha_1^* \mu_f Vc / k_1(\kappa + 1)$  of the flat punch ( $h_1/h_2 = 1.0$ ,  $c/h_2 = 0.2$ ,  $\mu_f = 0.5$ ,  $\sigma_o = P/2c$ ).

behavior and, at the same time, to satisfy the condition of perfect contact between the punch and the coating, the heat flux should remain below certain levels, which, in this case, depends on the sliding speed of the punch as well as the friction coefficient. For the current analysis, the sliding speed of the punch should thus be properly chosen to maintain such a magnitude of the frictional heat flow that ensures the complete thermoelastic contact condition during the indentation of the punch (Joachim-Ajao and Barber, 1998).

For the graded interlayer,  $h_1/h_2 = 1.0$ , the resulting distributions of normalized contact stress  $\sigma_{1xx}(0,y)/\sigma_o$  and in-plane stress  $\sigma_{1yy}(0,y)/\sigma_o$  are plotted in Figs. 2a and b, respectively, where  $\sigma_o = P/2c$  is the average contact pressure for different values of nondimensional sliding speed of the punch defined as  $V_o = \mu_1 \alpha_1^* \mu_f Vc / k_1(\kappa + 1)$ . In this case, the powers of stress singularity at the leading ( $y = c$ ) and the trailing ( $y = -c$ ) edges of the flat punch as determined from Eq. (60) are  $\chi = -0.4548$  and  $\omega = -0.5452$ , respectively. The fact that the stronger singularity prevails at the trailing edge than at the leading edge is reflected in Fig. 2a such that there are greater stress concentrations around the trailing end of the punch. When the nonisothermal effect is taken into account accompanied by the increase in the sliding speed of the punch, the contact stress distributions tend to be markedly skewed near the leading edge of the punch and become attenuated. The in-plane component of the stress as illustrated in Fig. 2b clearly depicts that its magnitude is also unbounded and discontinuous at both edges of the punch. In particular, when the punch is stationary as  $V_o = 0.0$  corresponding to the isothermal contact, the in-plane stress just behind the trailing edge ( $y < -c$ ) exhibits tensile and unbounded behavior as well, which is recovered by the present solution. This has relevance to the well-known experimental findings (Suresh et al., 1999) that the trailing edge is a more likely location of surface crack initiation and propagation in load transfer components. When the sliding speed of the punch becomes greater, however, it is predicted that the in-plane stress is rendered less tensile behind the trailing edge and more compressive in the other region of the coating surface ( $y > -c$ ), indicating the suppression of the aforementioned tendency of surface cracking at the trailing edge.

