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Dependence of crack tip singularity on loading functions

Youn-Sha Chan [a,*], Glaucio H. Paulino [b], Bao-Feng Feng [c], Alok Sutradhar [d]

[a] Department of Computer and Mathematical Sciences, University of Houston–Downtown, One Main Street, Houston, Texas 77002, USA
[b] Department of Civil and Environmental Engineering, University of Illinois, 3129e Newmark Laboratory, 205 North Mathews Avenue, Urbana, IL 61801, USA
[c] Department of Mathematics, The University of Texas–Pan American, 1201 W. University Drive, Edinburg, Texas 78541, USA
[d] Department of Surgery, The Ohio State University, 410 West 10th Avenue, Columbus, OH 43210, USA

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ABSTRACT

Under the theory of classical linear fracture mechanics, a finite crack sitting in an isotropic and homogeneous medium is considered. We find that the well-known crack tip singularity, the inverse square-root singularity $1/\sqrt{r}$, may disappear under certain type of loading traction functions. More specifically, depending on the crack-surface loading function, the behavior of the crack tip field may be shown to be as smooth as possible. The singular integral equation method is used to study the dependence of the crack tip singularity on the mode III loading traction functions. Exact crack opening displacements, stress fields, and their corresponding loading traction functions are provided. Although the method used is somewhat mathematically elementary, the outcome seems to be new and useful.

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1. Introduction

The stress/strain singular behavior at the crack tip have attracted the attention of researchers for a long time (Hui and Ruina, 1995), and we also revisit the topic. One of the simplest crack problems in classical linear elastic fracture mechanics (LEFM) is a finite crack located in an isotropic and homogeneous material. Under mode III (anti-plane) loading (Erdogan, 1978, 1985; Sneddon, 1966), the problem can be formulated as a boundary value problem with Laplace's equation being the governing partial differential equation (PDE):

$$\begin{cases} \nabla^2 w(x,y) = 0, & -\infty < x < \infty, \quad y > 0, \\ w(x,0) = 0, & x \notin [c,d], \\ \sigma_{yz}(x,0^+) = -p(x), & x \in (c,d), \end{cases} \tag{1}$$

where $w(x,y)$ is the z-component of the displacement vector and $\sigma_{yz}(x,0)$ is the crack surface traction given by the function $p(x)$ along the crack surfaces (c,d)—see Fig. 1. (It is worth noting that the superposition principle by Bueckner (1958) and Erdogan (1978) has been applied in Fig. 1.)

The above boundary value problem in its strong (differential) form can be reduced to a weak (integral) form by a process of integral transform (Erdogan, 1995; Sneddon, 1972; Sneddon and Lowengrub, 1969):

$$\frac{\mu}{\pi} \fint_c^d \frac{\phi(t)}{t-x}\,dt = -p(x), \quad c < x < d \quad \text{with} \quad \phi(x) = \frac{\partial w(x,0)}{\partial x}, \tag{2}$$

where μ is the material shear modulus (constant). The integral in equation (2) is singular at $t = x$, and $\fint_c^d$ denotes the Cauchy principal value (Muskhelishvili, 1953, 1963). The strain function $\phi(x)$ in integral equation (2) has a closed form solution (Sölingen, 1939),

$$\phi(x) = \frac{\gamma_0}{\sqrt{(x-c)(d-x)}} + \frac{1}{\pi\mu\sqrt{(x-c)(d-x)}} \times \fint_c^d \frac{p(t)\sqrt{(t-c)(d-t)}}{t-x}\,dt, \quad c < x < d, \tag{3}$$

where γ_0 is some constant determined by the boundary condition.[1] Under the condition

$$\int_c^d \phi(x)\,dx = 0,$$

the constant γ_0 in Eq. (3) turns to be zero (see Appendix A); thus, integral equation (2) has a unique solution (Muskhelishvili, 1953, 1963; Sölingen, 1939):

$$\phi(x) = \frac{1}{\pi\mu\sqrt{(x-c)(d-x)}} \fint_c^d \frac{p(t)\sqrt{(t-c)(d-t)}}{t-x}\,dt, \quad c < x < d. \tag{5}$$

[1] If one considers displacement $w(x,0)$ to be the unknown function, then the Cauchy singular integral equation (2) becomes a hypersingular integral equation

$$\frac{\mu}{\pi} \fint_c^d \frac{w(t,0^+)}{(t-x)^2}\,dt = p(x), \quad c < x < d, \tag{4}$$

where $\fint_c^d$ denotes the Hadamard finite part integral (Hadamard, 1952; Kaya and Erdogan, 1987; Martin, 1991). A closed form solution to (4) can be found in reference Martin (1992). A boundary element method applicable to mode III cracks in nonhomogeneous materials can be found in Paulino and Sutradhar (2006).

* Corresponding author. Tel.: +1 (713) 221 8414.
E-mail address: chany@uhd.edu (Y.-S. Chan).

