rspa.royalsocietypublishing.org Proc R Soc A 0000000

Supplementary Material for the paper "Topology Optimization with Local Stress Constraints and Continuously Varying Load Direction and Magnitude: Towards Practical Applications"

Fernando V. Senhora[1,2], Ivan F. M. Menezes[1] and Glaucio H. Paulino[3,4]

[1]Pontifical Catholic University of Rio de Janeiro (PUC-Rio), Rua Marquês de São Vicente, 225, Rio de Janeiro, R.J. 22453, Brazil
[2]School of Civil and Environmental Engineering, Georgia Institute of Technology, 790 Atlantic Drive, Atlanta, GA 30332, USA
[3]Department of Civil and Environmental Engineering, Princeton University, Princeton, New Jersey 08544, USA
[4]Princeton Institute for the Science and Technology of Materials, Princeton University, Princeton, New Jersey 08544, USA

S1. Implementation details

This section presents the details of the implementation used to obtain the numerical results displayed in Section 8 of the main manuscript. The values of the numerical parameters used in the implementation are displayed in Table S1. We update the AL multipliers every 5 iterations using Eq. (S1.1):

$$\lambda_i^{(k+1)} = \max(\lambda_i^{(k)} + \mu g_i,\ 0) \tag{S1.1}$$

in which $\lambda_i^{(k)}$ is the AL multiplier associated to constraint g_i at iteration (k), and μ is the AL penalization parameter. Furthermore, we define a measure of "change" according to Eq. (S1.2) of the design variables that help us quantify the stagnation of the optimization

$$\text{change} = \text{mean}\left(\left|\mathbf{z}^{(k+1)} - \mathbf{z}^{(k)}\right|\right) \tag{S1.2}$$

in which $\mathbf{z}$ is the vector of design variables. We define a value of tolerance represented by Tol, and we run 200 iterations of the optimization, after that, when the "change" reaches $0.2\,Tol$, we restart the value of λ back to its initial value, and restart the iteration counter (k) back to zero, as proposed by [S1]. We perform this restart procedure 4 times. After we reach the final restart, if the "change" is below the Tol for two consecutive iterations we update the AL multipliers, $\lambda_i^{(k)}$ following Eq. (S1.1), and multiply the AL penalization parameter μ by a factor of 2, in order to accelerate convergence towards a feasible solution. At this point, if the "change" is below the Tol, we also multiply the value of the Heaviside parameter, β, by 1.25, with a minimal interval of 20 iterations between increments, until it reaches the maximum value of 15 to achieve a solution closer to 0/1, i.e. minimize the intermediate values of density caused by the filter. We stop the optimization if the maximum stress is below the stress limit with a tolerance of 1%, and β reaches its maximum value of 15, and the "change" is below the tolerance (Tol) indicated in Table S1 .

Table S1. Values of the numerical parameters used in the optimization.

Parameter	Description	Value
p	SIMP penalization factor	4
β	Heaviside parameter	3
$\lambda^{(0)}$	Initial AL multiplier	0
μ	Initial AL penalization	1
Tol	Optimization tolerance	0.0001
$move$	Optimization move limit	0.1

S2. Optimization Algorithm

We developed an adaptive gradient descent-based approach that is efficient, and simple to implement. Notice that we handle the constraints of the optimization problem with the Augmented Lagrangian method, and, therefore, we are left with an unconstrained optimization problem. Algorithm 1 describes the main steps of the proposed algorithm.

Algorithm 1 Adaptive Gradient Descent

procedure MODIFIED_GRADIENT_DESCENT($\mathbf{z}$, f, $df/d\mathbf{z}$, f_{old1}, f_{old2}, $\mathbf{z}_{min}$, $\mathbf{z}_{max}$, α, $\texttt{move}$, $\texttt{norm}_0$)
  if $(f_{\text{old1}} - f_{\text{old2}})(f - f_{\text{old1}}) < 0$ **then**
    $\alpha \leftarrow \max(\, 0.25\alpha,\ 10^{-6})$
  else
    $\alpha \leftarrow \min(\, 1.25\alpha,\ 1)$
  end if
  $\mathbf{m}_r \leftarrow \texttt{move}\,(\mathbf{z}_{max} - \mathbf{z}_{min})$
  $\mathbf{d}_f \leftarrow \dfrac{\alpha}{\texttt{norm}_0} \dfrac{df}{d\mathbf{z}}$
  $\mathbf{z} \leftarrow \max(\max(\min(\min(\mathbf{z} - \mathbf{d}_f, \mathbf{z} + \mathbf{m}_r), \mathbf{z}_{max}), \mathbf{z} - \mathbf{m}_r), \mathbf{z}_{min})$
  $f_{\text{old2}} \leftarrow f_{\text{old1}}$
  $f_{\text{old1}} \leftarrow f$
end procedure