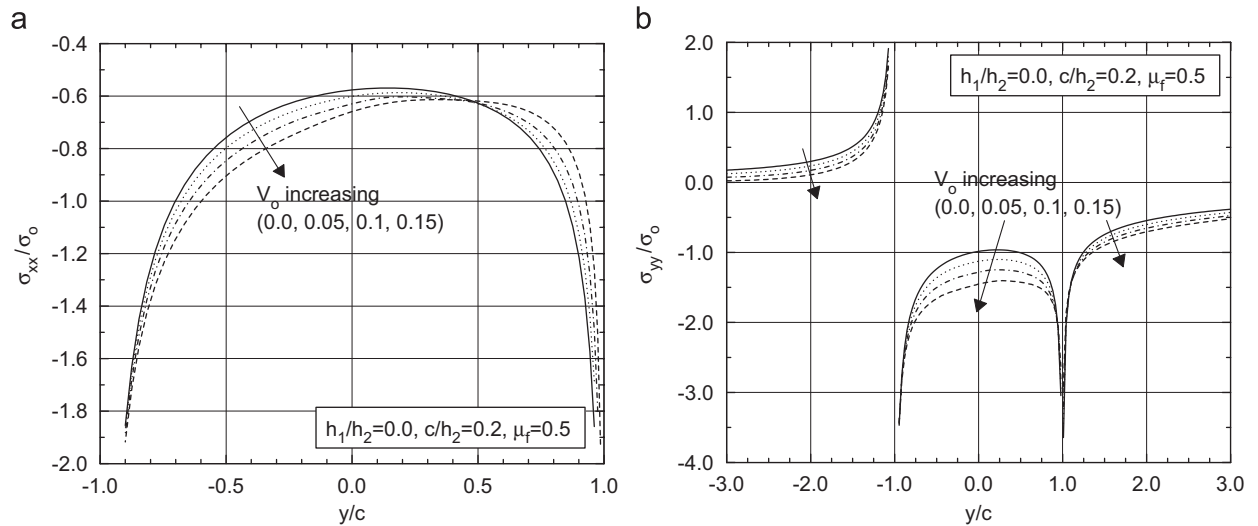


Fig. 3. (a) Contact stress  $\sigma_{1,xx}(0,y)/\sigma_0$  and (b) in-plane stress  $\sigma_{1,yy}(0,y)/\sigma_0$  distributions on the surface of the coated medium for different values of the nondimensional sliding speed  $V_0 = \mu_1 \alpha_1 \mu_f V c / k_1 (\kappa + 1)$  of the flat punch ( $h_1/h_2 = 0.0, c/h_2 = 0.2, \mu_f = 0.5, \sigma_0 = P/2c$ ).

Table 2

Normalized stress intensity factors  $K_T/K_{0T}$  and  $K_L/K_{0L}$  at the edges of the flat punch for different values of  $V_0$  and  $h_1/h_2$  ( $c/h_2 = 0.2, \mu_f = 0.5, K_{0T} = P(2c)^\nu, K_{0L} = P(2c)^\omega$ )

$V_0$	$h_1/h_2 = 1.0$		$h_1/h_2 = 0.0$	
	$K_T/K_{0T}$	$K_L/K_{0L}$	$K_T/K_{0T}$	$K_L/K_{0L}$
0.0	0.3340	0.2990	0.3723	0.3110
0.05	0.3139	0.2432	0.3609	0.2761
0.1	0.2965	0.1786	0.3503	0.2380
0.15	0.2867	0.0993	0.3411	0.1958

The problem of the graded coating,  $h_1/h_2 = 0.0$ , is next considered with the aid of Figs. 3a and b. A notable feature to be found from the comparison between Figs. 2 and 3 is that, despite the same strength of stress singularities as for the graded interlayer model, the near-edge stress state for the graded coating appears to become somewhat intensified and the leading edge is experiencing less skewed stress distributions as the sliding speed  $V_0$  increases. In addition, the in-plane stress component behind the trailing edge remains tensile for the given values of  $V_0$ , with the implication that the two-layer graded coating/substrate system, to some extent, would be more vulnerable to the sliding-contact surface damage near the trailing edge of the punch than the three-layer coating/substrate system with a graded interlayer.

Although the results in Figs. 2 and 3 confirm the existence of the severe stress field in the close vicinity of the punch edges, they are not explicit enough to measure the criticality of the singular nature of the pressure distributions inherent in all rigid flat punch problems. In this respect, the normalized stress intensity factors,  $K_T/K_{0T}$  and  $K_L/K_{0L}$ , are evaluated from Eqs. (69) and (70) and provided in Table 2 for different values of  $V_0$  and  $h_1/h_2$ , from which it is obvious that those at the trailing edge are of greater magnitude than at the leading edge. Moreover, it is demonstrated that the stress intensities decrease at both edges of the punch as the sliding speed  $V_0$  is increased and the presence of the homogeneous coating over the graded interlayer, e.g.,  $h_1/h_2 = 1$ , tends to alleviate the severity of stress state at the edges.

Figs. 4a and b illustrate how the width of the sliding punch,  $c/h_2$ , affects the contact stress field, where in these figures and the others that follow, the results are for the coating/substrate system with a graded interlayer ( $h_1/h_2 = 1.0$ ) and the nondimensional sliding speed is set equal to  $V_0 = 0.1$ . A comment to be made