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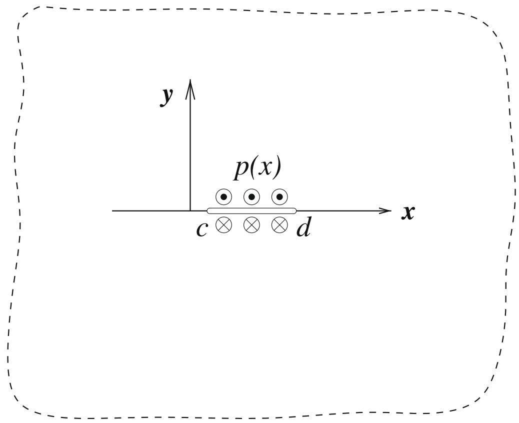

Fig. 1. Illustration of the mode III crack problem.

Clearly, one may see that $1/\sqrt{(x-c)(d-x)}$ provides the singularity, $1/\sqrt{r}$, of $\phi(x)$ around the crack tips as $r \to 0$, where r denotes the distance to the two crack tip, *i.e.* $r = |c-x|$ or $r = |d-x|$. However, the solution in (5) also clearly states that the strain function $\phi(x)$ depends on the loading function $p(x)$. Can some loading function $p(x)$ be chosen so that the inverse square-root singularity no longer exists? As the fracture process takes place at the crack tip, the influence of the loading function $p(x)$ on the crack tip singularity is relevant to engineering applications in fracture mechanics. The influence of the loading function on the stress and displacement fields near the crack tip can be significant—we have found that for some special loading functions, the crack tip singularity may disappear. Furthermore, one can choose the loading functions so that the stress and displacement fields near the crack tip may be as smooth as possible. This paper presents a detailed derivation of how the loading functions influence the behavior around the crack tip so that the crack tip singularity disappears. The derivation gives the exact solutions to the crack opening displacements and the stress fields under the selected loading traction functions.

2. Green's function

The influence of the loading functions $p(x)$ on the strain function $\phi(x)$ will be easier explained from the point view of Green's function. Without loss of generality, we first take a step of normalization. Let

$$s = \frac{2}{d-c}\left(t - \frac{c+d}{2}\right) \quad \text{and} \quad r = \frac{2}{d-c}\left(x - \frac{c+d}{2}\right), \tag{6}$$

then the integral equation (2) may be written as

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\Phi(s)}{s-r}\,ds = -P(r), \quad -1 < r < 1 \quad \text{with} \quad \int_{-1}^{1} \Phi(r)\,dr = 0, \tag{7}$$

where $P(r)$ and $\Phi(r)$ are normalized versions of $p(x)$ and $\phi(x)$, respectively. According to Eq. (5), the solution to the Cauchy singular integral equation (7) above is

$$\Phi(r) = \frac{1}{\pi\sqrt{1-r^2}}\int_{-1}^{1} \frac{P(s)\sqrt{1-s^2}}{s-r}\,ds, \quad -1 < r < 1. \tag{8}$$

The solution $\Phi(r)$ can be expressed as

$$\Phi(r) = \frac{1}{\pi}\int_{-1}^{1} G(s,r)P(s)\,ds, \quad -1 < r < 1, \tag{9}$$

where the Green's function (Sneddon, 1972; Widder, 1971)

$$G(s,r) = \frac{1}{\sqrt{1-r^2}}\frac{\sqrt{1-s^2}}{s-r}.$$

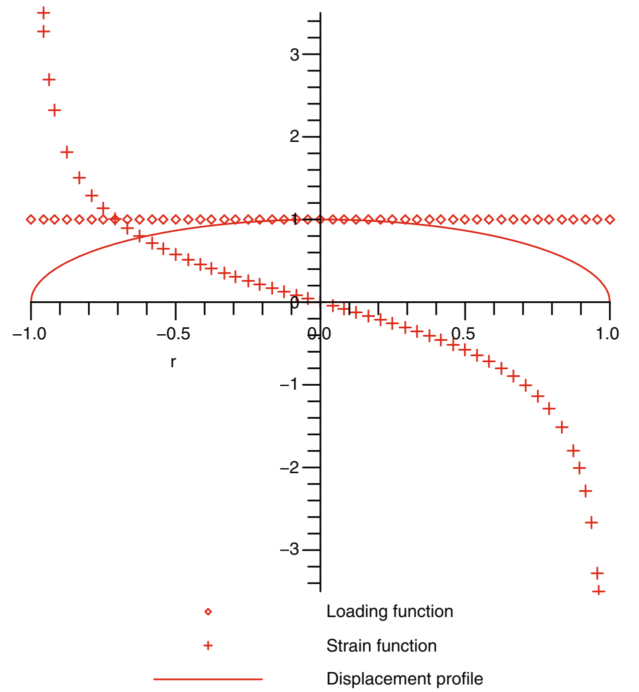

Fig. 2. Strain function $\Phi(r) = -r/\sqrt{1-r^2}$ and displacement function $W(r) = \sqrt{1-r^2}$ under uniform loading $P(r) = 1$, $-1 < r < 1$.