In Algorithm 1, $\mathbf{z}$ are the design variables, f is the value of the objective function, $df/d\mathbf{z}$ is the gradient of the objective function in respect to the design variables, f_{old1} and f_{old2} are the values of the objective function in the last two iterations, $\mathbf{z}_{min}$ and $\mathbf{z}_{max}$ are the lower and upper bound of the design variables respectively, α is an adaptive parameter that damps oscillation by controlling the size of the step taken by the optimizer, $\texttt{move}$ is move limit, and $\texttt{norm}_0$ is a normalizing parameter for the gradient. The value of this normalizing parameter is set to $\texttt{norm}_0 = \left\|(df/d\mathbf{z})_0\right\|_2 / \texttt{move}$, in which $(df/d\mathbf{z})_0$ is the gradient at the first iteration of the optimization. For some problems, it might be beneficial to update the value of $\texttt{norm}_0$ every 25 iterations or so to accelerate convergence.

S3. Convergence of the Optimization Algorithm

The convergence of the optimization algorithm is displayed in Figs. S1, S2, and S3, in which we show the volume (objective function), and the maximum stress of the structure as a function of the optimization iterations. The peaks (downward peaks in the volume, and upper peaks in the maximum stress) are caused by the restart of the AL method penalty terms ($\lambda_j^{(k)}$ and $\mu^{(k)}$) as described in Section S1 of this supplementary material (for more details see [S1]). Notice that, disregarding the aforementioned peaks the convergence is fairly smooth. The plots also display the number of optimization iterations necessary for each example, which ranges from 983 to 1523 iterations.

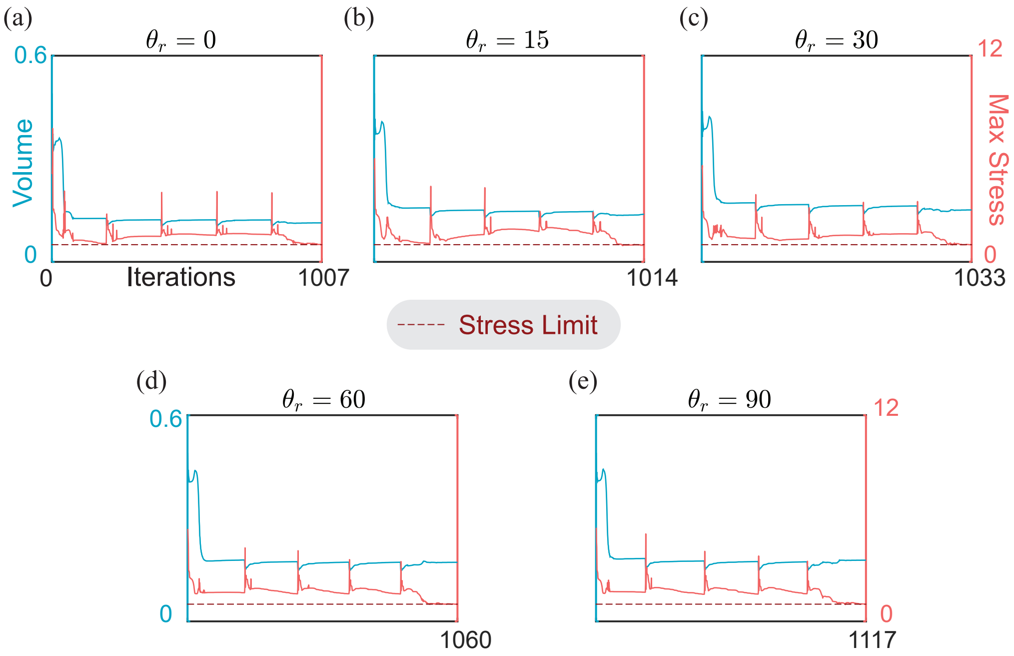

Figure S1. Optimization convergence plots, displaying the volume (objective function) in blue, and maximum stress on the structure (constraints) in red, for the Double L-bracket considering the two loads varying with the same angle (case 1 presented in Section 5(a) of the main manuscript) for different ranges of admissible angle (θ_r), corresponding to the results displayed in Fig. 12 of the main manuscript.