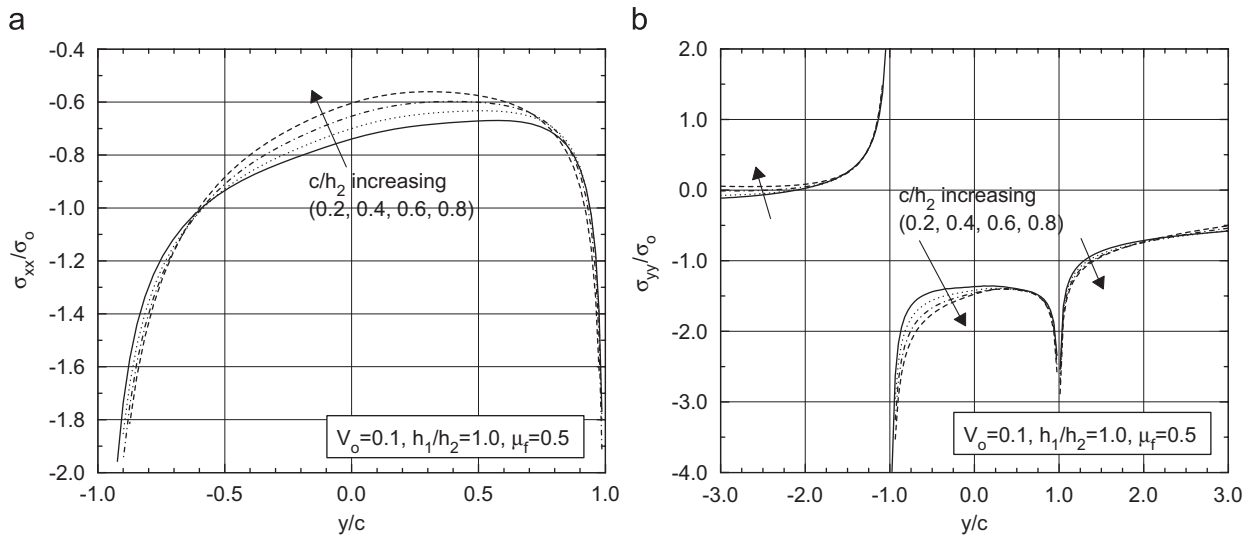


Fig. 4. (a) Contact stress  $\sigma_{1,xx}(0,y)/\sigma_0$  and (b) in-plane stress  $\sigma_{1,yy}(0,y)/\sigma_0$  distributions on the surface of the coated medium for different values of the width  $c/h_2$  of the flat punch ( $V_0 = 0.1$ ,  $h_1/h_2 = 1.0$ ,  $\mu_f = 0.5$ ,  $\sigma_0 = P/2c$ ).

Table 3

Normalized stress intensity factors  $K_T/K_{0T}$  and  $K_L/K_{0L}$  at the edges of the flat punch for different values of  $c/h_2$  ( $V_0 = 0.1$ ,  $h_1/h_2 = 1.0$ ,  $\mu_f = 0.5$ ,  $K_{0T} = P(2c)^{\lambda}$ ,  $K_{0L} = P(2c)^{\omega}$ )

	$c/h_2 = 0.2$	$c/h_2 = 0.4$	$c/h_2 = 0.6$	$c/h_2 = 0.8$
$K_T/K_{0T}$	0.2965	0.3242	0.3500	0.3735
$K_L/K_{0L}$	0.1786	0.1820	0.1948	0.2125

for Fig. 4a is that some reduction in the magnitude of contact pressure (within the contact area) in proportion to  $c/h_2$  is offset by the greater stress concentrations near the edges of the punch for the enlarged punch width. Such a near-edge response is also predictable, especially at the trailing edge of the punch, from the values of stress intensity factors given in Table 3. The results in Fig. 4b, however, depict the less significant effect of the punch width on the in-plane stress component.

With the coefficient of friction  $\mu_f$  ranging from 0.1 to 0.8 and  $c/h_2 = 0.2$ , the variations of contact traction distributions are examined in Figs. 5a and b. When the sliding contact between the punch and the coating surface is more frictional, it can be pointed out from Fig. 5a that the stress concentration becomes greater in the region close to the trailing edge of the punch. On the other hand, the reverse trend of stress relaxation is observed near the leading edge. Note that the above descriptions regarding the near-edge behavior are in parallel with those of the stress intensity factors evaluated and specified in Table 4 for different values of  $\mu_f$ . In addition, such a response near the trailing edge in Fig. 5a is well correlated with that of the in-plane stress component shown in Fig. 5b such that the increase in the friction coefficient may cause the in-plane stress to be totally tensile and unbounded behind the trailing edge of the contact area.