From this point of view, it is clear that the solution $\Phi(r)$ depends on the loading function $P(r)$.

3. Uniform loading

Under uniform loading,

$$P(r) = 1, \tag{10}$$

the solution $\Phi(r)$ is

$$\Phi(r) = \frac{1}{\pi\sqrt{1-r^2}}\int_{-1}^{1} \frac{\sqrt{1-s^2}}{s-r}\,ds = \frac{-r}{\sqrt{1-r^2}}, \quad -1 < r < 1, \tag{11}$$

where we have used

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-s^2}}{s-r}\,ds = -r. \tag{12}$$

The case of uniform loading is the most common applied loading in fracture mechanics, e.g. using the complex stress function by Sneddon and Elliott (1946) and Westergaard (1939). As a mathematical fact, the solution $\Phi(r)$ in Eq. (11) clearly shows the $1/\sqrt{r}$ crack tip singularity. This situation was also investigated by Gray and Paulino (1998). In Fig. 2 the strain function $\Phi(r)$ and the displacement function $W(r) = \sqrt{1-r^2}$ are plotted under uniform loading function. Note that the tangent lines of the displacement function at $r = 1$ and $r = -1$ have infinite slope.

4. Linear loading

4.1. Singularity suppression at the right crack tip

By choosing different loading function $P(r)$, one can write out the exact solution for $\Phi(r)$ such that there is no longer $1/\sqrt{r}$ singularity existing in the expression at one end of the crack tips, say, at $r = 1$. For instance, by choosing

$$P(r) = 1 - 2r, \tag{13}$$

then

$$\Phi(r) = \frac{1}{\pi\sqrt{1-r^2}} \int_{-1}^{1} \frac{(1-2s)\sqrt{1-s^2}}{s-r}\,ds = \frac{2r^2-r-1}{\sqrt{1-r^2}}, \quad -1 < r < 1, \tag{14}$$

where we have used Eq. (12) and

$$\frac{1}{\pi}\int_{-1}^{1} \frac{s\sqrt{1-s^2}}{s-r}\,ds = -r^2 + \frac{1}{2} \quad \text{(see Appendix B.1)}. \tag{15}$$

Clearly, one can see that Eq. (14) becomes

$$\Phi(r) = \frac{(2r+1)(r-1)}{\sqrt{1-r^2}} = -\frac{(2r+1)\sqrt{1-r}}{\sqrt{1+r}},$$

and there is no more $1/\sqrt{r}$ singularity at $r = 1$. Fig. 3 shows the strain function $\Phi(r)$ and displacement profile

$$W(r) = \sqrt{1+r}\,(1-r)^{3/2},$$

under the linear loading function $P(r) = 1 - 2r$.

It seems that the disappearance of singularity at the right crack tip is due to the effect of the loading function (compression) around the neighborhood of the right end crack tip. However, if the loading function is changed to be

$$P(r) = 1 - \frac{3r}{2}, \tag{16}$$

then the right crack tip still experiences $1/\sqrt{r}$ singularity, even it is under the compression from the loading function (see Fig. 4). Thus one cannot conclude that the disappearance of singularity at the right crack tip is purely due to the compression of the loading function. For instance, if the compression at the right crack tip is increased to

$$P(r) = 1 - \frac{5r}{2}, \tag{17}$$

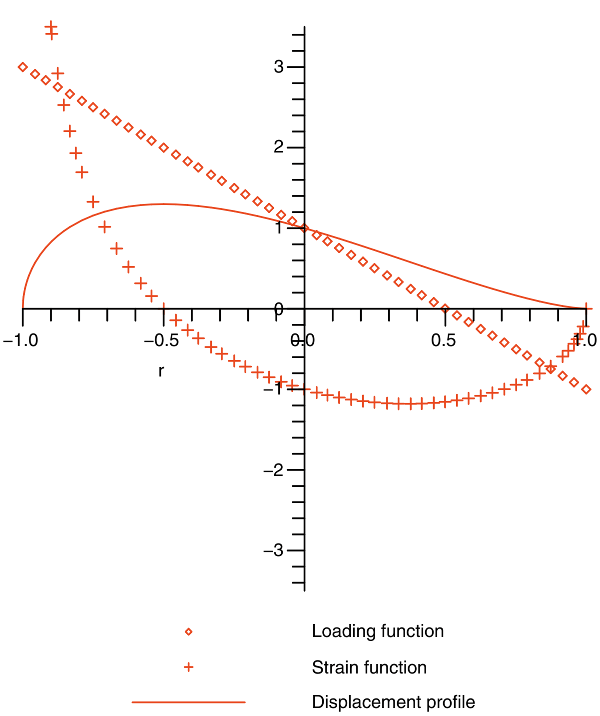

Fig. 3. Strain function $\Phi(r) = -(2r+1)\sqrt{1-r}/\sqrt{1+r}$ and displacement function $W(r) = \sqrt{1+r}\,(1-r)^{3/2}$ under linear loading $P(r) = 1 - 2r$, $-1 < r < 1$.