S4. Critical Stress Sensitivity Analysis

In this section, we present the sensitivity analysis of the expressions derived in Section 5 of the main manuscript for the worst-case multiple load directions formulation with respect to the design variables. The sensitivity information is necessary because the topology optimization problem is solved using gradient-based optimization algorithms [S2]. We will also provide proof of the differentiability of the worst-case stress analytical expressions presented previously. To compute the sensitivity information, we will use the chain rule starting by differentiating the AL function:

$$\frac{\mathrm{d}J^{(k)}}{\mathrm{d}z_j} = \frac{\partial J^{(k)}}{\partial M(\mathbf{z})}\frac{\partial M(\mathbf{z})}{\partial z_j} + \sum_{i=1}^{N_e}\frac{\partial J^{(k)}}{\partial g_i}\frac{\partial g_i}{\partial z_j} = \frac{\mathrm{d}M(\mathbf{z})}{\mathrm{d}z_j} + \sum_{i=1}^{N_e}(\lambda_i^{(k)} + \mu^{(k)} g_i)\frac{\mathrm{d}g_i}{\mathrm{d}z_j} \tag{S4.1}$$

We will now focus on the derivative of the critical stress constraints because their formulation is one of the main contributions of this work:

$$\frac{\mathrm{d}g_i}{\mathrm{d}z_j} = \frac{\partial g_i}{\partial z_j} + \frac{\partial g_i}{\partial \widetilde{\sigma}_i^v}\frac{\partial \widetilde{\sigma}_i^v}{\partial z_j} \tag{S4.2}$$

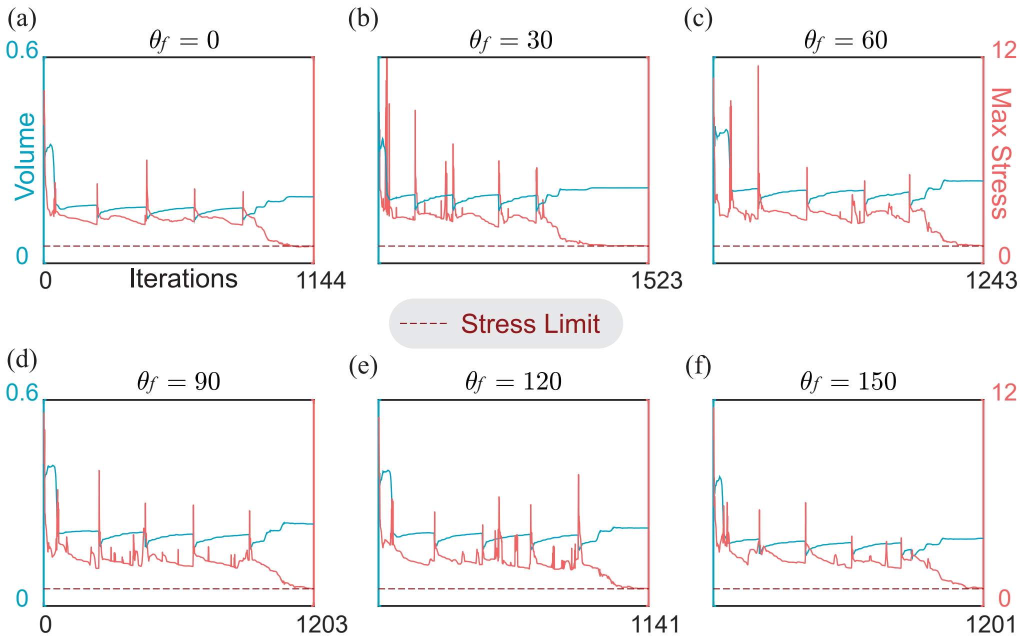

Figure S2. Optimization convergence plots, displaying the volume (objective function) in blue, and maximum stress on the structure (constraints) in red, for the Double L-bracket considering one load varying in direction with angle θ_1 combined with a simultaneous fixed load (case 2 presented in Section 5(b) of the main manuscript), corresponding to the results displayed in Fig. 13 of the main manuscript.