### 5.2. Effects of thermoelastic parameters

In the sequel, additional results are presented in Figs. 6–8 in order to gain an insight into the influences of material property variations on the contact stress field that develops during the heat-generating frictional sliding of the punch. To this end, one of the three thermoelastic parameters ( $k_1/k_3$ ,  $\mu_1/\mu_3$ ,  $\alpha_1/\alpha_3$ ) is taken to be variable and the other two parameters remain the same as those specified in the beginning of this section. Table 5 provides the corresponding values of the normalized stress intensity factors.

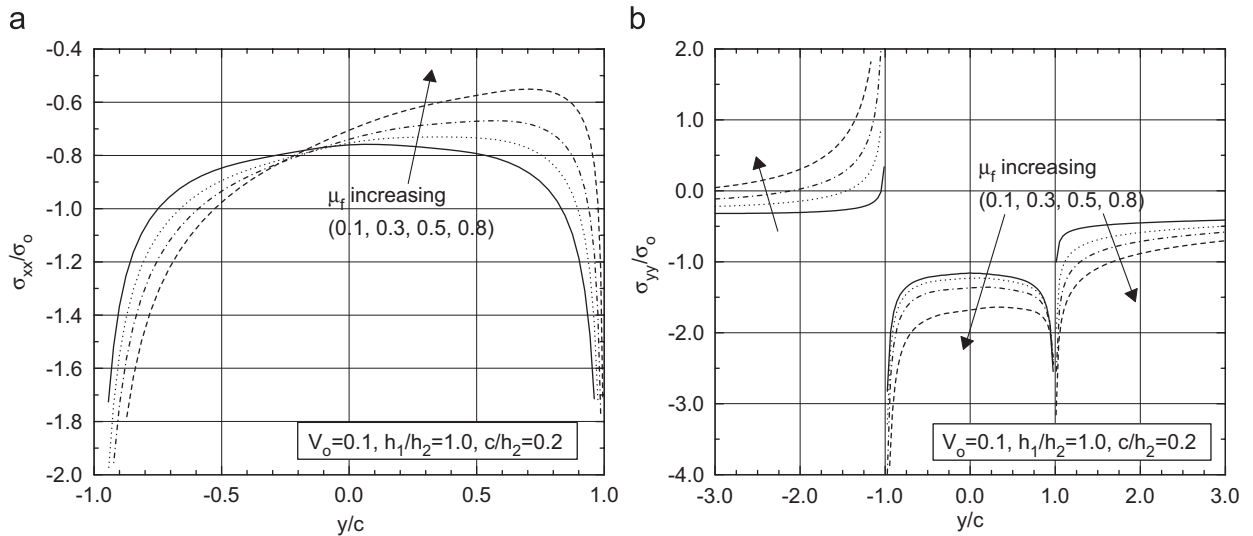


Fig. 5. (a) Contact stress  $\sigma_{1,xx}(0,y)/\sigma_o$  and (b) in-plane stress  $\sigma_{1,yy}(0,y)/\sigma_o$  distributions on the surface of the coated medium for different values of the friction coefficient  $\mu_f$  ( $V_o = 0.1, h_1/h_2 = 1.0, c/h_2 = 0.2, \sigma_o = P/2c$ ).