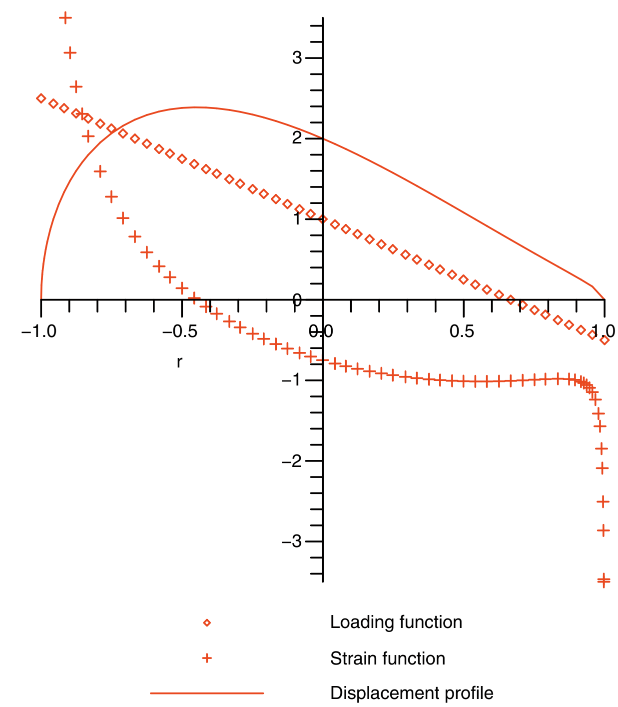

Fig. 4. Strain function $\Phi(r) = (6r^2 - 4r - 3)/\left(4\sqrt{1-r^2}\right)$ and displacement function $W(r) = (4-3r)\sqrt{1-r^2}/2$ under linear loading $P(r) = 1 - 3r/2$, $-1 < r < 1$.

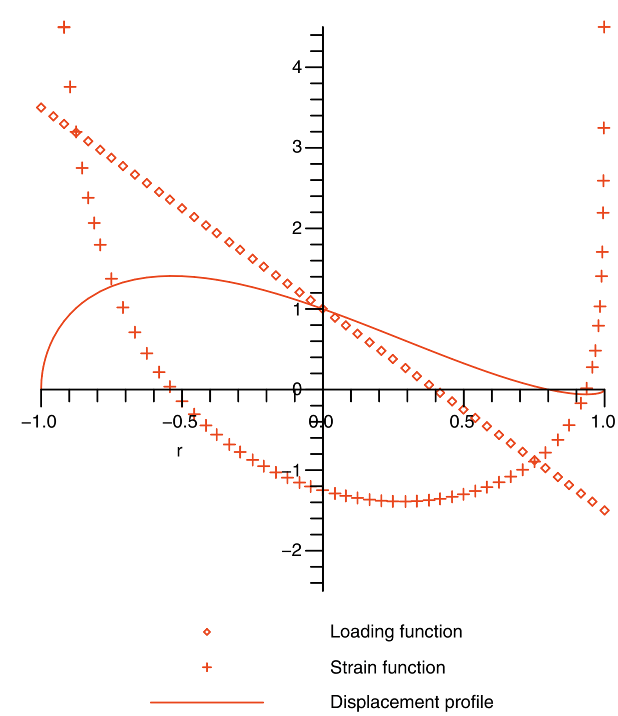

Fig. 5. Strain function $\Phi(r) = (10r^2 - 4r - 5)/\left(4\sqrt{1-r^2}\right)$ and displacement function $W(r) = (4-5r)\sqrt{1-r^2}/4$ under linear loading $P(r) = 1 - 5r/2$, $-1 < r < 1$.

then there still exists $1/\sqrt{r}$ singularity at $r = 1$ (see Fig. 5). The difference is that in Fig. 5 one can observe that the displacement profile penetrates under the crack surface at the right crack tip, which is simply a mathematical outcome reflecting the higher compressive load.

4.2. Singularity suppression at the left crack tip

By the same token one can choose

$$P(r) = 1 + 2r, \tag{18}$$

such that the crack tip singularity at $r = -1$ disappear. That is,

$$\Phi(r) = \frac{1}{\pi\sqrt{1-r^2}} \int_{-1}^{1} \frac{(1+2s)\sqrt{1-s^2}}{s-r}\, ds = \frac{-2r^2 - r + 1}{\sqrt{1-r^2}} = \frac{(1-2r)\sqrt{1+r}}{\sqrt{1-r}}, \quad -1 < r < 1.$$

The strain function $\Phi(r)$ and displacement profile

$$W(r) = \sqrt{1-r}\,(1+r)^{3/2}$$

are plotted in Fig. 6 under linear loading $P(r) = 1 + 2r$.