The partial derivative of the constraint with respect to the design variable is simple to compute, and it can be written as:

$$\frac{\partial g_i}{\partial z_j} = p\tilde{\rho}_i^{p-1}\left[(\sigma_j^v/\sigma_{\text{lim}} - 1)^2 + 1(\sigma_j^v/\sigma_{\text{lim}} - 1)\right]\frac{\partial \rho_i}{\partial z_j} \tag{S4.3}$$

Next, if we expand the second term with the derivation of the constraint in respect to the von Mises stress, we obtain:

$$\frac{\partial g_i}{\partial \tilde{\sigma}_i^v}\frac{\partial \tilde{\sigma}_i^v}{\partial z_j} = \frac{\rho_j^p}{\sigma_{\text{lim}}}\left[2(\sigma_j^v/\sigma_{\text{lim}} - 1) + 1\right]\frac{\partial \tilde{\sigma}_i^v}{\partial (\sigma_c)_i}\frac{\partial (\sigma_c)_i}{\partial z_j}, \tag{S4.4}$$

where the term $(\sigma_c)_i$ represent the stress components of each sub-case, in which the index c represent the associated stress component, and the index i represent the associated constraint. For each different case displayed in Section 5 of the main manuscript, we have a different expression for $\partial\tilde{\sigma}_i^v/\partial(\sigma_c)_i$, therefore, Sections S5(a), (b), and (c), will present the derivation of this term for each of the cases 1, 2, and 3. The derivation of the term $\partial(\sigma_c)_i/\partial z_j$ is analogous for all cases, and, because of this, its derivation will be presented in Section S5(d).

(a) Sensitivity of case 1: Planar load varying in an arbitrary range

In this Section, we will display the sensitivity for case 1 (Section 5(a) of the main manuscript). We decompose the derivative of $\tilde{\sigma}^v$ using the chain rule as:

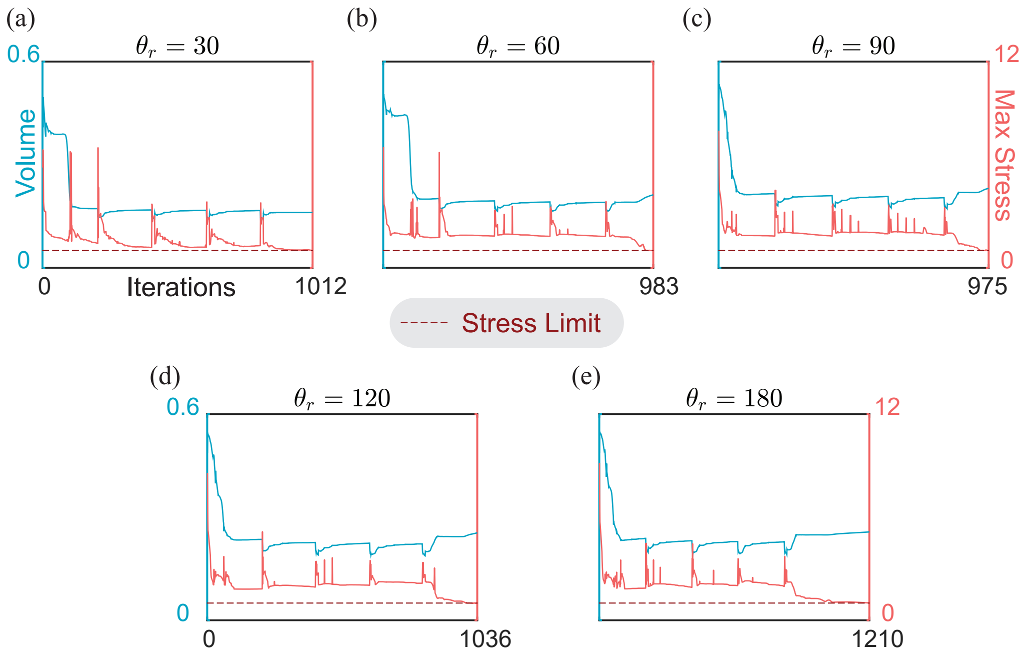

Figure S3. Optimization convergence plots, displaying the volume (objective function) in blue, and maximum stress on the structure (constraints) in red, for the Double L-bracket considering two loads varying independently in direction (case 3 presented in Section 5(c) of the main manuscript), corresponding to the results displayed in Fig. 14 of the main manuscript.