Table 4

Normalized stress intensity factors  $K_T/K_{0T}$  and  $K_L/K_{0L}$  at the edges of the flat punch for different values of  $\mu_f$  ( $V_o = 0.1, h_1/h_2 = 1.0, c/h_2 = 0.2, K_{0T} = P(2c)^{1/2}, K_{0L} = P(2c)^{1/2}$ )

	$\mu_f = 0.1$	$\mu_f = 0.3$	$\mu_f = 0.5$	$\mu_f = 0.8$
$K_T/K_{0T}$	0.2541	0.2763	0.2965	0.3217
$K_L/K_{0L}$	0.2296	0.2043	0.1786	0.1399

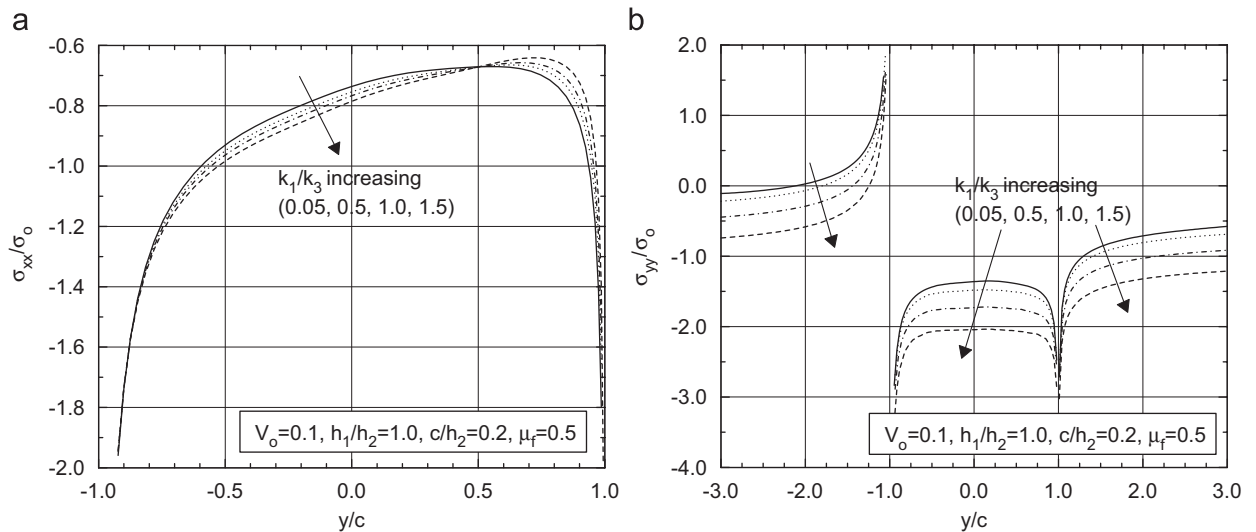


Fig. 6. (a) Contact stress  $\sigma_{1,xx}(0,y)/\sigma_o$  and (b) in-plane stress  $\sigma_{1,yy}(0,y)/\sigma_o$  distributions on the surface of the coated medium for different values of the thermal conductivity ratio  $k_1/k_3$  ( $V_o = 0.1, h_1/h_2 = 1.0, c/h_2 = 0.2, \mu_f = 0.5, \sigma_o = P/2c$ ).

From Fig. 6a, it appears that the increase in the thermal conductivity of the coating layer,  $k_1/k_3$ , relieves the stress concentration in the vicinity of the leading edge of the punch. The fact that the coating with the higher thermal conductivity can facilitate the heat transfer within the coated system is understood to be accountable