Notice the disappearance of the singularity at the left crack tip and compare with Fig. 3, which shows no singularity at the right crack tip.

Remark 1. The detailed derivation of how to find the loading function $P(r)$ such that the crack tip singularity no longer exists is given in Appendix B.2. An example for finding a higher order of $P(r)$ such that the $1/\sqrt{r}$ singularity disappears at both crack tips is demonstrated there. The basic procedures include that $\Phi(r)$ [and/or $\Phi'(r), \Phi''(r), \ldots$] are to be evaluated as zero at $r = -1$ and/or $r = 1$.

Remark 2. There are actually infinitely many solutions of the loading function $P(r)$ in each case. For instance, any scalar multiple of $P(r) = 1 - 2r$ in Eq. (14) will also be a solution such that the $1/\sqrt{r}$ singularity disappears at right crack tip. In order to have a unique solution and be able to compare the magnitude of the displacement profile in each case, the loading function $P(r)$ is normalized such that

$$\int_{-1}^{1} P(r)\, dr = 2. \tag{19}$$

So far in each case (see Eqs. (10), (13), (16), (17), and (18)) Eq. (19) is satisfied.

5. Quadratic loading

One can also choose the loading function $P(r)$ to be a quadratic polynomial such that the inverse square-root singularity disappears at both ends. For instance, by choosing

$$P(r) = 3 - 6r^2 \quad \left(\text{so that} \quad \int_{-1}^{1} P(r) dr = 2\right),$$

then

$$\Phi(r) = \frac{1}{\pi\sqrt{1-r^2}} \int_{-1}^{1} \frac{(3-6s^2)\sqrt{1-s^2}}{s-r}\, ds = \frac{6(r^3 - r)}{\sqrt{1-r^2}}, \quad -1 < r < 1, \tag{20}$$

where we have used Eq. (12) and

$$\frac{1}{\pi} \int_{-1}^{1} \frac{s^2\sqrt{1-s^2}}{s-r}\, ds = -r^3 + \frac{r}{2} \quad (\text{see Appendix B.1}). \tag{21}$$

Obviously, Eq. (20) becomes

$$\Phi(r) = \frac{6r(r^2-1)}{\sqrt{1-r^2}} = -6r\sqrt{1-r^2},$$

and there is no more $1/\sqrt{r}$ singularity at both crack tips. Fig. 7 illustrates the strain $\Phi(r)$ and displacement profile

$$W(r) = 2(1-r)(1+r)\sqrt{1-r^2}$$

under the quadratic loading function $P(r) = 3 - 6r^2$. Note that the tangent lines of the displacement profile at $r = 1$ and $r = -1$ have

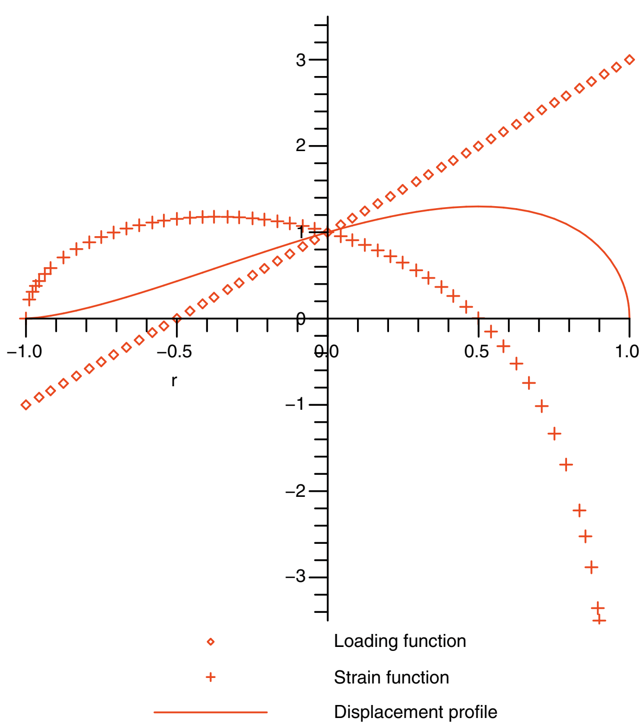

Fig. 6. Strain function $\Phi(r) = (1-2r)\sqrt{1+r}/\sqrt{1-r}$ and displacement function $W(r) = \sqrt{1-r}\ (1+r)^{3/2}$ under linear loading $P(r) = 1 + 2r$, $-1 < r < 1$.