$$\frac{\mathrm{d}\widetilde{\sigma}_i^v}{\mathrm{d}(\sigma_c)_i} = \frac{\partial\widetilde{\sigma}_i^v}{\partial(\sigma_c)_i} + \frac{\partial\widetilde{\sigma}_i^v}{\partial\theta^\star}\frac{\partial\theta^\star}{\partial(\sigma_c)_i} \tag{S4.5}$$

However, notice that $(\partial\widetilde{\sigma}_i^v/\partial\theta^\star)(\partial\theta^\star/\partial(\sigma_c)_i) = 0$, because if $\theta^\star = -\theta_r$ or $\theta^\star = \theta_r$ then $\theta^\star$ does not depend on z_j, and if $\theta^\star = \theta_{max}^{cr}$ then $\theta^\star$ is a critical point of the expression for the von Mises stress, and, consequently, $\partial\widetilde{\sigma}_i^v/\partial\theta^\star = 0$ by the definition of θ_{max}^{cr}. Therefore, we only need to compute:

$$\frac{\partial\widetilde{\sigma}_i^v}{\partial(\sigma_c)_i} = \frac{1}{2\widetilde{\sigma}_i^v}\left[\sin(2\,\theta^\star)\frac{\partial t_{xy}}{\partial(\sigma_c)_i} + 0.5(1+\cos(\theta^\star))\frac{\partial t_{xx}}{\partial(\sigma_c)_i} + 0.5(1-\cos(\theta^\star))\frac{\partial t_{yy}}{\partial(\sigma_c)_i}\right] \tag{S4.6}$$

(b) Sensitivity of case 2: Planar load varying in direction plus a fixed load

This Section presents the derivation of the sensitivity of case 2 (Section 5(b) of the main manuscript), a planar load varying 360^o degrees in direction plus a fixed load. We decompose the derivative of $\widetilde{\sigma}^v$ using the chain rule:

$$\frac{\mathrm{d}\widetilde{\sigma}_i^v}{\mathrm{d}(\sigma_c)_i} = \frac{\partial\widetilde{\sigma}_i^v}{\partial(\sigma_c)_i} + \frac{\partial\widetilde{\sigma}_i^v}{\partial u^\star}\frac{\partial u^\star}{\partial(\sigma_c)_i} \tag{S4.7}$$

Similar to the previous case, we have that$\partial\widetilde{\sigma}_i^v/\partial u^\star = 0$, because $u^\star$ is a critical point of the expression for the von Mises stress by definition. Therefore, the sensitivity for this case reduces to:

$$\frac{\partial \widetilde{\sigma}_i^v}{\partial (\sigma_c)_i} = \frac{1}{2\widetilde{\sigma}_i^v \left[(u^\star)^2+1\right]^2} \left\{ \left[(u^\star)^4 - 2\,(u^\star)^2 + 1\right] \frac{\partial t_{xx}}{\partial (\sigma_c)_i} + 4\,(u^\star)^2 \frac{\partial t_{yy}}{\partial (\sigma_c)_i} \right.$$
$$+ \left[(u^\star)^4 + 2\,(u^\star)^2 + 1\right] \frac{\partial t_{ff}}{\partial (\sigma_c)_i} - 4\,\left[(u^\star)^3 - (u^\star)\right] \frac{\partial t_{xy}}{\partial (\sigma_c)_i} \tag{S4.8}$$
$$\left. -2\,\left[(u^\star)^4 - 1\right] \frac{\partial t_{xf}}{\partial (\sigma_c)_i} + 4\,\left[(u^\star)^3 + (u^\star)\right] \frac{\partial t_{yf}}{\partial (\sigma_c)_i} \right\}$$

(c) Sensitivity of case 3: Multiple Planar loads varying independently

This Section presents the derivation of the sensitivity of case 3 (Section 5(c) of the main manuscript), in which we have multiple loads varying independently in direction. In order to simplify the expression, we will split the computation of the sensitivity, and consider the sensitivity of ξ_1, ξ_2 and ξ_{12} separately. The expressions for ξ_1 and ξ_2 are identical to the expression for a single load varying in direction for which the sensitivity has already been computed in Section S5(a). Therefore, we refrain from repeating it here, and we will focus on the sensitivity of ξ_{12}. We have that:

$$\frac{\mathrm{d}\xi_{12}}{\mathrm{d}(\sigma_c)_i} = \frac{\partial \xi_{12}}{\partial (\sigma_c)_i} + \frac{\partial \xi_{12}}{\partial u^\star}\frac{\partial u^\star}{\partial (\sigma_c)_i} + \frac{\partial \xi_{12}}{\partial v^\star}\frac{\partial v^\star}{\partial (\sigma_c)_i} \tag{S4.9}$$