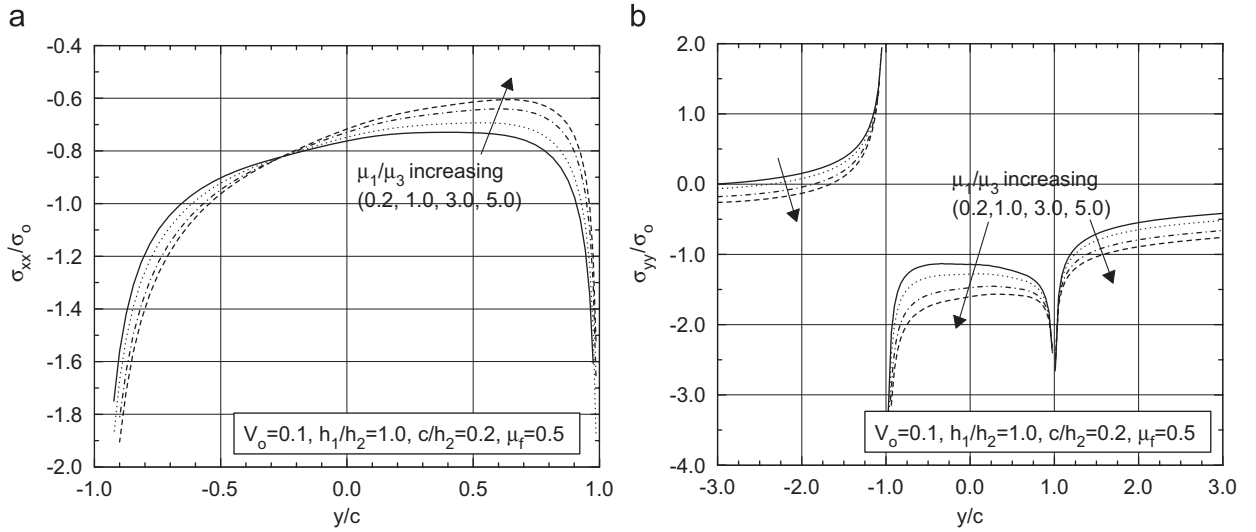


Fig. 7. (a) Contact stress  $\sigma_{1,xx}(0,y)/\sigma_o$  and (b) in-plane stress  $\sigma_{1,yy}(0,y)/\sigma_o$  distributions on the surface of the coated medium for different values of the shear modulus ratio  $\mu_1/\mu_3$  ( $V_o = 0.1, h_1/h_2 = 1.0, c/h_2 = 0.2, \mu_f = 0.5, \sigma_o = P/2c$ ).

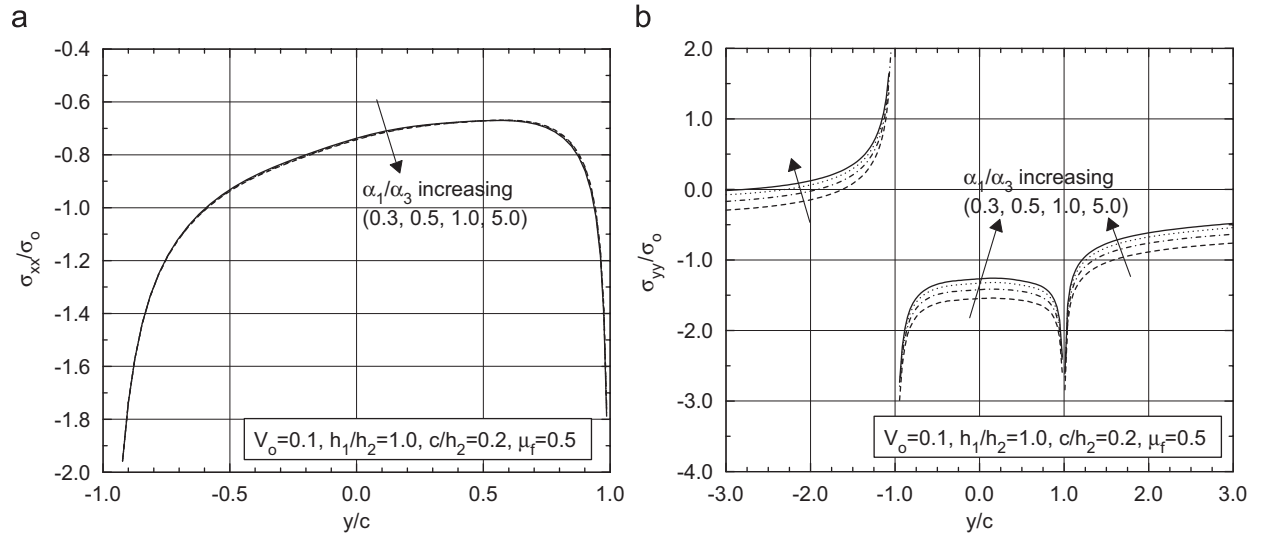


Fig. 8. (a) Contact stress  $\sigma_{1,xx}(0,y)/\sigma_o$  and (b) in-plane stress  $\sigma_{1,yy}(0,y)/\sigma_o$  distributions on the surface of the coated medium for different values of the thermal expansion coefficient ratio  $\alpha_1/\alpha_3$  ( $V_o = 0.1, h_1/h_2 = 1.0, c/h_2 = 0.2, \mu_f = 0.5, \sigma_o = P/2c$ ).