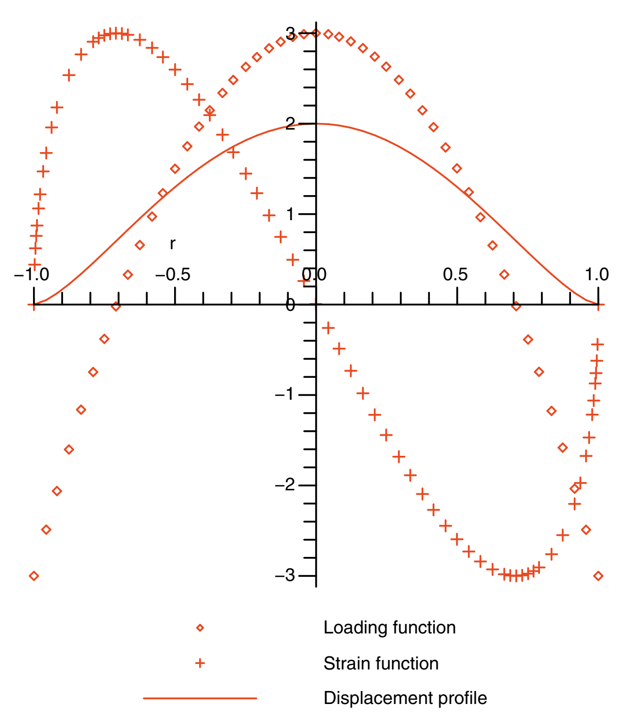

Fig. 7. Strain function $\Phi(r) = -6r\sqrt{1-r^2}$ and displacement function $W(r) = 2(1-r)(1+r)\sqrt{1-r^2}$ under quadratic loading $P(r) = 3 - 6r^2$, $-1 < r < 1$.

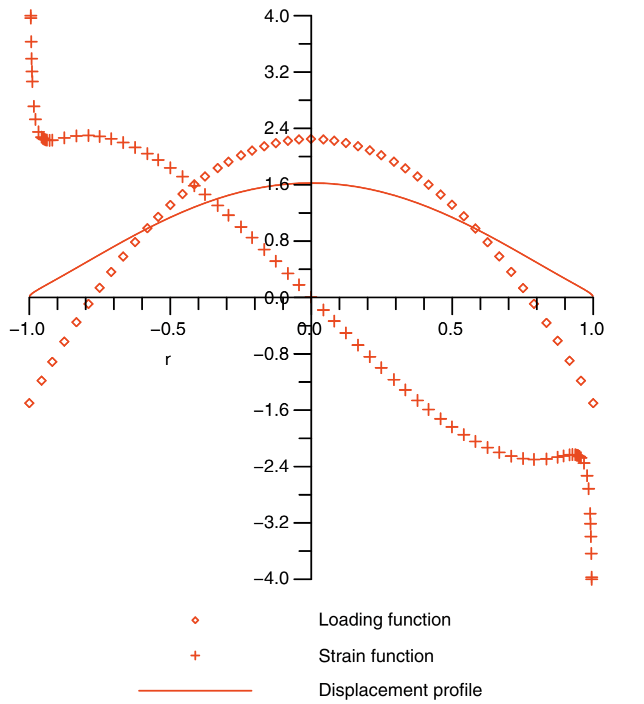

Fig. 8. Strain function $\Phi(r) = r(\frac{15}{4}r^2 - \frac{33}{8})/\sqrt{1-r^2}$ and displacement profile $W(r) = (13-10r^2)\sqrt{1-r^2}/8$ under quadratic loading function $P(r) = (9-15r^2)/4$, $-1 < r < 1$.

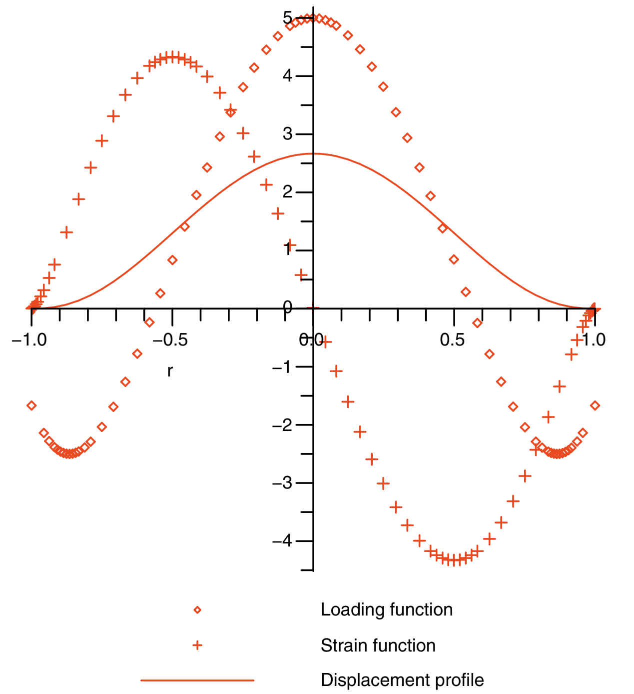

Fig. 9. Strain function $\Phi(r) = -40r(1-r^2)\sqrt{1-r^2}/3$ and displacement profile $W(r) = 8(1-2r^2+r^4)\sqrt{1-r^2}/3$ under the 4th degree polynomial loading function $P(r) = 5(3-12r^2+8r^4)/3$, $-1 < r < 1$.

slope zero. This displacement profile resembles a cusping crack as obtained in strain gradient theory (Fannjiang et al., 2002; Paulino et al., 2003) or cohesive zone models (Zhang and Paulino, 2005).