Similar to previous cases, $u^\star$ and $v^\star$ are critical points of the expression for ξ_{12}, and, consequently, $\partial\xi_{12}/\partial u^\star = 0$ and $\partial\xi_{12}/\partial v^\star = 0$. The remaining term reads:

$$\frac{\mathrm{d}\xi_{12}}{\mathrm{d}(\sigma_c)_i} = \frac{\partial \xi_{12}}{\partial (\sigma_c)_i} = \left[\cos\left(u^\star\right) + \cos\left(v^\star\right)\right] \frac{\partial s_{xx}}{\partial (\sigma_c)_i} + \left[\cos\left(v^\star\right) - \cos\left(u^\star\right)\right] \frac{\partial s_{yy}}{\partial (\sigma_c)_i} +$$
$$\left[\sin\left(u^\star\right) - \sin\left(v^\star\right)\right] \frac{\partial s_{xy}}{\partial (\sigma_c)_i} + \left[\sin\left(u^\star\right) + \sin\left(v^\star\right)\right] \frac{\partial s_{yx}}{\partial (\sigma_c)_i} \tag{S4.10}$$

The full expression of the sensitivity for this case is:

$$\frac{\mathrm{d}\widetilde{\sigma}_i^v}{\mathrm{d}(\sigma_c)_i} = \frac{1}{2\widetilde{\sigma}_i^v} \left[\frac{\partial \xi_1}{\partial (\sigma_c)_i} + \frac{\partial \xi_2}{\partial (\sigma_c)_i} + \frac{\partial \xi_{12}}{\partial (\sigma_c)_i}\right] \tag{S4.11}$$

where the expressions for $\frac{\partial \xi_1}{\partial (\sigma_c)_i}$ and $\frac{\partial \xi_2}{\partial (\sigma_c)_i}$ are the same as the expression for the sensitivity of case 1 (Section S5(a)).

(d) Sensitivity of the Stress Components

In this section, we will compute the sensitivity of the quadratic stress terms (e.g. $\partial\, t_{xx}/\partial\, \sigma_c$, $\partial\, t_{yy}/\partial\, \sigma_c$ and $\partial\, t_{xy}/\partial\, \sigma_c$ for case 1). The sensitivity of the quadratic stress terms follows a basic formula for all the cases displayed in this work. Therefore, the sensitivity for an arbitrary quadratic stress term is derived here. Let $\varUpsilon_{ab}$ be the quadratic stress terms generated by $\varUpsilon_{ab} = \sigma_a \mathbf{V} \sigma_b$, e.g. in case 1, $\varUpsilon_{xy} = t_{xy} = \sigma_x \mathbf{V} \sigma_y$. Then we have that:

$$\frac{\partial \varUpsilon_{ab}}{\partial \sigma_c} = \frac{\partial \sigma_a}{\partial \sigma_c}\mathbf{V}\sigma_b + \sigma_a \mathbf{V}\frac{\partial \sigma_b}{\partial \sigma_c} \tag{S4.12}$$

in which σ_c is a stress component. Now we focus on the derivation of the stress component in respect to the design variable, written as:

$$\frac{\partial (\sigma_c)_i}{\partial z_j} = \frac{\partial (\sigma_c)_i}{\partial U_{lm}}\frac{\partial U_{lm}}{\partial z_j} \tag{S4.13}$$

where the index c of the variable σ_c represents the stress component associated with that variable, and the outermost index i of $(\sigma_c)_i$ represents the constraint associated with the i-th stress constraint. Furthermore, the first index l of the term U_{lm} represents the displacement generated by the load basis vector F_l, while the second index m represents the degrees of freedom of the displacement vector. The term $\frac{\partial U_{lm}}{\partial z_k}$ is computed using the adjoint method. In the adjoint method, we differentiate the equilibrium equations associated with our problem considering all the load basis vectors:

$$\frac{\partial}{\partial z_j}\left(K_{nl}U_{lm} - F_{nm}\right) = \frac{\partial K_{nl}}{\partial z_j}U_{lm} + K_{nl}\frac{\partial U_{lm}}{\partial z_j} = 0 \tag{S4.14}$$