Table 5

Normalized stress intensity factors  $K_T/K_{0T}$  and  $K_L/K_{0L}$  at the edges of the flat punch for different values of  $k_1/k_3, \mu_1/\mu_3$ , and  $\alpha_1/\alpha_3$  ( $V_o = 0.1, h_1/h_2 = 1.0, c/h_2 = 0.2, \mu_f = 0.5, K_{0T} = P(2c)^{3/2}, K_{0L} = P(2c)^{3/2}$ )

$k_1/k_3$	$K_T/K_{0T}$	$K_L/K_{0L}$	$\mu_1/\mu_3$	$K_T/K_{0T}$	$K_L/K_{0L}$	$\alpha_1/\alpha_3$	$K_T/K_{0T}$	$K_L/K_{0L}$
0.05	0.2972	0.1816	0.2	0.2599	0.2067	0.3	0.2953	0.1738
0.5	0.2937	0.1665	1.0	0.2828	0.1890	0.5	0.2963	0.1778
1.0	0.2914	0.1554	3.0	0.3126	0.1668	1.0	0.2966	0.1793
1.5	0.2898	0.1466	5.0	0.3316	0.1533	5.0	0.2969	0.1807

for such stress relaxation. The trailing edge is observed, however, to be quite insensitive to the variations of  $k_1/k_3$ , which is also clear from the stress intensity factors given in Table 5. Equally notable are the in-plane stress distributions shown in Fig. 6b that become substantially compressive with the increase in the values of  $k_1/k_3$ , especially behind the trailing edge, which implies that the use of coating materials with the higher thermal conductivity may reduce the likelihood of contact-induced surface cracking.

The effect of variations of the shear modulus of the coating,  $\mu_1/\mu_3$ , is demonstrated in Figs. 7a and b. As can be inferred from Fig. 7a and the values of stress intensity factors in Table 5, the increase in the stiffness of the coating results in the greater stress concentration around the trailing edge of the punch, but the opposite trend is seen to exist near the leading edge. Fig. 7b further shows that under the current thermoelastic condition involving frictional heating, the stiffer coating may render the in-plane stress component on its surface more compressive, counteracting the brittle failure of the coating materials in general. Another interesting feature to be revealed is the effect of the thermal expansion coefficient,  $\alpha_1/\alpha_3$ , of the coating layer as provided in Fig. 8a and Table 5, where the dependence of contact pressure distributions on the variations of  $\alpha_1/\alpha_3$  is observed to be negligible. The results in Fig. 8b indicate, however, that the distributions of the in-plane stress component may be affected rather significantly by the values of  $\alpha_1/\alpha_3$  such that the larger values of  $\alpha_1/\alpha_3$  may make the coating material more susceptible to surface damage when subjected to the sliding contact with frictional heat generation.

## 6. Closure

The thermoelastic contact analysis has been performed for the coating/substrate system with graded properties using the framework of steady-state plane thermoelasticity, where the rigid flat punch was assumed to slide slowly over the surface of the coating generating the frictional heat. The graded material was modeled as a nonhomogeneous interlayer between the dissimilar, homogeneous phases of the coated system or as a nonhomogeneous coating deposited directly on the substrate, with continuous variations of its thermoelastic moduli expressed in exponential form. As a result, the distributions of the contact pressure and the in-plane surface stress component were obtained for various combinations of geometric, loading, and material parameters of the coated medium under prescribed thermoelastic contact loading conditions. The stress intensity factors were also evaluated at the edges of the flat punch to quantify the degree of severity of singular behavior of contact stresses that are inherent at such locations.