To reiterate that the disappearance of singularity is not purely due to any arbitrary loading function which is compressive around the neighborhood of both crack tips, Fig. 8 is provided. It employs the quadratic loading (recall $\int_{-1}^{1} P(r)\,dr = 2$)

$$P(r) = \frac{9}{4} - \frac{15}{4}r^2$$

which leads to the strain function

$$\Phi(r) = \frac{r}{\sqrt{1-r^2}}\left(\frac{15}{4}r^2 - \frac{33}{8}\right)$$

and displacement profile

$$W(r) = \frac{1}{8}(13-10r^2)\sqrt{1-r^2}.$$

One can clearly observe that the $1/\sqrt{r}$ singularity appears at $r = 1$ and $r = -1$ even though the loading function renders compression at both crack tips.

6. Higher degree loading functions

It is certainly possible that one can choose the loading function $P(r)$ to be a polynomial of higher degree such that not only the inverse square-root singularity disappears at both ends, but also the derivative of the strain function $\Phi(r)$ does NOT possess any singularity at the crack tips. The detail of how to find those polynomials is given in Appendix B.2 .

As an additional example, a loading function of 4th degree is examined next. By choosing

$$P(r) = \frac{5}{3}(3-12r^2+8r^4) \quad \left(\text{recall} \int_{-1}^{1} P(r)\,dr = 2\right), \quad \text{then,} \tag{22}$$

$$\Phi(r) = \frac{5}{\pi\sqrt{1-r^2}} \int_{-1}^{1} \frac{(3-12r^2+8r^4)\sqrt{1-s^2}}{3(s-r)}\,ds = \frac{-40r(1-2r^2+r^4)}{3\sqrt{1-r^2}}, \quad -1 < r < 1, \tag{23}$$

where we have used formulas (12), (21), and

$$\frac{1}{\pi}\int_{-1}^{1} \frac{s^4\sqrt{1-s^2}}{s-r}\,ds = -r^5 + \frac{r^3}{2} + \frac{r}{8} \quad (\text{see Appendix B.1}). \tag{24}$$

Obviously, Eq. (23) becomes

$$\Phi(r) = \frac{-40r(r+1)^2(r-1)^2}{3\sqrt{1-r^2}} = -\frac{40}{3}r(1-r^2)\sqrt{1-r^2}$$

and one can see that not only the $1/\sqrt{r}$ singularity disappear at $r = 1$ and $r = -1$, but the strain function also has zero slope tangent line at both crack tips. Fig. 9 shows the strain function $\Phi(r)$ and displacement profile

$$W(r) = \frac{8}{3}(1-2r^2+r^4)\sqrt{1-r^2},$$

under the loading function $P(r) = \frac{5}{3}(3-12r^2+8r^4)$.

7. Concluding remarks

Under the setting of LEFM, the stress singularity near the crack tip shows a $1/\sqrt{r}$ behavior, as confirmed by Eq. (3). However, Eq. (3) clearly affirms that the solution to the strain function $\phi(x)$ depends on the traction loading function $p(x)$, and it seems that little attention has been drawn on how the loading function influences the stress singularity near the crack tip. This paper presents a detailed derivation of the exact solution to different traction loading functions. By judicious choice of particular loading functions one can obtain smooth stress and displacement fields.

As expected, our work reassures the standard $1/\sqrt{r}$ crack tip singularity. However, our point is that, among an infinite family of loading functions, there is one and only one function for a polynomial with given degree [normalized in terms of Eq. (19)] that will make the crack tip singularity disappear—we show how to find such functions exactly.

This work deals with a theoretical investigation involving mode III cracks. A practical extension of this work is the investigation of mode I cracks. Similarly to the mode III case, the crack displacements may be negative in some instances, which indicate interpenetration of the crack faces. Thus the study may be conducted in conjunction with contact mechanics to suppress interpenetration of crack surfaces (Anderson, 2005).

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Appendix A. Unique Solution to Integral Equation (2)

Recall that all formulas have been normalized through a change of variables by Eq. (6). Let $W(r)$ represent the normalized displacement, then the condition

$$\int_{-1}^{1} \Phi(r)\,dr = W(1) - W(-1) = 0$$

implies that

$$\int_{-1}^{1}\left[\frac{\gamma_0}{\sqrt{1-r^2}} + \frac{1}{\pi\mu\sqrt{1-r^2}}\int_{-1}^{1}\frac{P(s)\sqrt{1-s^2}}{s-r}\,ds\right]dr = 0, \quad |r|<1. \tag{25}$$

After a switch of the double integral and the following formula

$$\int_{-1}^{1}\frac{1}{(s-r)\sqrt{1-s^2}}\,ds = 0, \quad |r|<1, \tag{26}$$

one can readily see that constant γ_0 must be zero. Thus the unique solution is that

$$\Phi(r) = \frac{1}{\pi\mu\sqrt{1-r^2}}\int_{-1}^{1}\frac{P(s)\sqrt{1-s^2}}{s-r}\,ds.$$