Because the above Eq. is equal to zero, we can multiply it by a factor $\boldsymbol{\xi}$, and add it to our sensitivity equation (Eq. S4.1), without altering its value. For simplicity, we only show the relevant terms of Eq. (S4.1), i.e. we ignore the terms related to the objective function and the partial derivative $\frac{\partial g_i}{\partial z_j}$, and focus only on the terms related to $\frac{\partial g_i}{\partial \tilde{\sigma}_i^v}\frac{\partial \tilde{\sigma}_i^v}{\partial z_j}$. So, we have:

$$\frac{\partial J^{(k)}}{\partial \tilde{\sigma}_i^v}\frac{\partial \tilde{\sigma}_i^v}{\partial z_j} = \sum_{i=1}^{N_e}(\lambda_i^{(k)} + \mu^{(k)}g_i)\frac{\partial g_i}{\partial \tilde{\sigma}_i^v}\frac{\partial \tilde{\sigma}_i^v}{(\partial \sigma_c)_i}\frac{\partial (\sigma_c)_i}{\partial U_{lm}}\frac{\partial U_{lm}}{\partial z_j} + \boldsymbol{\xi}_{nm}\left(\frac{\partial K_{nl}}{\partial z_j}U_{lm} + K_{nl}\frac{\partial U_{lm}}{\partial z_j}\right) \tag{S4.15}$$

If we collect the terms containing $\frac{\partial U_{lm}}{\partial z_j}$:

$$\frac{\partial J^{(k)}}{\partial \tilde{\sigma}_i^v}\frac{\partial \tilde{\sigma}_i^v}{\partial z_j} = \left[\sum_{i=1}^{N_e}(\lambda_i^{(k)} + \mu^{(k)}g_i)\frac{\partial g_i}{\partial \tilde{\sigma}_i^v}\frac{\partial \tilde{\sigma}_i^v}{(\partial \sigma_c)_i}\frac{\partial (\sigma_c)_i}{\partial U_{lm}} + \boldsymbol{\xi}_{nm}K_{nl}\right]\frac{\partial U_{lm}}{\partial z_j} + \boldsymbol{\xi}_{nm}\frac{\partial K_{nl}}{\partial z_j}U_{lm} \tag{S4.16}$$

We can then set the value of factor $\boldsymbol{\xi}_{nm}$ so that the term in brackets becomes zero. In order to do that we have to solve the Eq.:

$$K_{nl}\boldsymbol{\xi}_{nm} = -\sum_{i=1}^{N_e}(\lambda_i^{(k)} + \mu^{(k)}g_i)\frac{\partial g_i}{\partial \tilde{\sigma}_i^v}\frac{\partial \tilde{\sigma}_i^v}{(\partial \sigma_c)_i}\frac{\partial (\sigma_c)_i}{\partial U_{lm}} \tag{S4.17}$$

Therefore, we have to solve the system K_{nl} of linear equations for m right-hand sides, where m is the number of load basis vectors, which depends on the load case of interest. Once we have calculated $\boldsymbol{\xi}_{nm}$, we can compute this part of the sensitivity as:

$$\frac{\partial J^{(k)}}{\partial \tilde{\sigma}_i^v}\frac{\partial \tilde{\sigma}_i^v}{\partial z_j} = \boldsymbol{\xi}_{nm}\frac{\partial K_{nl}}{\partial z_j}U_{lm} \tag{S4.18}$$

S5. Minimization of compliance with multiple load directions

This section presents a brief description of how to extend the formulation for compliance minimization. The compliance minimization optimization statement is presented in Eq. (S5.1):

$$\begin{aligned}
\min_{\mathbf{z}} \quad & C(\mathbf{z},\boldsymbol{\theta}) = \mathbf{U}^T\mathbf{K}(\mathbf{z})\mathbf{U} \\
\text{s.t.} \quad & g_V(\mathbf{z}) \leq 0 \\
& 0 \leq z_e \leq 1, \quad e = 1,\ldots,N_e \\
\text{with:} \quad & \mathbf{K}(\mathbf{z})\mathbf{U} = \mathbf{F}(\boldsymbol{\theta})
\end{aligned} \tag{S5.1}$$