Specifically, it was manifested that the highly concentrated stresses that are largely responsible for the sliding-contact surface damage, especially near the trailing edge of the punch, tend to be relieved by the frictional heating effect that is generated during the sliding of the punch. Another noteworthy feature was that the two-layer graded coating/substrate system is more likely to be vulnerable to the contact damage when compared with the three-layer coated system with the graded interlayer. Furthermore, the coating material with the enhanced thermal conductivity was shown to lower the likelihood of failure initiation that may occur at the coating surface. On the other hand, increase in the stiffness of the coating resulted in greater stress concentrations around the trailing edge of the punch, while the in-plane stress component was rendered more compressive, offsetting the tendency of the brittle failure of the coating materials. Although the contact pressure distribution was affected to a negligible extent by the variations of the thermal expansion coefficient of the coating material, it was predicted that the increased thermal expansion could exert some adverse influence such that the in-plane stresses behind the trailing edge may become tensile, implying greater susceptibility of the coating damage during the contact process.

The work presented herein offers room for further viable and promising extensions. For instance, the assumptions set forth in Section 2 could influence the contact stress field and could thus be relaxed requiring extra elaborations in order to cover a broader range of applications. The current solutions could also be valuable resources in verifying and calibrating computational models (e.g., finite and boundary element methods) for the thermoelastic contact of graded coating/substrate systems with frictional heat generation. Besides, in the sense that the frictional contact on the surface of a brittle coating may eventually give rise to damage patterns in the form of surface and/or interface cracking (Suresh et al., 1999), the feasible coupled crack/contact analysis in the graded coatings incorporating the contact-induced frictional heating effect would

pose another interesting research topic of technological significance. Such issues need to be addressed and will be reported in the forthcoming papers.

## Appendix A

The unknowns,  $A_{kj}(s)$ ,  $k = 1, 2$ ,  $j = 1, 2$ , and  $A_{31}(s)$ , involved in the general solutions for the temperature field are determined in terms of the transformed heat flux  $\bar{q}_f(s)$  by applying the thermal boundary and interface conditions in Eqs. (6) and (7) to be expressed as

$$A_{11}(s) = \frac{1}{k_1} \left[ \left( 1 + \frac{\lambda_2}{s} \right) \left( \frac{|s| + \lambda_1}{s} \right) e^{\lambda_1 h_2} - \left( 1 + \frac{\lambda_1}{s} \right) \left( \frac{|s| + \lambda_2}{s} \right) e^{\lambda_2 h_2} \right] \frac{e^{-sh_1}}{\Delta_h} \bar{q}_f, \quad (\text{A.1})$$

$$A_{12}(s) = \frac{1}{k_1} \left[ \left( 1 - \frac{\lambda_2}{s} \right) \left( \frac{|s| + \lambda_1}{s} \right) e^{\lambda_1 h_2} - \left( 1 - \frac{\lambda_1}{s} \right) \left( \frac{|s| + \lambda_2}{s} \right) e^{\lambda_2 h_2} \right] \frac{e^{sh_1}}{\Delta_h} \bar{q}_f, \quad (\text{A.2})$$

$$A_{21}(s) = -\frac{2}{k_1} \left( \frac{|s| + \lambda_2}{s} \right) \frac{e^{\lambda_2 h_2}}{\Delta_h} \bar{q}_f, \quad (\text{A.3})$$

$$A_{22}(s) = \frac{2}{k_1} \left( \frac{|s| + \lambda_1}{s} \right) \frac{e^{\lambda_1 h_2}}{\Delta_h} \bar{q}_f, \quad (\text{A.4})$$

$$A_{31}(s) = \frac{2(\lambda_1 - \lambda_2)}{k_1 s} \frac{e^{(\lambda_1 + \lambda_2) h_2}}{\Delta_h} \bar{q}_f, \quad (\text{A.5})$$

$$\begin{aligned} \Delta_h(s) = & \left[ \left( 1 + \frac{\lambda_1}{s} \right) e^{-sh_1} - \left( 1 - \frac{\lambda_1}{s} \right) e^{sh_1} \right] (|s| + \lambda_2) e^{\lambda_2 h_2} \\ & - \left[ \left( 1 + \frac{\lambda_2}{s} \right) e^{-sh_1} - \left( 1 - \frac{\lambda_2}{s} \right) e^{sh_1} \right] (|s| + \lambda_1) e^{\lambda_1 h_2}, \end{aligned} \quad (\text{A.6})$$

where  $\bar{q}_f(s)$  is defined in Eq. (38b).

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