Appendix B. Some Cauchy integral formulas

B.1. Polynomials: s^n

Let

$$I_n(r) = \frac{1}{\pi}\int_{-1}^{1}\frac{s^n\sqrt{1-s^2}}{s-r}\,ds, \tag{27}$$

then (Kaya and Erdogan, 1987)

$$I_0(r) = -r, \quad I_1(r) = -r^2 + \frac{1}{2}, \quad I_2(r) = -r^3 + \frac{r}{2}, \tag{28}$$

$$I_3(r) = -r^4 + \frac{r^2}{2} + \frac{1}{8}, \quad I_4(r) = -r^5 + \frac{r^3}{2} + \frac{r}{8},$$
$$I_5(r) = -r^6 + \frac{r^4}{2} + \frac{r^2}{8} + \frac{1}{16}. \tag{29}$$

The general formula for evaluating $I_n(r)$ is

$$I_n(r) = \frac{1}{2\sqrt{\pi}}\sum_{k=0}^{n+1} c_k r^k,$$
$$c_k = \begin{cases} \Gamma\left(\frac{n-k}{2}\right)/\Gamma\left(\frac{n-k+3}{2}\right), & \text{if} \quad n-k \quad \text{is odd,} \\ 0, & \text{if} \quad n-k \quad \text{is even,} \end{cases} \tag{30}$$

where

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,dy = (\alpha-1)! \ \text{ and } \ \Gamma\left(-\frac{1}{2}\right) = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}.$$

B.2. Tchebyshev polynomials

The main formula that has been used in the derivation of finding the loading function is

$$\frac{1}{\pi}\int_{-1}^{1}\frac{U_n(s)\sqrt{1-s^2}}{s-r}\,ds = -T_{n+1}(r), \ n \geqslant 0 \quad \text{and} \quad |r|<1, \tag{31}$$

where the $T_n(r)$ and $U_n(r)$ are the Tchebyshev polynomials of the first and second type, respectively. For example, to find the loading function $P(r)$ in Eq. (22), one can write $P(r)$ as a linear combination of the Tchebyshev polynomials of the second type

$$P(r) = \sum_{k=0}^{4} c_k U_k(r),$$

then by Eq. (8)

$$\begin{aligned}\Phi(r) &= \frac{1}{\pi\sqrt{1-r^2}}\int_{-1}^{1}\frac{P(s)\sqrt{1-s^2}}{s-r}\,ds \\ &= \frac{1}{\pi\sqrt{1-r^2}}\int_{-1}^{1}\frac{\sqrt{1-s^2}\sum_{k=0}^{4}c_kU_k(s)}{s-r}\,ds \\ &= \frac{1}{\sqrt{1-r^2}}\sum_{k=0}^{4}c_kT_{k+1}(r), \quad -1<r<1,\end{aligned}$$

where we have applied formula (31). As $\Phi(r)$ is required to cancel out the singularity as $r \to 1^-$ and $r \to -1^+$, it must have factors $(1-r)$ and $(1+r)$. For the example that has been demonstrated in Eq. (22), $\Phi(r)$ has factors $(1-r)^2$ and $(1+r)^2$, that is,

$$\Phi(1) = 0, \quad \Phi(-1) = 0, \quad \Phi'(1) = 0, \quad \text{and} \quad \Phi'(-1) = 0.$$

It leads to solve the following system of linear equation of c_i's:

$$c_1 + c_3 = 0, \quad c_0 + c_2 + c_4 = 0, \quad c_1 + 4c_3 = 0, \quad \text{and}$$
$$c_0 + 9c_2 + 25c_4 = 0.$$

There are infinite many solutions (five unknowns with four restrictions), and one solution is that

$$c_0 = 2, \quad c_1 = 0, \quad c_2 = -3, \quad c_3 = 0, \quad \text{and} \quad c_4 = 1.$$

A simple calculation leads to

$$P(r) \quad \text{is a multiple of} \quad 2U_0(r) - 3U_2(r) + U_4(r) = 6 - 24r^2 + 16r^4$$

and

$$\Phi(r) \quad \text{has a factor of} \quad 2T_1(r) - 3T_3(r) + T_5(r)$$
$$= 16r - 32r^3 + 16r^5 = 16r(1-r^2)^2.$$

After normalization $\int_{-1}^{1} P(r)\,dr = 2$, we find that

$$P(r) = \frac{5}{3}(3 - 12r^2 + 8r^4) \quad \text{and}$$
$$\Phi(r) = \frac{1}{\pi\sqrt{1-r^2}}\int_{-1}^{1}\frac{P(s)\sqrt{1-s^2}}{s-r}\,ds = \frac{-40r(1-2r^2+r^4)}{3\sqrt{1-r^2}},$$
$$-1<r<1.$$

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