in which, $C(\mathbf{z})$ is the compliance, and $g_V(\mathbf{z})$ is the volume constraint. Similar to the stress constraint formulation, the load direction in the compliance formulation also depends on a variable θ. However, instead of stress constraints, which limit the worst-case stress for any load direction possible, now we minimize the worst-case compliance for any load direction possible. Here, we will derive the formulation for the equivalent of the previously described case 1 of load variation (Section 4(a) of the main manuscript), in which we have a single load that can vary 360^o in direction. The other load cases can be derived following the same approach. The derivations for the worst-case compliance start in the same way as the stress, by decomposing the load into linearly independent components:

$$\mathbf{F}(\theta) = \mathbf{F}_x \cos(\theta) + \mathbf{F}_y \sin(\theta) \tag{S5.2}$$

we then replace this expression for the loads in the equilibrium equation:

$$\mathbf{U} = \mathbf{K}^{-1}\mathbf{F}(\theta) = \left(\mathbf{K}^{-1}\mathbf{F}_x\right)\cos(\theta) + \left(\mathbf{K}^{-1}\mathbf{F}_y\right)\sin(\theta) \tag{S5.3}$$

By defining $\mathbf{U}_x = \left(\mathbf{K}^{-1}\mathbf{F}_x\right)$ and $\mathbf{U}_y = \left(\mathbf{K}^{-1}\mathbf{F}_y\right)$, we can compute the compliance as:

$$C(\mathbf{z}, \boldsymbol{\theta}) = \left[(\mathbf{U}_x)\cos(\theta) + (\mathbf{U}_y)\sin(\theta)\right]^T \mathbf{K}(\mathbf{z}) \left[(\mathbf{U}_x)\cos(\theta) + (\mathbf{U}_y)\sin(\theta)\right] \tag{S5.4}$$

To simplify the expression in Eq. S5.4 we define the quadratic compliance terms $t_{xx} = \mathbf{U}_x^T\mathbf{K}(\mathbf{z})\mathbf{U}_x$, $t_{yy} = \mathbf{U}_y^T\mathbf{K}(\mathbf{z})\mathbf{U}_y$, and $t_{xy} = \mathbf{U}_x^T\mathbf{K}(\mathbf{z})\mathbf{U}_y$, which we then substitute in Eq. (S5.4):

$$C(\mathbf{z}, \boldsymbol{\theta}) = t_{xx}\ \cos^2(\theta) + t_{yy}\ \sin^2(\theta) + 2t_{xy}\ \cos(\theta)\sin(\theta) \tag{S5.5}$$

We simplify this equation even further using trigonometric identities:

$$C(\mathbf{z}, \boldsymbol{\theta}) = t_{xy}\sin(2\,\theta) + 0.5\left[\left(t_{xx} - t_{yy}\right)\cos(2\,\theta) + t_{xx} + t_{yy}\right] \tag{S5.6}$$

and we finally obtain the optimization problem for the worst-case compliance:

$$\begin{aligned} \max_{\theta\in\Gamma}\ \ & C(\mathbf{z}, \boldsymbol{\theta}) = t_{xy}\sin(2\,\theta) + 0.5\left[\left(t_{xx} - t_{yy}\right)\cos(2\,\theta) + t_{xx} + t_{yy}\right] \\ \text{with:}\ \ & \mathbf{K}(\mathbf{z})\mathbf{U}_x = \mathbf{F}_x \\ & \mathbf{K}(\mathbf{z})\mathbf{U}_y = \mathbf{F}_y \end{aligned} \tag{S5.7}$$

which is exactly the same as the one in Eq. (5.9) of the main manuscript, for which the solution is:

$$\theta^\star = \theta^{cr}_{max} = \frac{1}{2}\tan^{-1}\left(2\,t_{xy},\ t_{xx} - t_{yy}\right) \tag{S5.8}$$

References

[S1] F. V. Senhora, O. Giraldo-Londono, I. F. Menezes and G. H. Paulino, Topology optimization with local stress constraints: a stress aggregation-free approach, *Structural and Multidisciplinary Optimization*, 2020, **62**, 1639–1668.

[S2] K. K. Choi and N.-H. Kim, *Structural sensitivity analysis and optimization 1: linear systems*, Springer Science & Business Media, 2004.